PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 13, Fasc. 2 (1992), pp. 269–276

EQUIVALENT CONDITIONS FOR THE CONSISTENCY OF NONPARAMETRIC SPLINE DENSITY ESTIMATORS

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Abstract. We study the nonparametric spline density estimators of probability density. The equivalence of weak convergence for L_1-consistency of one density and completely for L_1-consistency of all densities is proved. It is equivalent also to suitable rates of convergence of window parameter.

1. Introduction. For $r \geqslant 1$, let $N^{(r)}(x) = r\cdot[0, \ldots, r; (\cdot - x)_+^{r-1}]$, $x \in \boldsymbol{R}$, be the r-th order B-spline associated with the knots $0, \ldots, r$. Here, for $f\colon \boldsymbol{R}^1 \to \boldsymbol{R}^1$, $[s_0, \ldots, s_r; f]$ denotes the divided difference of f taken at the points $s_0, \ldots, s_r$. The B-spline $N^{(r)}$ has the support $[0, r]$, it is a piecewise polynomial of degree $r-1$, and it is of class C^{r-2} for $r \geqslant 2$. By translation and scaling of $N^{(r)}$ we can obtain a B-spline basis for any equally spaced set of knots on $\boldsymbol{R}$. Let $t_i = (i+\theta)h$ for $i \in \boldsymbol{Z} = \{0, \mp 1, \mp 2, \ldots\}$, $h \in \boldsymbol{R}_+ = (0, +\infty)$, and $\theta = 0$ if r is even and $\theta = \frac{1}{2}$ if r is odd. We set

$$N_{i,h}^{(r)}(x) = N^{(r)}\big((x-t_i)/h\big) \quad \text{for } x \in \boldsymbol{R} \text{ and } i \in \boldsymbol{Z}.$$

The spline operator considered in the note will be defined by the kernel $Q_h^{(r)}\colon \boldsymbol{R}^2 \to \boldsymbol{R}^1$, introduced by Ciesielski [1], where

$$Q_h^{(r)}(x, y) = \sum_{s \in \boldsymbol{Z}} h^{-1} N_{s,h}^{(r)}(x)\cdot N_{s,h}^{(r)}(y), \quad (x, y) \in \boldsymbol{R}^2,\ r \geqslant 1,\ h \in \boldsymbol{R}_+.$$

The kernels $Q_h^{(r)}$ are local due to the following property:

$$Q_h^{(r)}(x, y) = 0 \quad \text{if } |x-y| > rh. \tag{1.1}$$

They are also bounded:

$$0 \leqslant Q_h^{(r)}(x, y) \leqslant 1/h, \quad (x, y) \in \boldsymbol{R}^2. \tag{1.2}$$

We now assume that we are given a probability space $(\Omega, \mathscr{F}, \Pr)$ and a simple sample of size n, i.e., a sequence $X_1, \ldots, X_n$ of i.i.d. real-valued random variables such that their common distribution has density f.

The *spline density estimator* is defined by

$$(1.3)\qquad f_{n,h}(x) = n^{-1} \sum_{j=1}^{n} Q_h^{(r)}(x, X_j), \qquad x \in \mathbf{R}^1.$$

The estimator $f_{n,h}$ is neither of kernel nor of series type, but it has some properties of estimators of both these types. In particular, it is local like kernel estimators (cf. (1.1)). We can see the relationship with the series type estimators if we write the definition formula in the following form:

$$(1.4)\qquad f_{n,h}(x) = \sum_{s \in \mathbf{Z}} a_{s,h} N_{s,h}^{(r)}(x), \qquad \text{where } a_{s,h} = n^{-1} \sum_{j=1}^{n} h^{-1} N_{s,h}^{(r)}(x).$$

In other words, $f_{n,h}$ is a linear combination of B-splines.

Asymptotic properties of a large family of such estimators in several variables are discussed in [3].

Another look at the linear combination of B-splines as the density estimators can be found in [7].

2. The Theorem. Before formulating the main result of our paper (Theorem A) let us introduce the following notation:

Random variables J_n, $n \in N$, are said to be *exponentially convergent to zero*, in symbols $J_n \to 0$ exponentially, if for each positive $\varepsilon > 0$ there exist $b > 0$ and n_0 such that for all n greater than n_0

$$\Pr\{|J_n| > \varepsilon\} \leqslant \exp\{-bn\}.$$

Theorem A. *Let $f_{n,h}$ be the spline density estimator defined by* (1.3). *Moreover, let $J_n = \int |f_{n,h} - f|$, $h = h_n$, $n \in N$, be the sequence of $\mathbf{L}_1$-distances between f and f_{n,h_n}. Then the following statements are equivalent*:

(i) $h_n \to 0$ *and* $nh_n \to \infty$;

(ii) *for some density f, $J_n \to 0$ in probability*;

(iii) *$J_n \to 0$ in probability, for all densities f*;

(iv) *$J_n \to 0$ with probability one, for all f*;

(v) *$J_n \to 0$ exponentially, for all f.*

An analogue of Theorem A for kernel density estimators was obtained by Devroye [4].

We will try to extract the key facts used in the proof of Theorem A. They are collected in several lemmas, which are of independent interest.

Lemma 1. *Let Ψ be the family of disjoint intervals in $\mathbf{R}^1$. Suppose that, for given $m > 0$ and for all $B \in \Psi$, $|B| = m > 0$, where $|B|$ denotes the Lebesgue measure of the set B. Let μ_n be the empirical probability measure for the sample $X_1, \ldots, X_n$, and let μ be the probability measure of density f. For every $\varepsilon > 0$ there exist positive λ_1, λ_2 and λ_3 such that for all $n \in N$*

$$\Pr\Big\{\sum_{B \in \Psi} |\mu_n(B) - \mu(B)| > \varepsilon\Big\} \leqslant \exp\{-n\lambda_1 + m^{-1}\lambda_2 + \lambda_3\}.$$

Proof. Denote by $S_{x,R}$ the closed interval $[x-R, x+R]$, where $x \in \mathbf{R}$, $R \in \mathbf{R}_+$, and by $S_{x,R}^c$ its complement in $\mathbf{R}^1$. Choose an $R > 0$ such that $\mu(S_{0,R}^c) < \varepsilon$. Divide Ψ into two subfamilies:

$$\Psi_1 = \{B \in \Psi\colon B \cap S_{0,R} \neq \emptyset\} \quad \text{and} \quad \Psi_2 = \{B \in \Psi\colon B \cap S_{0,R} = \emptyset\}.$$

Note that Ψ_1 is finite. Now,

$$\sum_{B \in \Psi_2} |\mu_n(B) - \mu(B)| = 2 \sum_{B \in \Psi_2, \mu(B) \geqslant \mu_n(B)} (\mu(B) - \mu_n(B)) + \sum_{B \in \Psi_2} (\mu_n(B) - \mu(B)).$$

Let us estimate the second term:

$$\sum_{B \in \Psi_2} (\mu_n(B) - \mu(B)) \leqslant \mu_n(S_{0,R}^c) - \sum_{B \in \Psi_2} \mu(B) \pm \mu(S_{0,R}^c)$$

$$\leqslant |\mu_n(S_{0,R}^c) - \mu(S_{0,R}^c)| + \mu(S_{0,R}^c).$$

Hence

$$\sum_{B \in \Psi_2} |\mu_n(B) - \mu(B)| \leqslant 3\mu(S_{0,R}^c) + |\mu_n(S_{0,R}^c) - \mu(S_{0,R}^c)|.$$

The elementary inequality $\ln(1+x) \geqslant 2x/(2+x)$ for $x > 0$ implies the modification of Hoeffding's inequality [5, Theorem 3]: If $\xi_1, \xi_2, \ldots, \xi_n$ are independent, $E\xi_i = 0$, $\xi_i \leqslant b$, $E\xi_i^2 = \sigma^2$ $(i = 1, 2, \ldots, n)$, then for $0 < \varepsilon < b$

$$\text{(2.1)} \qquad \Pr\left\{\left|\frac{1}{n}\sum_{i=1}^{n} \xi_i\right| \geqslant \varepsilon\right\} \leqslant 2\exp\left\{-\frac{n\varepsilon^2}{2\sigma^2 + b\varepsilon}\right\}.$$

By this inequality for Bernoulli random variables we have

$$\text{(2.2)} \qquad \Pr\{\sum_{B \in \Psi_2} |\mu_n(B) - \mu(B)| > 4\varepsilon\} \leqslant \Pr\{|\mu_n(S_{0,R}^c) - \mu(S_{0,R}^c)| > \varepsilon\}$$

$$\leqslant 2\exp\left\{\frac{-2n\varepsilon^2}{2+\varepsilon}\right\}.$$

Let $\Psi_1 = \{B_i\}_{i=1}^k$. Since $|\bigcup_{i=1}^k B_i| \leqslant 2R + 2m$, we have $k \leqslant 2R/m + 2$. For $i \in \{1, \ldots, k\}$ we denote by Y_i the number of elements of the sample $X_1, \ldots, X_n$ contained in the set B_i. It is easy to see that

$$\Pr\{\sum_{B \in \Psi_1} |\mu_n(B) - \mu(B)| > \varepsilon\} = \Pr\{n^{-1} \sum_{j=1}^{k} |Y_j - E(Y_j)| > \varepsilon\}.$$

From the equality

$$\sum_{i=1}^{k} |Y_i/n - E(Y_i/n)| = 2\sup_A \left|\sum_{i \in A} [Y_i/n - E(Y_i/n)]\right|,$$

where A is the set of all subsets of the set $\{1, \ldots, k\}$, we obtain

$$\Pr\{\sum_{B \in \Psi_1} |\mu_n(B) - \mu(B)| > \varepsilon\} \leqslant 2^k \sup_A \Pr\{\left|\sum_{i \in A} [Y_i - E(Y_i)]\right| > n\varepsilon/2\}.$$

The random variable $\sum_{i\in A} Y_i$ has the binomial (n, p_A) distribution, where $p_A = \Pr\{X_1 \in \bigcup_{i\in A} B_i\}$. Then, again, by inequality (2.1) for the Bernoulli random variable we have

$$\Pr\{\sum_{B\in\Psi_1} |\mu_n(B)-\mu(B)| > \varepsilon\} \leqslant 2^{k+1}\exp\left\{-\frac{n\varepsilon^2}{2(1+\varepsilon)}\right\}.$$

Since $k \leqslant 2R/m+2$, we come to the statement

$$\Pr\{\sum_{B\in\Psi_1} |\mu_n(B)-\mu(B)| > \varepsilon\} \leqslant \exp\{-n\varepsilon_1+\varepsilon_2\}, \tag{2.3}$$

where $\varepsilon_1 = \varepsilon^2/2(1+\varepsilon)$ and $\varepsilon_2 = (2R/m+3)\ln(2)$. Combining (2.2) and (2.3) we get the desired conclusion.

The next two lemmas concern the behavior of the spline operator $Q_h^{(r)}$: $L_1 \to L_1$ given by the formula

$$Q_h^{(r)}(f)(x) = \int f(y)\, Q_h^{(r)}(x, y)\,dy \quad \text{for } x\in \mathbf{R}. \tag{2.4}$$

LEMMA 2 (Ciesielski [2]). *Let $Q_h^{(r)}$ be the above spline operator. Then*

$$\|Q_h^{(r)}(f)-f\|_1 \leqslant 8(2r^2+1)\,\omega_{2,1}(f, h) \quad \text{for } 0<h<1,$$

where $\omega_{2,1}(f, h) = \sup_{|t|<h} \|\Delta_t^2 f\|_1$ and Δ_t^2 is the 2-nd order progressive difference with step t.

LEMMA 3 (nonexistence of unbiased spline density estimators). *Let $f\in L^1(\mathbf{R})$, $f \neq 0$, and let $Q_h^{(r)}$ be the spline operator given by formula* (2.4) *with $r \geqslant 2$. Then $\int |Q_h^{(r)}(f)-f| > 0$ for each $h\in \mathbf{R}_+$, and if $\{h_n\colon n\in \mathbf{N}\}$ is a sequence of positive numbers, then*

$$\lim_{n\to\infty} \int |Q_{h_n}^{(r)}(f)-f| = 0 \text{ implies } h_n \to 0.$$

Proof. Suppose to the contrary that $\|Q_h^{(r)}(f)-f\|_1 = 0$ for some $h\in\mathbf{R}_+$ and probability density f. Thus, for almost all $x\in \mathbf{R}^1$,

$$f(x) = \sum_{s\in\mathbf{Z}} a_{s,h} N_{s,h}^{(r)}(x), \quad \text{where } a_{s,h} = h^{-1}\int f(y) N_{s,h}^{(r)}(y)\,dy.$$

By the stability inequality (see [6]) we have $\{a_{s,h};\ s\in\mathbf{Z}\} \in l_1(\mathbf{Z})$. Let us compare $Q_h^{(r)}(f)$ to f:

$$f(x) = Q_h^{(r)}(f)(x) = Q_h^{(r)}\big(\sum_{s\in\mathbf{Z}} a_{s,h} N_{s,h}^{(r)}\big)(x) = \sum_{s'\in\mathbf{Z}} \big(\sum_{s\in\mathbf{Z}} a_{s,h}\, g_{s'-s,h}\big) N_{s',h}^{(r)}(x),$$

where $g_{s'-s,h} = h^{-1}(N_{s,h}^{(r)}, N_{s',h}^{(r)})$ depends on $s'-s$ and h, only. Since the sequence of splines $\{N_{s,h}^{(r)}\}_{s\in\mathbf{Z}}$ forms a basis in the space of splines of order r with simple knots $\{t_s;\ s\in\mathbf{Z}\}$, we get for all $s'\in\mathbf{Z}$

$$a_{s',h} = \sum_{s\in\mathbf{Z}} a_{s,h}\, g_{s'-s,h}. \tag{2.5}$$

Let us introduce, for $t \in \mathbf{R}^1$,

$$T_h(t) = \sum_{s \in \mathbf{Z}} a_{s,h} \exp\{ist\}.$$

By (2.5) we obtain

$$T_h(t) = \sum_{s \in \mathbf{Z}} \Big(\sum_{s' \in \mathbf{Z}} a_{s',h} g_{s-s',h} \Big) \exp\{ist\} = \Big(\sum_{s' \in \mathbf{Z}} a_{s',h} \exp\{is't\} \Big) G(t) = T_h(t) G(t),$$

where $G(t) = \sum_{u=-r+1}^{r-1} g_{u,h} \exp\{itu\}$. Then for each $t \in \mathbf{R}^1$ we get $T_h(t)(1-G(t)) = 0$. Since G is a nontrivial trigonometric polynomial and $G(t) \neq 1$ for almost all $t \in \mathbf{R}^1$, it follows that $T_h = 0$ and, consequently, that $f = 0$, which is impossible.

For the second statement of Lemma 3 suppose that $\|Q_{h_n}^{(r)}(f) - f\|_1$ converges to zero as $n \to \infty$. In the case where h_n tends to infinity as $n \to \infty$, using Fatou's lemma and (1.2) we have

$$\liminf_{n \to \infty} \|Q_{h_n}^{(r)}(f) - f\|_1 \geqslant \|\liminf_{n \to \infty} |Q_{h_n}^{(r)}(f) - f|\,\|_1 = \|f\|_1 = 1.$$

If $\{h_n\colon n \in \mathbf{N}\}$ has a subsequence convergent to a finite limit, for simplicity, let $h_n \to \delta > 0$; then

$$\liminf_{n \to \infty} \|Q_{h_n}^{(r)}(f) - f\|_1 \geqslant \|Q_\delta^{(r)}(f) - f\|_1 > 0,$$

and the proof is complete.

3. Proof of Theorem A. The proof will be established by proving two lemmas. The first states the implication (i) $\Rightarrow$ (v), the second one the implication (ii) $\Rightarrow$ (i). The remaining implications are clear.

LEMMA 4. *Let $\{h_n\colon n \in \mathbf{N}\}$ be a sequence such that*

$$\lim_{n \to \infty} h_n + (nh_n)^{-1} = 0$$

and let $f_{n,h}$ be given by formula (1.3). *Suppose that the simple sample has a density f. Then $J_n = \|f_{n,h_n} - f\|_1$ tends to zero exponentially.*

Proof of Lemma 4. For given $\varepsilon > 0$ find finite positive constants $v, a_1, \ldots, a_v$ and v disjoint finite intervals $A_1, \ldots, A_v$ in $[0, r]$ of equal length such that the function $\bar{N}^{(r)} = \sum_{i=1}^{v} a_i \chi_{A_i}$ satisfies

$$\|N^{(r)} - \bar{N}^{(r)}\|_{\sup} \leqslant \varepsilon, \qquad \|N^{(r)} - \bar{N}^{(r)}\|_1 \leqslant \varepsilon \quad \text{and} \quad \|\bar{N}^{(r)}\|_1 \leqslant 1.$$

For $x \in \mathbf{R}$ and $h \in \mathbf{R}_+$, we introduce the following notation:

$$\bar{f}_{n,h}(x) = n^{-1} \sum_{j=1}^{n} \bar{Q}_h^{(r)}(x, X_j) \quad \text{and} \quad \bar{Q}_h^{(r)}(f)(y) = \int f(x) \bar{Q}_h^{(r)}(x, y)\,dx,$$

where

$$\bar{Q}_h^{(r)}(x, y) = h^{-1} \sum_{s\in \boldsymbol{Z}} \bar{N}^{(r)}(x/h-s)\,\bar{N}^{(r)}(y/h-s) \qquad \text{for } (x, y)\in \boldsymbol{R}^2.$$

Now,

$$\text{(3.1)} \qquad \|f_{n,h_n}-f\|_1 \leqslant \|f_{n,h_n}-\bar{f}_{n,h_n}\|_1+\|\bar{f}_{n,h_n}-\bar{Q}_{h_n}^{(r)}(f)\|_1 + \|\bar{Q}_{h_n}^{(r)}(f)-Q_{h_n}^{(r)}(f)\|_1+\|Q_{h_n}^{(r)}(f)-f\|_1.$$

Let F_n be the empirical distribution of the simple sample $X_1, \ldots, X_n$. Then

$$\begin{aligned}\text{(3.2)} \quad \|f_{n,h_n}-\bar{f}_{n,h_n}\|_1 &\leqslant \int\Big(\int|Q_{h_n}^{(r)}(x, y)-\bar{Q}_{h_n}^{(r)}(x, y)|\,dx\Big)\,dF_n(y)\\ &\leqslant \int\Big[\int \sum_{s\in \boldsymbol{Z}} N^{(r)}(y/h_n-s)\,|N^{(r)}(x-s)-\bar{N}^{(r)}(x-s)|\\ &\quad + \bar{N}^{(r)}(x-s)\,|N^{(r)}(y/h_n-s)-\bar{N}^{(r)}(y/h_n-s)|\,dx\Big]\,dF_n(y)\\ &\leqslant \int \|N^{(r)}-\bar{N}^{(r)}\|_1\Big[\sum_{s\in \boldsymbol{Z}} N^{(r)}(y/h_n-s)\Big]\,dF_n(y)\\ &\quad + \int \|\bar{N}^{(r)}\|_1\,\|N^{(r)}-\bar{N}^{(r)}\|_{\sup}\,dF_n(y) \leqslant 2\varepsilon.\end{aligned}$$

In the same way, replacing the empirical distribution F_n by the distribution function of the sample F, we obtain

$$\text{(3.3)} \qquad \|Q_h^{(r)}(f)-\bar{Q}_h^{(r)}(f)\|_1 \leqslant 2\varepsilon.$$

Let us estimate $\|\bar{f}_{n,h}-\bar{Q}_h^{(r)}(f)\|_1$:

$$\begin{aligned}\text{(3.4)} \quad \|\bar{f}_{n,h}(x)-\bar{Q}_h^{(r)}(f)\|_1 &\leqslant \int \sum_{s\in \boldsymbol{Z}} h^{-1}\bar{N}^{(r)}(x/h-s)\Big|\int \bar{N}^{(r)}(y/h-s)\,d[F_n(y)-F(y)]\Big|\,dx\\ &\leqslant \sum_{s\in \boldsymbol{Z}} \Big|\int \bar{N}^{(r)}(y/h-s)\,d[F_n(y)-F(y)]\Big|\\ &= \sum_{s\in \boldsymbol{Z}} \Big|\sum_{j=1}^{\nu} a_j \int \chi_{A_j}(y/h-s)\,d[F_n(y)-F(y)]\Big|\\ &= \sum_{j=1}^{\nu} a_j \sum_{s\in \boldsymbol{Z}} \big|\mu_n\big(h(s+A_j)\big)-\mu\big(h(s+A_j)\big)\big|,\end{aligned}$$

where μ_n and μ are the measures having distributions F_n and F, respectively. Combining inequalities (3.1)–(3.4) and Lemma 2 we have

$$\Pr\{\|f_{n,h_n}-f\|_1 \geqslant (\nu a^*+6)\varepsilon\} \leqslant \sum_{j=1}^{\nu} \Pr\Big\{\sum_{s\in \boldsymbol{Z}} \big|\mu_n\big(h_n(s+A_j)\big)-\mu\big(h_n(s+A_j)\big)\big| > \varepsilon\Big\},$$

where $a^* = \max\{a_1, \ldots, a_\nu\}$.

By Lemma 1 there exist positive b_1, b_2, b_3 and $n_0\in \boldsymbol{N}$ such that

(3.5) $$\Pr\{\|f_{n,h_n}-f\|_1 \geqslant \varepsilon\} \leqslant \exp\{-nb_1+h_n^{-1}b_2+b_3\}$$

for $n > n_0$. Lemma 4 is thus proved.

LEMMA 5. *Let $f_{n,h}$ be given by formula* (1.3). *Suppose that $\{h_n: n \in N\}$ is a sequence such that $\|f_{n,h_n}-f\|_1 \to 0$ in probability as $n \to \infty$. Then $h_n \to 0$ and $nh_n \to \infty$.*

Proof. Since $J_n \leqslant 2$ for all n, we have $\lim_{n\to\infty} E(J_n) = 0$. By Jensen's inequality we obtain

$$E(J_n) \geqslant \|Ef_{n,h_n}-f\|_1 = \|Q_{h_n}^{(r)}(f)-f\|_1.$$

So $\|Q_{h_n}^{(r)}(f)-f\|_1 \to 0$ as $n \to \infty$ and, by Lemma 3, $h_n \to 0$. From the inequality

$$E(J_n) \geqslant E\|f_{n,h_n}-Q_{h_n}^{(r)}(f)\|_1 - \|f-Q_{h_n}^{(r)}(f)\|_1$$

and Lemma 2 we have

(3.6) $$\lim_{n\to\infty} E\|f_{n,h_n}-Q_{h_n}^{(r)}(f)\|_1 = 0.$$

Let, for given $x \in R$, $A_x = \{\omega: \forall i \in \{1, \ldots, n\}, X_i \notin S_{x,hr}\}$. Hence

$$\begin{aligned} E\|f_{n,h}-Q_h^{(r)}(f)\|_1 &\geqslant \int E[|f_{n,h}(x)-Q_h^{(r)}(f)(x)|\cdot\chi_{A_x}]\,dx \\ &\geqslant \int E(Q_h^{(r)}(f)(x)\cdot\chi_{A_x})\,dx - \int E(f_{n,h}(x)\cdot\chi_{A_x})\,dx. \end{aligned}$$

It is clear that $f_{n,h}(x)\cdot\chi_{A_x} = 0$ for all $x \in R^1$ and for each $\omega \in \Omega$. Thus

$$\int E(f_{n,h}(x)\cdot\chi_{A_x})\,dx = 0.$$

To estimate the first term we note that if μ is the probability measure with density f, then

$$\begin{aligned} \int E(Q_h^{(r)}(f)(x)\cdot\chi_{A_x})\,dx &= \int Q_h^{(r)}(f)(x)\cdot\Pr\{A_x\}\,dx \\ &= \int Q_h^{(r)}(f)(x)\cdot\Pr\Big\{\bigcap_{j=1}^{n}\{X_j \in S_{x,hr}^c\}\Big\}\,dx \\ &= \int Q_h^{(r)}(f)(x)\cdot\exp\{n\ln\mu(S_{x,hr}^c)\}\,dx \\ &\geqslant \int Q_h^{(r)}(f)(x)\cdot\exp\left\{-n\frac{\mu(S_{x,hr})}{1-\mu(S_{x,hr})}\right\}dx. \end{aligned}$$

If $\liminf_{n\to\infty} nh_n = s < \infty$, then by Fatou's lemma we have

$$\begin{aligned} \liminf_{n\to\infty}\int E(Q_{h_n}^{(r)}(f)(x)\cdot\chi_{A_x})\,dx &\geqslant \int \liminf_{n\to\infty} Q_{h_n}^{(r)}(f)(x)\cdot\exp\left\{-n\frac{\mu(S_{x,h_nr})}{1-\mu(S_{x,h_nr})}\right\}dx \\ &= \int f(x)\cdot\exp\{-2rsf(x)\}\,dx > 0, \end{aligned}$$

which contradicts (3.6).

Acknowledgement. I wish to thank Professor Z. Ciesielski for many useful suggestions resulted in substantial improvements in the present work.

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Received on 19.12.1990;
revised version on 13.2.1991