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A GENERALIZATION OF KAWADA AND ITÔ'S THEOREM

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Abstract. For a locally compact group G, a continuous automorphism T, and a probability measure λ on G the sequence given by $\varrho_n := \lambda T(\lambda) \ldots T^{n-1}(\lambda)$ is considered. Under the assumption that the set $\{T^{-n}: n \in N\}$ is equicontinuous it is shown that $(\varrho_n)_{n\in N}$ converges, and then necessarily to an idempotent probability measure, if and only if the support of λ is not contained in a proper coset $Hx = xT(H) \neq H$ of a compact subgroup H of G and the support of λ is contained in a compact T-invariant subgroup of G.

1. Introduction. The sequence $(\mu^n)_{n\in N}$ of the n-th convolution powers of a probability measure μ on a compact group G converges, and then necessarily to an idempotent probability measure, if and only if the support of μ is not contained in a proper coset $Hx = xH \neq H$ of a compact subgroup H of G. This is the conclusion of a well-known theorem of Kawada and Itô [6]. Clearly, this result remains true for a not necessarily compact locally compact group G if the condition is completed by requiring that the support of μ generates a compact group. In the present paper we shall provide a necessary and sufficient condition (similar to that in Kawada and Itô's theorem) on the support of a regular probability measure λ on a locally compact group G ($\lambda \in \mathscr{P}(G)$) for the convergence of the sequence given by

$$\varrho_n := \varrho_n(\lambda, T) := \lambda T(\lambda) \ldots T^{n-1}(\lambda).$$

Here T is an element of the group $\operatorname{Aut}(G)$ of continuous automorphisms of G. Clearly, for $T = \mathrm{Id}$ we are back in the situation of Kawada and Itô's theorem. We need the following

1.1. DEFINITION. For a locally compact group G, $\lambda \in \mathscr{P}(G)$, and $T \in \operatorname{Aut}(G)$ let us define

$$\mathscr{H}(\lambda, T) := \{H < G:\ H \text{ is compact, and there is a coset } Hx \text{ of } H \text{ such that } \operatorname{supp}(\lambda) \subset Hx = xT(H)\}.$$

If $\mathscr{H}(\lambda, T) \neq \emptyset$, let

$$H(\lambda, T) := \bigcap \{H \subset G:\ H \in \mathscr{H}(\lambda, T)\}.$$

Note that if $\mathscr{H}(\lambda, T) \neq \emptyset$, which is always true for compact G, $H(\lambda, T) \in \mathscr{H}(\lambda, T)$ is the smallest compact subgroup H of G such that $\operatorname{supp}(\lambda) \subset Hg = gT(H)$ for any $g \in \operatorname{supp}(\lambda)$.

1.2. LEMMA. *If $\mathscr{H}(\lambda, T) \neq \emptyset$, then the following statements are equivalent:*
(1) $\operatorname{supp}(\lambda) \subset H(\lambda, T)$.
(2) *The support of λ is not contained in a proper coset $Hx = xT(H) \neq H$ of a compact subgroup H of G.*

Proof. Clearly, if (1) does not hold, then by definition

$$\operatorname{supp}(\lambda) \subset H(\lambda, T)g = gT(H(\lambda, T)) \neq H(\lambda, T)$$

for any $g \in \operatorname{supp}(\lambda)$. Hence (2) does not hold. Conversely, if (2) does not hold, then there exists $H < G$ such that

$$\operatorname{supp}(\lambda) \subset Hg = gT(H) \neq H.$$

Since $H(\lambda, T) \subset H$ and $\operatorname{supp}(\lambda) \cap H = \emptyset$, this implies that (1) does not hold. ∎

2. The compact case for inner T. As a first step we go back to the case of compact G. We want to point out a result due to Maurer [7] or Tortrat [8] which is a slight extension of Kawada and Itô's theorem and written down in a readable way in [5]. For $\mu \in \mathscr{P}(G)$ let $H(\mu)$ denote the smallest compact subgroup of G which contains $\operatorname{supp}(\mu)$ in a coset $H(\mu)y = yH(\mu)$. Then the sequence given by $\mu^n x^{-n}$ converges to an idempotent measure, namely the normed Haar measure $\omega_{H(\mu)}$ on $H(\mu)$ for all $x \in H(\mu)y$. For our purposes we need a little bit more.

2.1. LEMMA. *Under the above assumptions $(\mu^n x^{-n})_{n\in N}$ converges, and then necessarily to $\omega_{H(\mu)}$, if and only if $x \in H(\mu)y$.*

Proof. Only the forward implication remains to prove. To do this let $x \in G$ and assume $(\mu^n x^{-n})_{n\in N}$ to be convergent. For $y \in \operatorname{supp}(\mu)$ the sequence $(y^n x^{-n})_{n\in N}$ has the limit point e. Indeed, as is well known the sequence of the n-th powers of an element of any compact group has the unit element as a limit point. In particular, the sequence $((y, x^{-1})^n)_{n\in N} \subset G \times G$ has the limit point (e, e), from which the assertion follows by continuity. Consequently, the sequence given by

$$\mu^n x^{-n} = \mu^n y^{-n} x^n y^{-n}$$

has $\omega_{H(\mu)}$ as a limit point. Hence $\mu^n x^{-n} \xrightarrow{n} \omega_{H(\mu)}$. Since

$$\mu\omega_{H(\mu)}x^{-1} = \mu\big(\lim_{n\to\infty} \mu^n x^{-n}\big)x^{-1} = \lim_{n\to\infty} \mu^{n+1} x^{-(n+1)} = \omega_{H(\mu)},$$

we obtain $yx^{-1} \in H(\mu)$ from the support formula, and hence $x \in H(\mu)y$ as required. ∎

Now we are ready to prove our theorem for an inner automorphism $T \in \operatorname{In}(G)$.

2.2. THEOREM. *Let G be a compact group, $T \in \mathrm{In}(G)$, $\lambda \in \mathscr{P}(G)$, and $\varrho_n := \lambda T(\lambda) \dots T^{n-1}(\lambda)$. Then the following statements are equivalent:*

(1) *$(\varrho_n)_{n\in N}$ converges in $\mathscr{P}(G)$.*

(2) *$(\varrho_n)_{n\in N}$ converges to $\omega_{H(\lambda,T)}$.*

(3) *The support of λ is not contained in a proper coset $Hx = xT(H) \neq H$ of a compact subgroup H of G.*

Proof. Let $T = (x \to txt^{-1})$. Then we have $\varrho_n = (\lambda t)^n t^{-n}$. From Lemma 2.1 we infer for $y \in \mathrm{supp}(\lambda)$ that $\lim_{n\to\infty} (\lambda t)^n (yt)^{-n} = \omega_{H(\lambda t)}$ and that the sequence $((\lambda t)^n x^{-n})_{n\in N}$ converges if and only if $x \in H(\lambda t) yt$. Hence $(\varrho_n)_{n\in N}$ converges if and only if $t \in H(\lambda t) yt$, which is equivalent to $y \in H(\lambda t)$. Since y was an arbitrary element of $\mathrm{supp}(\lambda)$, $(\varrho_n)_{n\in N}$ converges if and only if $\mathrm{supp}(\lambda) \subset H(\lambda t)$. Moreover, in the case of convergence we have $\lim_{n\to\infty} \varrho_n = \omega_{H(\lambda t)}$. Since, by definition, $H(\lambda t)$ is minimal with $\mathrm{supp}(\lambda t) \subset H(\lambda t) yt = ytH(\lambda t)$, which is equivalent to $\mathrm{supp}(\lambda) \subset H(\lambda t) y = ytH(\lambda t)t^{-1} = yT(H(\lambda t))$, we have $H(\lambda t) = H(\lambda, T)$. Hence the assertion follows from Lemma 1.2. ∎

The group $\mathrm{Aut}(G)$ is a topological group with respect to the natural topology. For any subgroup L of $\mathrm{Aut}(G)$ the semidirect product $G \rtimes L$ is a topological group with respect to the product topology. For these and related facts see [4].

2.3. COROLLARY. *The conclusions of Theorem 2.2 remain true if "$T \in \mathrm{In}(G)$" is replaced by "$G \rtimes \langle T \rangle^-$ compact".*

Proof. Put $F := G \rtimes \langle T \rangle^-$, $\iota: G \to F: x \to (x, \mathrm{Id})$, and $\pi: F \to G: (x, R) \to x$. Then $\mu := \iota(\lambda) \in \mathscr{P}(F)$, and $S: (x, R) \to (e, T)(x, R)(e, T)^{-1} = (Tx, R)$ from F into F defines an element of $\mathrm{In}(F)$. Observing that $H(\lambda, T) = \pi(H(\mu, S))$ and applying Theorem 2.2 to the sequence

$$\sigma_n := \mu S(\mu) \dots S^{n-1}(\mu) = \iota(\varrho_n)$$

we obtain

$$\varrho_n \text{ converges (to } \omega_{H(\lambda,T)}) \Leftrightarrow \sigma_n \text{ converges (to } \omega_{H(\mu,S)}) \Leftrightarrow \mathrm{supp}(\mu) \subset H(\mu, S)$$
$$\Leftrightarrow \mathrm{supp}(\lambda) = \pi(\mathrm{supp}(\mu)) \subset \pi(H(\mu, S)) = H(\lambda, T). \quad ∎$$

3. The case of general G. At first we shall provide some facts about equicontinuous families of automorphisms of a topological group.

3.1. DEFINITION. Let G be a topological group, $\mathscr{S} \subset \mathrm{Aut}(G)$. $\mathscr{S}$ is called *equicontinuous* if every neighbourhood U of e admits a neighbourhood V of e such that $V \subset T^{-1}(U)$ for all $T \in \mathscr{S}$.

3.2. LEMMA. *Let $\mathscr{S} \subset \mathrm{Aut}(G)$. Then the following statements are equivalent:*

(1) *$\mathscr{S}$ is equicontinuous.*

(2) *If U is a neighbourhood of e, then $\bigcap_{T\in\mathscr{S}} T^{-1}(U)$ is a neighbourhood of e.*

If $\mathcal{S} = \{T^z\colon z \in \mathbf{Z}\}$, *then* (1) *is equivalent to*

(3) e *has a neighbourhood basis of T-invariant sets.*

The proof is clear.

3.3. LEMMA. *Let G be a compact group and* $T \in \mathrm{Aut}(G)$. *Then the following statements are equivalent:*

(1) $\{T^{-n}\colon n \in N\}$ *is equicontinuous.*

(2) $\{T^z\colon z \in \mathbf{Z}\}$ *is equicontinuous.*

(3) $\langle T \rangle$ *is relatively compact in* $\mathrm{Aut}(G)$.

(4) $G \rtimes \langle T \rangle^-$ *is compact.*

Proof. By the theorem of Arzela–Ascoli [9] a subset of $\mathrm{Aut}(G)$ is equicontinuous if and only if it is relatively compact with respect to the compact open topology. By [3] the compact open topology and the natural topology on $\mathrm{Aut}(G)$ coincide if G is compact. Since $\mathrm{Aut}(G)$ endowed with the natural topology is a topological group, the inversion is continuous, which implies that either all or none of the sets $\{T^{-n}\colon n \in N\}$, $\{T^n\colon n \in N\}$, and $\{T^z\colon z \in \mathbf{Z}\}$ are relatively compact in $\mathrm{Aut}(G)$. ■

3.4. LEMMA. *Let G be a locally compact group,* $T \in \mathrm{Aut}(G)$, $\lambda \in \mathscr{P}(G)$, *and assume* $\{T^{-n}\colon n \in N\}$ *is equicontinuous. If* $(\varrho_n)_{n\in N}$ *converges to* $\nu \in \mathscr{P}(G)$, *then* $\nu = \nu^2$, *and for* $G' := \mathrm{supp}(\nu)$ *we have*

$$T(G') = G' \quad \text{and} \quad \mathrm{supp}(\lambda) \subset G'.$$

Proof. Putting $\nu_{k,n} := T^k(\lambda) \ldots T^{n-1}(\lambda)$ for $k < n$ we obtain $\nu_{k,n} \xrightarrow{n} T^k(\nu)$ for all $k \in N$ and $\nu_{k,n}\nu_{n,m} = \nu_{k,m}$ for $k < n < m$. Consequently, $\nu_{k,n}T^n(\nu) = T^k(\nu)$. Moreover, we conclude from Theorem 1.2.21 in [5] that the sequence $T^n(\nu)$ is relatively compact. For any limit point ϱ of $T^n(\nu)$ we obtain $T^k(\nu)\varrho = T^k(\nu)$ and $\varrho^2 = \varrho$. For any further limit point ϱ' of $T^n(\nu)$ we get $\varrho'\varrho = \varrho'$ and $\varrho\varrho' = \varrho$. Since ϱ' as well as ϱ is idempotent, the support formula implies $\varrho = \varrho'$. So we have $T^k(\nu) \xrightarrow{k} \varrho^2 = \varrho = T(\varrho)$ and $\nu\varrho = \nu$. We show that $\varrho = \nu$. Let U be a neighbourhood of e. By Lemma 3.2, $V := \bigcap_{n\in N} T^n(U)$ is again a neighbourhood of e, and for all $k \in N$ we have $T^{-k}(V) \subset U$. Hence for $H := \mathrm{supp}(\varrho) = \mathrm{supp}(T(\varrho)) = T(H)$ we get

$$1 \geqslant \nu(UH) \geqslant \nu(T^{-k}(V)H) = \nu(T^{-k}(VH)) = T^k(\nu)(VH).$$

Since $T^k(\nu)$ converges to $\varrho = \omega_H$, we obtain $\liminf_{k\to\infty} T^k(\nu)(VH) \geqslant \varrho(H) = 1$. Thus we have $\nu(UH) = 1$ for all neighbourhoods U of e, which implies $\nu(H) = 1$ and, in particular, $\mathrm{supp}(\nu) \subset H$. From this we conclude that $\nu = \nu\varrho = \varrho$. It remains to show that $\mathrm{supp}(\lambda) \subset H = G'$. But this follows from the equalities $\omega_H = \lim_{n\to\infty} \varrho_n = \lim_{n\to\infty} \lambda T(\varrho_n) = \lambda T(\omega_H) = \lambda\omega_H$. ■

Now we are ready to present our main result.

3.5. THEOREM. *Let G be a locally compact group, $\lambda \in \mathscr{P}(G)$, $T \in \mathrm{Aut}(G)$, and assume that $\{T^{-n}: n \in N\}$ is equicontinuous. Then the following statements are equivalent:*

(1) *$(\varrho_n)_{n \in N}$ converges in $\mathscr{P}(G)$.*

(2) *$(\varrho_n)_{n \in N}$ converges to an idempotent measure in $\mathscr{P}(G)$.*

(3) *$\mathscr{H}(\lambda, T) \neq \emptyset$ and $\mathrm{supp}(\lambda) \subset H(\lambda, T)$.*

(4) *The support of λ is contained in a compact T-invariant subgroup of G but it is not contained in a proper coset $Hx = xT(H) \neq H$ of a compact subgroup H of G.*

Proof. All statements (1)–(4) do immediately lead to the case where the underlying group is compact, namely (1) and (2) by Lemma 3.4, (3) by the fact that $\mathrm{supp}(\lambda) \subset H(\lambda, T)$ implies $T(H(\lambda, T)) = H(\lambda, T)$, and (4) trivially. Hence the required equivalences follow from Corollary 2.3 together with Lemmas 3.3 and 1.2. ∎

4. Remarks. Without any conditions on the automorphism T Theorem 3.5 is false. Consider, for example, $G = \boldsymbol{R}$ and $T: \boldsymbol{R} \to \boldsymbol{R}: x \to \alpha x$, $|\alpha| < 1$. For $\lambda \in \mathscr{P}(\boldsymbol{R})$ the measure ϱ_n is just the distribution of the random variable $Z_n := \sum_{i=0}^{n-1} \alpha^i X_i$, where $(X_i)_{i \in N_0}$ is an i.i.d. sequence with distribution λ. By [10], Z_n converges in distribution if and only if $\mathbb{E}\log(|X_0|+1) < \infty$, while $\mathscr{H}(\lambda, T) = \emptyset$ except for $\lambda = \varepsilon_0$.

Theorem 3.5 remains valid if "$\{T^{-n}: n \in N\}$ is equicontinuous" is replaced by "$G \rtimes \langle T \rangle^-$ is a Tortrat group" in the sense of [2]. This is, for example, true if $G \rtimes \langle T \rangle^-$ is a SIN-group, a MAP-group, or an almost connected nilpotent group. For these and more subtle conditions see [1].

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