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CENTRAL LIMIT THEOREMS ON NILPOTENT LIE GROUPS*

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Abstract. The Lindeberg theorem is derived on stratified nilpotent Lie groups; that is a normal convergence theorem for a triangular system of probability measures in case of bounded (homogeneous) moments of second order. By using necessary and sufficient conditions for convergence of convolution semigroups of probability measures on Lie groups a Lindeberg–Feller theorem is proved on the Heisenberg group.

Introduction. One of the classical questions of the central limit problems for a sequence $(\mu_n)_{n \geqslant 1}$ of probability measures on a topological group G is to find appropriate automorphisms τ_n of the group in such a way that the sequence $(\tau_n(\mu_1 * \ldots * \mu_n))_{n \geqslant 1}$ of the standardized convolution products converges to some Gaussian limit. The limiting Gaussian measure should be stable as well. It is known that nilpotent Lie groups play an important role concerning stability of probability measures on a topological group. In [28] and [10] Hazod and Siebert showed that the investigation of stable measures on a locally compact group can be reduced to the case of a simply connected nilpotent Lie group whose Lie algebra admits a positive graduation. Therefore it is natural to study generalizations of the classical results connected with central limit problems in case of such groups.

For the sake of simplicity we restrict our attention to the class of stratified nilpotent Lie groups. Leaving the classical case of $\boldsymbol{R}^k$ out of consideration, these groups are non-commutative, non-compact, have infinite-dimensional irreducible representations, and the set of finite-dimensional representations does not separate the points of the group (thus they are not maximally almost periodic).

We consider a triangular system $(\mu_{nk})_{k=1,\ldots,k_n; n \geqslant 1}$ of probability measures. In the case of a Lie group, the convergence behaviour of the sequence $(\mu_{n1} * \ldots * \mu_{nk_n})_{n \geqslant 1}$ of row products has been studied in [33], [6], [29], [12], and

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[26]. Using the results of Wehn [33] we prove in the case of a stratified Lie group the analogue of the classical central limit theorem under the Lindeberg condition

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x|\geqslant\varepsilon} |x|^2 \mu_{nk}(dx) = 0 \quad \text{for all } \varepsilon > 0,$$

where $x \to |x|$ is an arbitrary homogeneous norm on the group. Sometimes the condition

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \mu_{nk}(G\backslash U) = 0 \quad \text{for all neighborhoods } U \text{ of } e$$

on a topological group is also called a (*generalized*) *Lindeberg condition* (see, e.g., [33], [29], and [12]), but it should be rather called a *Khinchin condition* (cf. [5] and [1]). As a corollary we obtain a central limit theorem for probability measures μ with $\int |x|^2 \mu(dx) < \infty$, that is, the convergence of the sequence $(\delta_{1/\sqrt{n}}(\mu^n))_{n\geqslant 1}$ to some Gaussian measure, where $(\delta_t)_{t>0}$ are the natural dilations (see [22]; in [3] and [32] stronger moment conditions are supposed). Another corollary is a Lindeberg central limit theorem for suitably standardized n-fold convolution products $\mu_1 * \ldots * \mu_n$ of probability measures on the Heisenberg group (the simplest non-commutative stratified nilpotent Lie group). The standardization is performed in such a way that the limit distribution will be the standard Gaussian measure.

Next we are concerned with necessary and sufficient conditions for the convergence of convolution semigroups of probability measures on Lie groups in terms of their generating functionals and characteristics of their canonical decompositions. (This problem plays an important role in investigation of the necessity of the Lindeberg condition.) Some partial results were obtained in [33] (see also the interpretation of Grenander [6]). Hazod [8] has proved that the convergence of the generating functionals of the convolution semigroups implies the convergence of the convolution semigroups themselves. The converse is contained in [26], though not explicitly stated, as remarked by Hazod and Scheffler [9] (who have also formulated, in the case of exponential Lie groups, necessary and sufficient conditions in terms of the corresponding generating functionals on the Lie algebra) and discovered in Khokhlov [14] (who applied it for stable measures). We give a complete proof of the above-mentioned converse part using the idea of Siebert [26], and prove the analogue of the classical results (see [5], § 19, Theorems 1 and 2, in the case of $\boldsymbol{R}$, and [30] in the case of $\boldsymbol{R}^k$).

Finally, we study Feller type central limit theorems, that is, the necessity of the Lindeberg condition for a triangular system $(\mu_{nk})_{k=1,\ldots,k_n;n\geqslant 1}$ of probability measures. Siebert [26] showed that under some boundedness condition on the Fourier transforms of the measures μ_{nk} the classical approximation with the accompanying Poisson system can be applied. (In the case of a commutative group it can always be achieved by appropriate shifts of μ_{nk} that the limit points of the row products of (μ_{nk}) and those of the accompanying Poisson system coincide; but for a general group this does not work.)

In the case of the Heisenberg group we can use explicit forms for the irreducible unitary representations, and we prove that the condition

$$\sup_{n\geqslant 1}\sum_{k=1}^{k_n}\int|x|^2\,\mu_{nk}(dx)<\infty$$

implies the above-mentioned condition of Siebert. Thus the convergence of an infinitesimal triangular system with bounded (homogeneous) moments of second order implies that the accompanying Poisson system is convergent to the same limit. In the classical situation of $\boldsymbol{R}^k$ this implies the convergence of the corresponding accompanying sequence of Poisson semigroups to the unique embedding semigroup of the limit distribution. But for the Heisenberg group the uniqueness of the embedding convolution semigroup of a Gaussian measure is not known (it is known only that the embedding of a Gaussian measure into a Gaussian semigroup is unique on a simply connected nilpotent Lie group; see [2] in the case of 2-step nilpotent Lie groups and [21] for general nilpotent Lie groups). So we impose an additional condition: we suppose that the system $(\mu_{nk})_{k=1,\ldots,k_n;n\geqslant 1}$ is normal in the sense that $\mu_{nj}*\tilde{\mu}_{nk}=\tilde{\mu}_{nk}*\mu_{nj}$ for all $1\leqslant j,\,k\leqslant k_n$ (where $\tilde{\mu}$ is the adjoint of the measure μ) and prove the classical necessary and sufficient conditions for the convergence of triangular systems to a given Gaussian measure (for the sake of simplicity we assume also that the measures μ_{nk} are centered). Supposing moreover the convergence of variances of the first two coordinates and the Lindeberg condition for the third coordinate we obtain the usual form of the Lindeberg–Feller theorem.

1. Preliminaries on Lie groups. In this section we introduce some terminology, notation, preliminary background, and recall a version of the central limit theorem for infinitesimal, commutative triangular systems of probability measures on Lie groups due to Wehn [33].

Let G be a Lie group of dimension $m\geqslant 1$ with neutral element e. Let $G^\times:=G\setminus\{e\}$. If B is a subset of G, then B^- and ∂B denote the closure and the boundary of B, respectively. Let $\mathscr{B}(G)$ denote the Borel σ-algebra of G. Let $\mathscr{U}(e)$ denote the system of all neighborhoods of e. By $\mathscr{C}^b(G)$ we denote the space of bounded continuous complex-valued functions on G equipped with the supremum norm $\|\cdot\|_\infty$. Let $\mathscr{C}_u(G)$ be the subspace of $\mathscr{C}^b(G)$ of uniformly continuous functions with respect to the left uniform structure on G. Let $\mathscr{D}(G)$ be the space of infinitely differentiable complex-valued functions with compact support on G. The space $\mathscr{E}(G)$ of bounded *regular functions* on G is defined by

$$\mathscr{E}(G):=\{f\in\mathscr{C}^b(G)\colon f\cdot g\in\mathscr{D}(G)\ \text{ for all }\ g\in\mathscr{D}(G)\}.$$

Let $\mathscr{G}$ be the Lie algebra of G, and $\exp\colon\mathscr{G}\mapsto G$ the exponential mapping.

An element $X \in \mathscr{G}$ can be regarded as a (left-invariant) differential operator on G: for $f \in \mathscr{D}(G)$ we put

$$Xf(x) = \lim_{t \to 0} \frac{f(x \exp tX) - f(x)}{t}.$$

If $\{X_1, \ldots, X_m\}$ is a basis of $\mathscr{G}$, then there is an associated triplet $(U_0, \{\zeta_1, \ldots, \zeta_m\}, \varphi)$ (cf. [12], p. 260, and [26]) such that

(i) $U_0 \in \mathscr{U}(e)$;

(ii) $\{\zeta_1, \ldots, \zeta_m\}$ is a system of *canonical coordinates of the first kind* in $\mathscr{D}(G)$ adapted to the basis $\{X_1, \ldots, X_m\}$ and valid in U_0, i.e., one has

$$x = \exp\Big(\sum_{i=1}^{m} \zeta_i(x) X_i\Big) \quad \text{for each } x \in U_0;$$

(iii) φ is a *Hunt function* for G adapted to the coordinate system $\{\zeta_1, \ldots, \zeta_m\}$, i.e., φ is in $\mathscr{E}(G)$, non-negative, bounded away from zero on $G \backslash U$ for any $U \in \mathscr{U}(e)$, and

$$\varphi(x) = \sum_{i=1}^{m} \zeta_i(x)^2 \quad \text{for all } x \in U_0.$$

We denote by $\mathscr{M}_+(G)$ the space of positive Radon measures on G, $\mathscr{M}_+^b(G)$ is the subspace of bounded measures, and $\mathscr{P}(G)$ the set of probability measures on G which, equipped with the operation of convolution $*$ and the weak topology, is a topological semigroup. The Dirac measure in $x \in G$ is denoted by ε_x.

For $\mu \in \mathscr{P}(G), f \in \mathscr{C}^b(G)$ and $x \in G$ we define

$$T_\mu f(x) := \int f(xy)\, \mu(dy).$$

We have $T_\mu f \in \mathscr{C}^b(G)$, and T_μ is called the *convolution operator* of μ. It is a bounded linear operator on $\mathscr{C}^b(G)$ with $\|T_\mu\| = 1$, $T_{\mu * \nu} = T_\mu T_\nu$ for all $\mu, \nu \in \mathscr{P}(G)$, and the correspondence $\mu \to T_\mu|_{\mathscr{C}_u(G)}$ is continuous (cf. [12], p. 64).

A family $(\mu_t)_{t \geq 0}$ in $\mathscr{P}(G)$ is said to be a *(continuous) convolution semigroup* if $\mu_s * \mu_t = \mu_{s+t}$ for all $s, t \geq 0$, and $\lim_{t \downarrow 0} \mu_t = \mu_0 = \varepsilon_e$. If $(\mu_t)_{t \geq 0}$ is a convolution semigroup, the family $(T_{\mu_t})_{t \geq 0}$ of convolution operators defines a strongly continuous semigroup of contractions on the Banach space $\mathscr{C}_u(G)$ whose *infinitesimal generator* is denoted by $(N, \mathscr{N})$. We have $\mathscr{D}(G) \subset \mathscr{N}$ and

$$(Nf)(x) = \lim_{t \downarrow 0} t^{-1} \int \big(f(xy) - f(x)\big)\, \mu_t(dy)$$

for all $x \in G$ and $f \in \mathscr{N}$. The *generating functional* $(A, \mathscr{A})$ of the convolution semigroup $(\mu_t)_{t \geq 0}$ is defined by

$$\mathscr{A} := \Big\{f \in \mathscr{C}^b(G):\ A(f) := \lim_{t \downarrow 0} t^{-1} \int \big(f(x) - f(e)\big)\, \mu_t(dx) \text{ exists}\Big\}.$$

We have $\mathscr{E}(G) \subset \mathscr{A}$ and $(Nf)(x) = A({}_x f)$, where the function ${}_x f$ is defined by

${}_xf(y) = f(xy)$. On $\mathscr{E}(G)$ the functional A admits the *canonical decomposition* (Lévy–Khinchin formula)

$$A(f) = \sum_{i=1}^{m} a_i (X_i f)(e) + \sum_{i,j=1}^{m} a_{ij} (X_i X_j f)(e)$$
$$+ \int_{G^\times} \Big[f(x) - f(e) - \sum_{i=1}^{m} \zeta_i(x)(X_i f)(e)\Big]\, \eta(dx),$$

where $a_1, \ldots, a_m$ are real numbers, $(a_{ij})_{1 \leqslant i,j \leqslant m}$ is a real symmetric positive semidefinite matrix, and η is a Lévy measure on G, i.e., $\eta \in \mathscr{M}_+(G^\times)$ with $\int_{G^\times} \varphi(x)\, \eta(dx) < \infty$ (see [13], [25], and [12], p. 268). We shall also say that the generating functional A admits the *canonical decomposition* $(a_i, a_{ij}, \eta)_{1 \leqslant i,j \leqslant m}$.

A convolution semigroup $(\mu_t)_{t \geqslant 0}$ of non-degenerated measures is called a *Gaussian semigroup* if we have $\lim_{t \downarrow 0} t^{-1} \mu_t(G \backslash U) = 0$ for all $U \in \mathscr{U}(e)$. A non-degenerated convolution semigroup $(\mu_t)_{t \geqslant 0}$ with canonical decomposition $(a_i, a_{ij}, \eta)_{1 \leqslant i,j \leqslant m}$ is a Gaussian semigroup if and only if $\eta = 0$. A non-degenerated measure $\mu \in \mathscr{P}(G)$ is called a *Gaussian measure* if there exists a Gaussian semigroup $(\mu_t)_{t \geqslant 0}$ such that $\mu_1 = \mu$. (For information on Gaussian semigroups cf. [12], [27].)

For $\gamma \in \mathscr{M}^b_+(G)$ the *Poisson measure* $\exp\big(\gamma - \gamma(G)\varepsilon_e\big) \in \mathscr{P}(G)$ with exponent γ is defined by

$$\exp\big(\gamma - \gamma(G)\varepsilon_e\big) := e^{-\gamma(G)} \sum_{k=0}^{\infty} \gamma^k / k!,$$

where γ^k is the k-th convolution power of γ, and $\gamma^0 := \varepsilon_e$. For $t \geqslant 0$, clearly, $\mu_t := \exp\big(t(\gamma - \gamma(G)\varepsilon_e)\big)$ is the Poisson measure with exponent $t\gamma$, and $(\mu_t)_{t \geqslant 0}$ is a convolution semigroup with generating functional $\big(\gamma - \gamma(G)\varepsilon_e, \mathscr{C}^b(G)\big)$; it is called a *Poisson semigroup*. Clearly, its canonical decomposition is $(a_i, 0, \gamma)_{1 \leqslant i \leqslant m}$, where $a_i = \int_{G^\times} \zeta_i(x)\, \gamma(dx)$ for $i = 1, \ldots, m$ (that is, it contains no Gaussian part).

A triangular system $(\mu_{nk})_{k=1,\ldots,k_n; n \geqslant 1}$ of probability measures on G is called *infinitesimal* if

$$\lim_{n \to \infty} \max_{1 \leqslant k \leqslant k_n} \mu_{nk}(G \backslash U) = 0 \quad \text{for all } U \in \mathscr{U}(e).$$

The system $(\mu_{nk})_{k=1,\ldots,k_n; n \geqslant 1}$ is said to be *commutative* if

$$\mu_{nj} * \mu_{nk} = \mu_{nk} * \mu_{nj} \quad \text{for all } 1 \leqslant j, k \leqslant k_n \text{ and } n \geqslant 1.$$

The system $(\mu_{nk})_{k=1,\ldots,k_n; n \geqslant 1}$ is said to be *convergent to the limit* μ if

$$\mu \in \mathscr{P}(G) \quad \text{and} \quad \mu_{n1} * \ldots * \mu_{nk_n} \to \mu \text{ as } n \to \infty.$$

We shall apply the following central limit theorem proved by Wehn [33] (see also [6] and [26]):

THEOREM 1.1. *Let $(\mu_{nk})_{k=1,\ldots,k_n;n\geqslant 1}$ be a commutative and infinitesimal system on a Lie group G. Suppose that*

(i) $\lim_{n\to\infty}\sum_{k=1}^{k_n}\int_{G\setminus U}\varphi(x)\,\mu_{nk}(dx)=0$,

(ii) $\sup_{n\geqslant 1}\sum_{k=1}^{k_n}\left|\int_U \zeta_i(x)\,\mu_{nk}(dx)\right|<\infty$ *for* $i=1,\ldots,m$,

(iii) $\lim_{n\to\infty}\sum_{k=1}^{k_n}\int_U \zeta_i(x)\,\mu_{nk}(dx)=a_i$ *for* $i=1,\ldots,m$,

(iv) $\lim_{n\to\infty}\sum_{k=1}^{k_n}\int_U \zeta_i(x)\,\zeta_j(x)\,\mu_{nk}(dx)=a_{ij}$ *for* $i,j=1,\ldots,m$

for all $U\in\mathscr{U}(e)$. Then $\mu_{n1}*\ldots*\mu_{nk_n}\to\nu$ as $n\to\infty$, where ν is the Gaussian measure with the infinitesimal generator

$$\sum_{i=1}^{m}a_iX_i+\tfrac{1}{2}\sum_{1\leqslant i,j\leqslant m}a_{ij}X_iX_j.$$

Remark 1. Condition (i) implies that the system $(\mu_{nk})_{k=1,\ldots,k_n;n\geqslant 1}$ is infinitesimal. The boundedness of the coordinate functions $\{\zeta_1,\ldots,\zeta_m\}$ implies that if $(\mu_{nk})_{k=1,\ldots,k_n;n\geqslant 1}$ is an infinitesimal system, then conditions (ii), (iii) and (iv) are satisfied for all $U\in\mathscr{U}(e)$ if and only if they are satisfied for at least one $U\in\mathscr{U}(e)$. The non-classical condition (ii) was replaced by Siebert [26] by a weaker one (formulated by Fourier transforms of the measures μ_{nk}), but it is not known whether it can be omitted.

2. Stratified nilpotent Lie groups and homogeneous norms. An algebra $\mathscr{G}$ has a *stratified decomposition of step s* if there exists a vector space decomposition $\mathscr{G}=\bigoplus_{j=1}^{s}V_j$ such that $[V_i,V_j]\subset V_{i+j}$ when $i+j\leqslant s$ and $[V_i,V_j]=0$ when $i+j>s$, and V_1 generates $\mathscr{G}$ as an algebra. A basis $\{X_1,\ldots,X_m\}$ of $\mathscr{G}$ is *adapted* to the above decomposition if the basis elements in V_j form a basis for V_j. Let $d_k=j$ when $X_k\in V_j$.

A *stratified Lie group of step s* is a simply connected Lie group whose Lie algebra has a stratified decomposition of step s. Clearly, a stratified Lie group of step s is nilpotent of step s. Moreover, $(\boldsymbol{R}^k,+)$, $k\geqslant 1$, are the only commutative stratified Lie groups, and the non-commutative stratified Lie groups are non-compact.

Let G be a stratified Lie group of step s. Let $\{X_1,\ldots,X_m\}$ be an adapted basis in its Lie algebra $\mathscr{G}$. It is known that the exponential mapping $\exp\colon\mathscr{G}\mapsto G$ is now an analytic diffeomorphism; thus it can be used to transfer coordinates from $\mathscr{G}$ to G. For $x\in G$ we denote by $\{x_1,\ldots,x_m\}$ the *canonical coordinates of the first kind* adapted to the basis $\{X_1,\ldots,X_m\}$ and valid on the whole G:

$$x=\exp\Big(\sum_{i=1}^{m}x_i(x)X_i\Big)\qquad\text{for each } x\in G.$$

We equip $\mathscr{G}$ as well as G with the natural *dilations* by extending $\mathring{\delta}_t(X):=t^jX$, $t>0$, $X\in V_j$, by linearity to $\mathscr{G}$ and putting $\delta_t(\exp X):=\exp(\mathring{\delta}_tX)$. (The family $(\delta_t)_{t>0}$ is a continuous one-parameter group of automorphisms of G and plays the role of multiplication by scalars $t>0$.)

A *homogeneous norm* on G is a function $x \to |x|$ from G to $[0, \infty)$ satisfying
(i) $x \to |x|$ is continuous on G and C^∞ on G^*;
(ii) $|x| = 0$ if and only if $x = e$;
(iii) $|\delta_t x| = t|x|$ for $t > 0$, $x \in G$.
Observe that homogeneous norms always exist (cf. [4]). Let us define

$$\varrho(x) = \sum_{i=1}^{m} |x_i|^{1/d_i} \quad \text{for } x \in G.$$

It is known (cf. [4] and [16]) that for any homogeneous norm $|\cdot|$ on G there exist $c_1, c_2 > 0$ such that

$$c_1 \varrho(x) \leqslant |x| \leqslant c_2 \varrho(x). \tag{1}$$

Consequently, any two homogeneous norms are equivalent in the usual sense. We shall frequently use the property that for all $x \in G$ and $i = 1, \ldots, m$

$$|x_i| \leqslant c|x|^{d_i} \tag{2}$$

with a suitable constant $c > 0$ depending only on the homogeneous norm $|\cdot|$ (cf. [16]).

Let $\mu \in \mathscr{P}(G)$. For $k \in N$ let us consider the *homogeneous moment of k-th order* of μ:

$$M_k(\mu) = \sum_{i=1}^{m} \int |x_i|^{k/d_i} \mu(dx).$$

Inequality (1) implies that for a homogeneous norm $|\cdot|$ on G there are constants $c_k^{(1)}, c_k^{(2)} > 0$ such that

$$c_k^{(1)} \int |x|^k \mu(dx) \leqslant M_k(\mu) \leqslant c_k^{(2)} \int |x|^k \mu(dx).$$

Thus for $\mu \in \mathscr{P}(G)$ and $k \in N$ the following assertions are equivalent:
(i) $M_k(\mu) < \infty$.
(ii) $\int |x|^k \mu(dx) < \infty$ for some homogeneous norm $|\cdot|$ on G.
(iii) $\int |x|^k \mu(dx) < \infty$ for arbitrary homogeneous norms $|\cdot|$ on G.

3. The Lindeberg theorem for triangular systems on stratified Lie groups. Let G be a stratified nilpotent Lie group of step s, and $|\cdot|$ a homogeneous norm on G. One can suppose that in the triplet $(U_0, \{\zeta_1, \ldots, \zeta_m\}, \varphi)$ the neighborhood U_0 is the unit ball $\{x \in G: |x| < 1\}$ (cf. [12], p. 254). Clearly, a triangular system $(\mu_{nk})_{k=1,\ldots,k_n; n \geqslant 1}$ in $\mathscr{P}(G)$ is infinitesimal if and only if for any $\varepsilon > 0$

$$\lim_{n \to \infty} \max_{1 \leqslant k \leqslant k_n} \mu_{nk}(x: |x| \geqslant \varepsilon) = 0 \quad \text{for any } \varepsilon > 0.$$

From Theorem 1.1 one can easily obtain the following consequence:

THEOREM 3.1. *Let $(\mu_{nk})_{k=1,\ldots,k_n; n \geqslant 1}$ be a commutative system on a stratified Lie group. Suppose that*

(i) $\lim_{n\to\infty}\sum_{k=1}^{k_n}\mu_{nk}(x\colon |x|\geqslant\varepsilon)=0$ *for all* $\varepsilon>0$,

(ii) $\sup_{n\geqslant 1}\sum_{k=1}^{k_n}\left|\int_{|x|<1}x_i\,\mu_{nk}(dx)\right|<\infty$ *for* $i=1,\ldots,m$,

(iii) $\lim_{n\to\infty}\sum_{k=1}^{k_n}\int_{|x|<1}x_i\,\mu_{nk}(dx)=a_i$ *for* $i=1,\ldots,m$,

(iv) $\lim_{n\to\infty}\sum_{k=1}^{k_n}\int_{|x|<1}x_i x_j\,\mu_{nk}(dx)=a_{ij}$ *for* $i,j=1,\ldots,m$.

Then $\mu_{n1}*\ldots*\mu_{nk_n}\to\nu$ *as* $n\to\infty$, *where* ν *is the Gaussian measure with the infinitesimal generator*

$$\sum_{i=1}^{m}a_iX_i+\tfrac{1}{2}\sum_{1\leqslant i,j\leqslant m}a_{ij}X_iX_j.$$

Now we derive Lindeberg's theorem, that is, the normal convergence theorem in the case of bounded (homogeneous) moments of second order. For the sake of simplicity we shall deal only with centered measures (condition (i) in the following theorem; cf. [3]).

THEOREM 3.2. *Let* $(\mu_{nk})_{k=1,\ldots,k_n;n\geqslant 1}$ *be a commutative system on a stratified Lie group. Suppose that*

(i) $\sup_{n\geqslant 1}\sum_{k=1}^{k_n}\int|x|^2\,\mu_{nk}(dx)<\infty$,

(ii) $\int x_i\,\mu_{nk}(dx)=0$ *for* $d_i=1$,

(iii) $\lim_{n\to\infty}\sum_{k=1}^{k_n}\int x_i\,\mu_{nk}(dx)=a_i$ *for* $d_i=2$,

(iv) $\lim_{n\to\infty}\sum_{k=1}^{k_n}\int x_ix_j\,\mu_{nk}(dx)=a_{ij}$ *for* $d_i=d_j=1$,

(v) $\lim_{n\to\infty}\sum_{k=1}^{k_n}\int_{|x|\geqslant\varepsilon}|x|^2\,\mu_{nk}(dx)=0$ *for all* $\varepsilon>0$.

Then $\mu_{n1}*\ldots*\mu_{nk_n}\to\nu$ *as* $n\to\infty$, *where* ν *is the Gaussian measure with the infinitesimal generator*

$$\sum_{d_i=2}a_iX_i+\tfrac{1}{2}\sum_{d_i=d_j=1}a_{ij}X_iX_j.$$

Proof. We shall show that the conditions of Theorem 3.1 are satisfied. Clearly,

$$\int_{|x|\geqslant\varepsilon}|x|^2\,\mu_{nk}(dx)\geqslant\varepsilon^2\,\mu_{nk}(x\colon |x|\geqslant\varepsilon),$$

so condition (v) implies $\lim_{n\to\infty}\sum_{k=1}^{k_n}\mu_{nk}(x\colon |x|\geqslant\varepsilon)=0$ for all $\varepsilon>0$.

If $d_i=1$, then using assumption (ii) and estimate (2) we have

$$\Big|\int_{|x|<1}x_i\,\mu_{nk}(dx)\Big|=\Big|\int_{|x|\geqslant 1}x_i\,\mu_{nk}(dx)\Big|\leqslant c\int_{|x|\geqslant 1}|x|\,\mu_{nk}(dx)\leqslant c\int_{|x|\geqslant 1}|x|^2\,\mu_{nk}(dx).$$

Thus from assumption (v) we conclude that

$$\sup_{n\geqslant 1}\sum_{k=1}^{k_n}\Big|\int_{|x|<1}x_i\,\mu_{nk}(dx)\Big|<\infty\quad\text{and}\quad\lim_{n\to\infty}\sum_{k=1}^{k_n}\int_{|x|<1}x_i\,\mu_{nk}(dx)=0.$$

If $d_i\geqslant 2$, then using again estimate (2) we get

$$\Big|\int_{|x|<1}x_i\,\mu_{nk}(dx)\Big|\leqslant c\int_{|x|<1}|x|^{d_i}\,\mu_{nk}(dx)\leqslant c\int|x|^2\,\mu_{nk}(dx).$$

Thus assumption (i) implies

$$\sup_{n\geqslant 1}\sum_{k=1}^{k_n}\Big|\int_{|x|<1} x_i\,\mu_{nk}(dx)\Big| < \infty.$$

In the case $d_i = 2$ the estimate

$$\Big|\int_{|x|\geqslant 1} x_i\,\mu_{nk}(dx)\Big| \leqslant c\int_{|x|\geqslant 1} |x|^2\,\mu_{nk}(dx)$$

together with assumptions (iii) and (v) gives

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}\int_{|x|<1} x_i\,\mu_{nk}(dx) = a_i.$$

If $d_i \geqslant 3$, then for $0 < \varepsilon < 1$ we have

$$\Big|\int_{|x|<\varepsilon} x_i\,\mu_{nk}(dx)\Big| \leqslant c\varepsilon^{d_i-2}\int |x|^2\,\mu_{nk}(dx) \text{ and } \Big|\int_{\varepsilon\leqslant|x|<1} x_i\,\mu_{nk}(dx)\Big| \leqslant c\mu_{nk}(x\colon |x|\geqslant\varepsilon).$$

Thus we obtain

$$\limsup_{n\to\infty}\sum_{k=1}^{k_n}\Big|\int_{|x|<1} x_i\,\mu_{nk}(dx)\Big| \leqslant c\varepsilon^{d_i-2}\sup_{n\geqslant 1}\sum_{k=1}^{k_n}\int |x|^2\,\mu_{nk}(dx).$$

Since $0 < \varepsilon < 1$ is arbitrary, we conclude that

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}\int_{|x|<1} x_i\,\mu_{nk}(dx) = 0.$$

Similarly, if $d_i+d_j \geqslant 3$, then from the inequality

$$\Big|\int_{|x|<\varepsilon} x_i x_j\,\mu_{nk}(dx)\Big| \leqslant c\varepsilon^{d_i+d_j-2}\int |x|^2\,\mu_{nk}(dx)$$

one can derive

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}\int_{|x|<1} x_i x_j\,\mu_{nk}(dx) = 0.$$

In the case $d_i = d_j = 1$ the estimate

$$\Big|\int_{|x|\geqslant 1} x_i x_j\,\mu_{nk}(dx)\Big| \leqslant c^2\int_{|x|\geqslant 1} |x|^2\,\mu_{nk}(dx)$$

together with assumptions (iv) and (v) gives

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}\int_{|x|<1} x_i x_j\,\mu_{nk}(dx) = a_{ij}.$$

Hence we obtain the assertion. ∎

Remark 2. It should be mentioned that moment conditions are needed only for coordinates x_i with $d_i = 1, 2$, and not every Gaussian measure can appear as a limit distribution, only those which are stable with respect to the natural dilations $(\delta_t)_{t>0}$. Condition (ii) assures that the measures μ_{nk} are centered. By the help of suitable shifts it is always possible to ensure that condition (ii) is satisfied (cf. [17]). The Lindeberg condition (v) of Theorem 3.2 implies the validity of the Feller condition

$$\lim_{n\to\infty} \max_{1\leqslant k\leqslant k_n} \int |x|^2 \mu_{nk}(dx) = 0$$

since for any $\varepsilon > 0$ we have

$$\int |x|^2 \mu_{nk}(dx) \leqslant \varepsilon^2 + \int_{|x|\geqslant\varepsilon} |x|^2 \mu_{nk}(dx) \leqslant \varepsilon^2 + \sum_{k=1}^{k_n} \int_{|x|\geqslant\varepsilon} |x|^2 \mu_{nk}(dx).$$

Thus

$$\limsup_{n\to\infty} \max_{1\leqslant k\leqslant k_n} \int |x|^2 \mu_{nk}(dx) \leqslant \varepsilon^2.$$

From Theorem 3.2 one can derive the standard version of the central limit theorem (cf. [32], [3], [22], and [17]).

THEOREM 3.3. *Let μ be a centered probability measure on a stratified Lie group with finite homogeneous moment of second order. (That is, $\int x_i \mu(dx) = 0$ when $d_i = 1$ and $\int |x|^2 \mu(dx) < \infty$.) Then*

$$\delta_{1/\sqrt{n}}(\mu^n) \to \nu,$$

where ν is the Gaussian measure with the infinitesimal generator

$$\sum_{d_i=2} a_i X_i + \tfrac{1}{2} \sum_{d_i=d_j=1} a_{ij} X_i X_j,$$

and $a_i = \int x_i \mu(dx)$ for $d_i = 2$ and $a_{ij} = \int x_i x_j \mu(dx)$ for $d_i = d_j = 1$.

Proof. An easy calculation shows that the triangular system $\mu_{nk} := \delta_{1/\sqrt{n}}(\mu)$, $1 \leqslant k \leqslant n$, $n \geqslant 1$, satisfies the conditions of Theorem 3.2, since

$$\int |x|^2 \mu_{nk}(dx) = n^{-1} \int |x|^2 \mu(dx),$$

$$\int x_i \mu_{nk}(dx) = n^{-1} \int x_i \mu(dx) \quad \text{for } d_i = 2,$$

$$\int x_i x_j \mu_{nk}(dx) = n^{-1} \int x_i x_j \mu(dx) \quad \text{for } d_i = d_j = 1,$$

$$\int_{|x|\geqslant\varepsilon} |x|^2 \mu_{nk}(dx) = n^{-1} \int_{|x|\geqslant\varepsilon\sqrt{n}} |x|^2 \mu(dx). \ \blacksquare$$

4. The Lindeberg theorem for standardized sequences on the Heisenberg group. In this section we investigate the convergence of suitably standardized n-fold convolution products $\mu_1 * \ldots * \mu_n$ of probability measures on the Heisenberg group to the standard Gaussian measure.

Let $\boldsymbol{R}^3$ be equipped with its natural topology and with the product

$$(x_1, x_2, x_3)(y_1, y_2, y_3) = \big(x_1+y_1, x_2+y_2, x_3+y_3+\tfrac{1}{2}(x_1 y_2 - x_2 y_1)\big).$$

Then we obtain a realization of the 3-dimensional Heisenberg group $\boldsymbol{H}$ (over $\boldsymbol{R}$).

The Lie algebra $\mathscr{L}(\boldsymbol{H})$ of the Lie group $\boldsymbol{H}$ can be realized as the vector space $\boldsymbol{R}^3$ with the multiplication

$$[(x_1, x_2, x_3), (y_1, y_2, y_3)] = (0, 0, x_1 y_2 - x_2 y_1).$$

Clearly, $\mathscr{L}(\boldsymbol{H}) = \boldsymbol{R}^2 \oplus \boldsymbol{R}$ is a stratified vector space decomposition of $\mathscr{L}(\boldsymbol{H})$, and the natural basis $\{X_1, X_2, X_3\}$ of $\mathscr{L}(\boldsymbol{H})$ is adapted to this decomposition. Thus $d_1 = d_2 = 1$ and $d_3 = 2$, and the Heisenberg group $\boldsymbol{H}$ is a stratified (nilpotent) Lie group of step 2. The exponential mapping exp: $\mathscr{L}(\boldsymbol{H}) \to \boldsymbol{H}$ is the identity mapping. The natural dilations are given by

$$\delta_t(x_1, x_2, x_3) = (tx_1, tx_2, t^2 x_3)$$

for $t > 0$ and $(x_1, x_2, x_3) \in \boldsymbol{H}$.

Let μ be a centered probability measure on $\boldsymbol{H}$, i.e., $\int x_1 \mu(dx) = \int x_2 \mu(dx) = 0$. Then it can be standardized by the help of an automorphism in the following way. For a real (2×2)-matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant 2}$ let

$$\delta_A(x_1, x_2, x_3) = \big(A(x_1, x_2)^{\mathrm{T}}, x_3 \det(A)\big) \quad \text{for } (x_1, x_2, x_3) \in \boldsymbol{H}.$$

Then δ_A is an automorphism of $\boldsymbol{H}$. Clearly, δ_A is represented by the matrix

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & \det(A) \end{bmatrix}.$$

For a centered probability measure μ on $\boldsymbol{H}$ with $A = \big(\int x_i x_j \mu(dx)\big)_{1 \leqslant i,j \leqslant 2}$ the measure $\delta_{A^{-1/2}}(\mu)$ (where $A^{-1/2}$ is the inverse of the positive definite square root of A) is *standard* in the sense that it is centered and the covariance matrix $\big(\int x_i x_j \delta_{A^{-1/2}}(\mu)(dx)\big)_{1 \leqslant i,j \leqslant 2}$ of the first two coordinates is the unit matrix.

THEOREM 4.1. *Let $(\mu_k)_{k \geqslant 1}$ be a sequence of commutative probability measures on the Heisenberg group $\boldsymbol{H}$ such that*

(i) $\int x_i \mu_k(dx) = 0$ *for* $i = 1, 2, 3$,

(ii) $\int |x|^2 \mu_k(dx) < \infty$.

For $n \geqslant 1$ let $A_n := \sum_{k=1}^n \big(\int x_i x_j \mu_k(dx)\big)_{1 \leqslant i,j \leqslant 2}$. Suppose that there exists $n_0 \in \boldsymbol{N}$ such that $A_{n_0} > 0$ (positive definite), and

(iii) $\sup_{n \geqslant n_0} \big(\det(A_n)\big)^{-1/2} \sum_{k=1}^n \int |x_3| \mu_k(dx) < \infty$,

(iv) $\lim_{n \to \infty} \operatorname{tr}(A_n^{-1}) \sum_{k=1}^n \int_{|x|^2 \geqslant \varepsilon/\operatorname{tr}(A_n^{-1})} |x|^2 \mu_k(dx) = 0$ *for all* $\varepsilon > 0$.

Then

$$\delta_{A_n^{-1/2}}(\mu_1 * \ldots * \mu_n) \to \nu \quad \textit{as } n \to \infty,$$

where $A_n^{-1/2}$ is the inverse of the positive definite square root of A_n, and ν is the standard Gaussian measure on $\boldsymbol{H}$, i.e., its infinitesimal generator is $\frac{1}{2}(X_1^2 + X_2^2)$.

Proof. We show that the triangular system $\mu_{nk} := \delta_{A_n^{-1/2}}(\mu_k)$, $1 \leqslant k \leqslant n$, $n \geqslant 1$, satisfies the conditions of Theorem 3.2.

Let $\langle\cdot,\cdot\rangle$, respectively $\|\cdot\|$, denote the ordinary scalar product and the norm of $\boldsymbol{R}^2$. If A is a positive definite symmetric (2×2)-matrix, then

$$\|A(x_1, x_2)^{\mathrm{T}}\|^2 = \langle A(x_1, x_2)^{\mathrm{T}}, A(x_1, x_2)^{\mathrm{T}}\rangle$$
$$= \langle A^2(x_1, x_2)^{\mathrm{T}}, (x_1, x_2)^{\mathrm{T}}\rangle \leqslant \|(x_1, x_2)\|^2 \operatorname{tr}(A^2)$$

and

$$\det(A) \leqslant \tfrac{1}{2}\operatorname{tr}(A^2).$$

Hence using the estimate $|(x_1, x_2, x_3)|^2 \leqslant c(x_1^2+x_2^2+|x_3|)$, $(x_1, x_2, x_3)\in\boldsymbol{H}$, valid with some $c>0$ depending on $|\cdot|$, we obtain

$$|\delta_A(x_1, x_2, x_3)|^2 \leqslant c\big(\|A(x_1, x_2)^{\mathrm{T}}\|^2+|x_3|\det(A)\big) \leqslant c\,|x|^2\operatorname{tr}(A^2).$$

Thus

$$\int_{|x|\geqslant\varepsilon} |x|^2\,\mu_{nk}(dx) = \int_{|\delta_{A_n^{-1/2}}x|\geqslant\varepsilon} |\delta_{A_n^{-1/2}}x|^2\,\mu_k(dx)$$
$$\leqslant c\operatorname{tr}(A_n^{-1}) \int_{|x|^2\geqslant\varepsilon^2/\operatorname{tr}(A_n^{-1})} |x|^2\,\mu_k(dx).$$

Clearly, for $i, j = 1, 2$ we have

$$\sum_{k=1}^{n}\int x_i x_j\,\mu_{nk}(dx) = \delta_{ij},$$

and this together with assumption (iii) implies

$$\sup_{n\geqslant 1}\sum_{k=1}^{n}\int |x|^2\,\mu_{nk}(dx) < \infty. \ \blacksquare$$

5. Convergence of convolution semigroups. For any $n\in\boldsymbol{N}$ let $S_n := (\mu_t^{(n)})_{t\geqslant 0}$ be a convolution semigroup in $\mathscr{P}(G)$ and let $S := (\mu_t)_{t\geqslant 0}$ be a further convolution semigroup in $\mathscr{P}(G)$. Then we write $S_n\to S$ if $\mu_t^{(n)}\to\mu_t$ uniformly in $t\in[0, d]$ for all $d>0$.

Hazod [8], p. 36, proved that $A_n(f)\to A(f)$ for all $f\in\mathscr{E}(G)$ implies $S_n\to S$. (Indeed, as mentioned in [9], it is sufficient to assume that $A_n(f)\to A(f)$ for all $f\in\mathscr{D}(G)$ and the Lévy measures η_n of A_n are uniformly tight outside a neighborhood of e.)

For the proof of the converse we shall use the following proposition due to Siebert [26], Propositions 6.3 and 6.4.

PROPOSITION 5.1. *Let G be a Lie group, $(S_n)_{n\geqslant 1}$ a sequence of convolution semigroups in $\mathscr{P}(G)$, and let S be a further convolution semigroup in $\mathscr{P}(G)$. Let A_n and A be the generating functionals, $(a_i^{(n)}, a_{ij}^{(n)}, \eta_n)_{1\leqslant i,j\leqslant m}$ and $(a_i, a_{ij}, \eta)_{1\leqslant i,j\leqslant m}$ the canonical decompositions of A_n and A, respectively. If $S_n\to S$, then*

$$\eta_n|(G\backslash U)\to\eta|(G\backslash U) \quad \textit{for all } U\in\mathscr{U}(e) \textit{ with } \eta(\partial U) = 0$$

and

$$\sup_{n\leqslant 1}\Big(\sum_{i=1}^{m}|a_i^{(n)}|+\sum_{i,j=1}^{m}|a_{ij}^{(n)}|+\int_{G^\times}\varphi\,d\eta_n\Big)<\infty.$$

Now we present the analogue of the classical necessary and sufficient conditions for convergence of convolution semigroups (see [5], § 19, Theorems 1 and 2, in case $\boldsymbol{R}$, and [30] in case $\boldsymbol{R}^k$).

THEOREM 5.1. *Let G be a Lie group, $(S_n)_{n\geqslant 1}$ a sequence of convolution semigroups in $\mathscr{P}(G)$, and let S be a further convolution semigroup in $\mathscr{P}(G)$. Let A_n and A be the generating functionals, $(a_i^{(n)}, a_{ij}^{(n)}, \eta_n)_{1\leqslant i,j\leqslant m}$ and $(a_i, a_{ij}, \eta)_{1\leqslant i,j\leqslant m}$ the canonical decompositions of A_n and A, respectively. Then the following assertions are equivalent*:

(i) $S_n\to S$.

(ii) $A_n(f)\to A(f)$ *for all* $f\in\mathscr{E}(G)$.

(iii) (a) $\eta_n(B)\to\eta(B)$ *for all* $B\in\mathscr{B}(G)$ *with* $e\notin B^-$ *and* $\eta(\partial B)=0$;

(b) $a_{ij}^{(n)}+\frac{1}{2}\int_{G^\times}\zeta_i(x)\zeta_j(x)\eta_n(dx)\to a_{ij}+\frac{1}{2}\int_{G^\times}\zeta_i(x)\zeta_j(x)\eta(dx)$ *for all* $1\leqslant i,\ j\leqslant m$;

(c) $a_i^{(n)}\to a_i$ *for all* $1\leqslant i\leqslant m$.

(iv) (a) $\eta_n(B)\to\eta(B)$ *for all* $B\in\mathscr{B}(G)$ *with* $e\notin B^-$ *and* $\eta(\partial B)=0$;

(b) $\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\big(a_{ij}^{(n)}+\frac{1}{2}\int_{0<\varphi(x)\leqslant\varepsilon}\zeta_i(x)\zeta_j(x)\eta_n(dx)\big)$ $=\lim_{\varepsilon\downarrow 0}\liminf_{n\to\infty}\big(a_{ij}^{(n)}+\frac{1}{2}\int_{0<\varphi(x)\leqslant\varepsilon}\zeta_i(x)\zeta_j(x)\eta_n(dx)\big)=a_{ij}$ *for all* $1\leqslant i,j\leqslant m$;

(c) $a_i^{(n)}\to a_i$ *for all* $1\leqslant i\leqslant m$.

Proof. (ii)$\Rightarrow$(i) has been proved by Hazod [8], p. 36.

(iii)$\Rightarrow$(iv). Let $\varepsilon>0$. Then there exist $\varepsilon_1,\varepsilon_2>0$ with $0<\varepsilon_1<\varepsilon<\varepsilon_2$ and

$$\eta(\{x\in G\colon\ \varphi(x)=\varepsilon_i\})=0\qquad\text{for } i=1,2$$

(cf. [26], p. 140). Clearly, there is a constant $c>0$ such that $|\zeta_i(x)\zeta_j(x)|\leqslant c\varphi(x)$ for all $x\in G$. Then

$$\Big|\int_{\varphi>\varepsilon}\zeta_i\zeta_j\,d\eta_n-\int_{\varphi>\varepsilon_2}\zeta_i\zeta_j\,d\eta_n\Big|=\Big|\int_{\varepsilon<\varphi\leqslant\varepsilon_2}\zeta_i\zeta_j\,d\eta_n\Big|\leqslant c\int_{\varepsilon<\varphi\leqslant\varepsilon_2}\varphi\,d\eta_n\leqslant c\int_{\varepsilon_1<\varphi\leqslant\varepsilon_2}\varphi\,d\eta_n.$$

Thus we obtain

$$\int_{\varphi>\varepsilon_2}\zeta_i\zeta_j\,d\eta_n-c\int_{\varepsilon_1<\varphi\leqslant\varepsilon_2}\varphi\,d\eta_n\leqslant\int_{\varphi>\varepsilon}\zeta_i\zeta_j\,d\eta_n\leqslant\int_{\varphi>\varepsilon_2}\zeta_i\zeta_j\,d\eta_n+c\int_{\varepsilon_1<\varphi\leqslant\varepsilon_2}\varphi\,d\eta_n.$$

Letting $n\to\infty$ it follows that

$$\int_{\varphi>\varepsilon_2}\zeta_i\zeta_j\,d\eta_n-c\int_{\varepsilon_1<\varphi\leqslant\varepsilon_2}\varphi\,d\eta\leqslant\liminf_{n\to\infty}\int_{\varphi>\varepsilon}\zeta_i\zeta_j\,d\eta_n$$
$$\leqslant\limsup_{n\to\infty}\int_{\varphi>\varepsilon}\zeta_i\zeta_j\,d\eta_n\leqslant\int_{\varphi>\varepsilon_2}\zeta_i\zeta_j\,d\eta+c\int_{\varepsilon_1<\varphi\leqslant\varepsilon_2}\varphi\,d\eta.$$

Consequently,

$$\begin{aligned}\limsup_{n\to\infty}\Big(a_{ij}^{(n)}+\tfrac12\int\limits_{0<\varphi\le\varepsilon}\zeta_i\zeta_j\,d\eta_n\Big) &\le \lim_{n\to\infty}\Big(a_{ij}^{(n)}+\tfrac12\int\limits_{G^\times}\zeta_i\zeta_j\,d\eta_n\Big)-\tfrac12\liminf_{n\to\infty}\int\limits_{\varphi>\varepsilon}\zeta_i\zeta_j\,d\eta_n\\ &\le a_{ij}+\tfrac12\int\limits_{G^\times}\zeta_i\zeta_j\,d\eta-\tfrac12\int\limits_{\varphi>\varepsilon_2}\zeta_i\zeta_j\,d\eta+\tfrac12 c\int\limits_{\varepsilon_1<\varphi\le\varepsilon_2}\varphi\,d\eta\\ &= a_{ij}+\tfrac12\int\limits_{0<\varphi\le\varepsilon_2}\zeta_i\zeta_j\,d\eta+\tfrac12 c\int\limits_{\varepsilon_1<\varphi\le\varepsilon_2}\varphi\,d\eta\le a_{ij}+c\int\limits_{0<\varphi\le\varepsilon_2}\varphi\,d\eta.\end{aligned}$$

Letting $\varepsilon\downarrow 0$ and $\varepsilon_2\downarrow 0$, we obtain

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\Big(a_{ij}^{(n)}+\tfrac12\int\limits_{0<\varphi(x)\le\varepsilon}\zeta_i(x)\zeta_j(x)\,\eta_n(dx)\Big)\le a_{ij}.$$

Similarly we have

$$\lim_{\varepsilon\downarrow 0}\liminf_{n\to\infty}\Big(a_{ij}^{(n)}+\tfrac12\int\limits_{0<\varphi(x)\le\varepsilon}\zeta_i(x)\zeta_j(x)\,\eta_n(dx)\Big)\ge a_{ij}.$$

Hence (iii) $\Rightarrow$ (iv) is proved.

(iv) $\Rightarrow$ (iii). Let $\varepsilon>0$ with $\eta(\{x\in G:\ \varphi(x)=\varepsilon\})=0$. Then

$$\begin{aligned}&\limsup_{n\to\infty}\Big(a_{ij}^{(n)}+\tfrac12\int\limits_{G^\times}\zeta_i\zeta_j\,d\eta_n\Big)\\ &\le \limsup_{n\to\infty}\Big(a_{ij}^{(n)}+\tfrac12\int\limits_{0<\varphi\le\varepsilon}\zeta_i\zeta_j\,d\eta_n\Big)+\tfrac12\lim_{n\to\infty}\int\limits_{\varphi>\varepsilon}\zeta_i\zeta_j\,d\eta_n\\ &= \limsup_{n\to\infty}\Big(a_{ij}^{(n)}+\tfrac12\int\limits_{0<\varphi\le\varepsilon}\zeta_i\zeta_j\,d\eta_n\Big)+\tfrac12\lim_{n\to\infty}\int\limits_{G^\times}\zeta_i\zeta_j\,d\eta_n-\tfrac12\lim_{n\to\infty}\int\limits_{0<\varphi\le\varepsilon}\zeta_i\zeta_j\,d\eta_n.\end{aligned}$$

Letting $\varepsilon\downarrow 0$ we obtain

$$\limsup_{n\to\infty}\Big(a_{ij}^{(n)}+\tfrac12\int\limits_{G^\times}\zeta_i\zeta_j\,d\eta_n\Big)\le a_{ij}+\tfrac12\int\limits_{G^\times}\zeta_i\zeta_j\,d\eta.$$

Similarly we get

$$\liminf_{n\to\infty}\Big(a_{ij}^{(n)}+\tfrac12\int\limits_{G^\times}\zeta_i\zeta_j\,d\eta_n\Big)\ge a_{ij}+\tfrac12\int\limits_{G^\times}\zeta_i\zeta_j\,d\eta.$$

Hence (iv) $\Rightarrow$ (iii) is proved.

(iii) $\Rightarrow$ (ii). Let $\varepsilon>0$ with $\eta(\{x\in G:\ \varphi(x)=\varepsilon\})=0$. Then conditions (a) and (b) of (iii) imply

$$a_{ij}^{(n)}+\tfrac12\int\limits_{0<\varphi\le\varepsilon}\zeta_i\zeta_j\,d\eta_n\to a_{ij}+\tfrac12\int\limits_{0<\varphi\le\varepsilon}\zeta_i\zeta_j\,d\eta$$

for all $1\le i,j\le n$. If moreover $\{x\in G:\ \varphi(x)\le\varepsilon\}\subset U_0$, then we obtain

$$\sum_{i=1}^m a_{ii}^{(n)}+\tfrac12\int\limits_{0<\varphi\le\varepsilon}\varphi\,d\eta_n\to\sum_{i=1}^m a_{ii}+\tfrac12\int\limits_{0<\varphi\le\varepsilon}\varphi\,d\eta.$$

Again from (iii) (a) we infer that for all $B\in\mathscr{B}(G^\times)$ with $\eta(\partial B)=0$

$$\sum_{i=1}^m a_{ii}^{(n)}+\tfrac12\int\limits_B\varphi\,d\eta_n\to\sum_{i=1}^m a_{ii}+\tfrac12\int\limits_B\varphi\,d\eta.$$

For every $n \in N$ we define the measure $\nu_n \in \mathscr{M}_+^b(G)$ by

$$\nu_n(\{e\}) := \sum_{i=1}^{m} a_{ii}^{(n)}, \qquad \nu_n(B) := \tfrac{1}{2}\int_B \varphi \, d\eta_n \text{ for all } B \in \mathscr{B}(G^\times).$$

Similarly, we define $\nu \in \mathscr{M}_+^b(G)$ by

$$\nu(\{e\}) := \sum_{i=1}^{m} a_{ii} \qquad \nu(B) := \tfrac{1}{2}\int_B \varphi \, d\eta \text{ for all } B \in \mathscr{B}(G^\times).$$

Then we have $\nu_n \to \nu$. Thus we can conclude that if $g \in \mathscr{E}(G)$ and $h := g/\varphi \in \mathscr{C}^b(G^\times)$ is continuously extended to G by $h(e) = 0$, then $\int_{G^\times} g\,d\eta_n \to \int_{G^\times} g\,d\eta$.

Now we use the idea which was pointed out by Grenander [6], p. 196. (See also [13].) For all $f \in \mathscr{E}(G)$ and $\varepsilon > 0$ we have the decomposition

$$A_n(f) = \sum_{i=1}^{m} a_i^{(n)}(X_i f)(e) + \sum_{i,j=1}^{m} \Big(a_{ij}^{(n)} + \tfrac{1}{2}\int_{0<\varphi\leqslant\varepsilon} \zeta_i\zeta_j \, d\eta_n\Big)(X_i X_j f)(e)$$

$$+ \int_{\varphi>\varepsilon} \Big[f(x) - f(e) - \sum_{i=1}^{m} \zeta_i(x)(X_i f)(e)\Big]\eta_n(dx) + \int_{0<\varphi\leqslant\varepsilon} g(x)\,\eta_n(dx),$$

where the function

$$g(x) := f(x) - f(e) - \sum_{i=1}^{m} \zeta_i(x)(X_i f)(e) - \tfrac{1}{2}\sum_{i,j=1}^{m} \zeta_i(x)\zeta_j(x)(X_i X_j f)(e)$$

lies in $\mathscr{E}(G)$, and the Taylor expansion in a neighborhood of $e \in G$ implies that the function $h := g/\varphi \in \mathscr{C}^b(G^\times)$ is continuously extended to G by $h(e) = 0$ since

$$|g(x)| \leqslant \tfrac{1}{6}\sum_{i,j,k=1}^{m} \big|\zeta_i(x)\zeta_j(x)\zeta_k(x)(X_i X_j X_k f)\big(\theta(x)\big)\big|$$

for all x in a suitable neighborhood $U \subset U_0$, where $\theta(x) \in U$. Thus

$$|g(x)| \leqslant c(f;\, m)\sum_{i=1}^{m} |\zeta_i(x)|^3 \leqslant c'(f;\, m)\,\varphi(x)^{3/2}$$

with some constants $c(f;\, m)$ and $c'(f;\, m)$ depending on $f \in \mathscr{E}(G)$ and on the dimension m. Taking into consideration the above decomposition of A_n we conclude (iii) $\Rightarrow$ (ii).

(i) $\Rightarrow$ (iii). Applying Proposition 5.1 we obtain (iii) (a) and

$$\sup_{n\geqslant 1}\Big(\sum_{i=1}^{m} |a_i^{(n)}| + \sum_{i,j=1}^{m} |a_{ij}^{(n)}| + \int_{G^\times} \varphi\, d\eta_n\Big) < \infty.$$

Now we use some idea of Siebert [26] (also applied in [14]). Let $(n_k)_{k\geqslant 1}$ be an arbitrary strictly monotone sequence in N. Then there is a subsequence $(n_{k_l})_{l\geqslant 1}$ of $(n_k)_{k\geqslant 1}$ such that there exist

$$b_i := \lim_{l\to\infty} a_i^{(n_{k_l})}, \qquad b_{ij} := \lim_{l\to\infty} \big(a_{ij}^{(n_{k_l})} + \tfrac{1}{2} \int_{G^\times} \zeta_i \zeta_j \, d\eta_{n_{k_l}}\big) - \tfrac{1}{2} \int_{G^\times} \zeta_i \zeta_j \, d\eta .$$

Obviously, $(b_{ij})_{1 \leqslant i,j \leqslant m}$ is a real symmetric positive semidefinite matrix. Let us define for all $f \in \mathscr{E}(G)$

$$B(f) := \sum_{i=1}^{m} b_i (X_i f)(e) + \sum_{i,j=1}^{m} b_{ij} (X_i X_j f)(e)$$

$$+ \int_{G^\times} \big[f(x) - f(e) - \sum_{i=1}^{m} \zeta_i(x)(X_i f)(e)\big] \, \eta(dx).$$

Then B is a generating functional of a convolution semigroup $\tilde{S}$. As in (iii) $\Rightarrow$ (ii), it follows that

$$\lim_{l\to\infty} A_{n_{k_l}}(f) = B(f) \qquad \text{for all } f \in \mathscr{E}(G),$$

and (ii) $\Rightarrow$ (i) implies $\lim_{l\to\infty} S_{n_{k_l}} = \tilde{S}$. Hence $S_n \to S$ yields $\tilde{S} = S$; thus $B = A$. Consequently, any strictly monotone sequence $(n_k)_{k\geqslant 1}$ in N has a subsequence $(n_{k_l})_{l\geqslant 1}$ for which (ii), (iii) (b) and (iii) (c) hold. This implies also (iii) (a) for this subsequence. Hence we obtain the assertion. ■

Remark 3. In the case of a stratified Lie group, conditions (iii) (a) and (iv) (a) can be replaced by

$$\eta_n\{|x| > \varepsilon\} \to \eta\{|x| > \varepsilon\} \qquad \text{for all } \varepsilon > 0 \text{ with } \eta\{|x| = \varepsilon\} = 0.$$

Condition (iii) (b) can be replaced by

$$a_{ij}^{(n)} + \tfrac{1}{2} \int_{0<|x|\leqslant\varepsilon} x_i x_j \eta_n(dx) \to a_{ij} + \tfrac{1}{2} \int_{0<|x|\leqslant\varepsilon} x_i x_j \eta(dx)$$

for all $1 \leqslant i, j \leqslant m$ and for every $\varepsilon > 0$ such that $\eta\{|x| = \varepsilon\} = 0$. (Obviously, it is enough to have the above relation for at least one $\varepsilon > 0$ with $\eta\{|x| = \varepsilon\} = 0$ because of (iii) (a).)

Similarly, (iv) (b) can be replaced by

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \big(a_{ij}^{(n)} + \tfrac{1}{2} \int_{0<|x|\leqslant\varepsilon} x_i x_j \eta_n(dx)\big)$$

$$= \lim_{\varepsilon\downarrow 0} \liminf_{n\to\infty} \big(a_{ij}^{(n)} + \tfrac{1}{2} \int_{0<|x|\leqslant\varepsilon} x_i x_j \eta_n(dx)\big) = a_{ij}$$

for all $1 \leqslant i, j \leqslant m$.

6. Unitary representations and Fourier transforms. A (continuous) *unitary representation* of a locally compact group G is a homomorphism D of G into the group of unitary operators on a complex Hilbert space $\mathscr{H}$ such that the mapping $x \to D(x)u$ of G into $\mathscr{H}$ is continuous for all $u \in \mathscr{H}$. The space $\mathscr{H}$ is called the *representation space* of D and is denoted by $\mathscr{H}(D)$. The inner product and the norm in $\mathscr{H}(D)$ are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. The class of all irreducible (continuous) unitary representations of G is denoted by $\mathrm{Irr}(G)$.

Let $D \in \mathrm{Irr}(G)$. The vector $u \in \mathscr{H}(D)$ is said to be *differentiable* for D if the coefficient function $x \to \langle D(x)u, v\rangle$ of G into $\boldsymbol{C}$ is in $\mathscr{E}(G)$ for any $v \in \mathscr{H}(D)$. By $\mathscr{H}_0(D)$ we denote the space of all vectors in $\mathscr{H}(D)$ differentiable for D.

Let $D \in \mathrm{Irr}(G)$. The vector $u \in \mathscr{H}(D)$ is said to be a C^∞*-vector* for D if the vector-valued function $x \to D(x)u$ of G into $\mathscr{H}(D)$ is infinitely differentiable. By $\mathscr{H}_\infty(D)$ we denote the space of all C^∞-vectors in $\mathscr{H}(D)$. Then $\mathscr{H}_\infty(D) \subseteq \mathscr{H}_0(D)$ is obvious; $\mathscr{H}_\infty(D) = \mathscr{H}_0(D)$ is known (cf. [26], p. 122, Remark).

It is known (see, e.g., [31]) that every irreducible (continuous) unitary representation of the Heisenberg group $\boldsymbol{H}$ is unitarily equivalent to one of the representations $D_{\alpha,\beta}$, $\alpha, \beta \in \boldsymbol{R}$, or $D_{\pm\lambda}$, $\lambda \in \boldsymbol{R}\setminus\{0\}$, defined in the following way:

$D_{\alpha,\beta}$, $\alpha, \beta \in \boldsymbol{R}$, are one-dimensional representations of $\boldsymbol{H}$ on $\boldsymbol{C}$ defined by

$$D_{\alpha,\beta}(x_1, x_2, x_3) := \exp[i(\alpha x_1 + \beta x_2)];$$

$D_{\pm\lambda}$, $\lambda \in \boldsymbol{R}\setminus\{0\}$, are infinite-dimensional representations of $\boldsymbol{H}$ on $L^2(\boldsymbol{R})$ defined by

$$D_{\pm\lambda}(x_1, x_2, x_3)u(s) := \exp[i(\pm\lambda x_3 \pm \lambda^{1/2} x_1 s + \lambda x_1 x_2/2)]\, u(s + \lambda^{1/2} x_2)$$

for $\lambda > 0$, $u \in L^2(\boldsymbol{R})$.

Moreover, $\mathscr{H}_0(D_{\pm\lambda}) = \mathscr{H}_\infty(D_{\pm\lambda}) = \mathscr{S}(\boldsymbol{R})$ for all $\lambda \in \boldsymbol{R}\setminus\{0\}$, where the Schwartz space $\mathscr{S}(\boldsymbol{R})$ is defined by (cf. [31])

$$\mathscr{S}(\boldsymbol{R}) := \left\{ u \in C^\infty(\boldsymbol{R}) \colon \left\| s^j \frac{d^k u}{ds^k} \right\|_\infty < \infty \text{ for all } j, k \in \boldsymbol{N} \cup \{0\} \right\}.$$

For a probability measure μ on a locally compact group G we define its *Fourier transform* $\hat{\mu}$ by

$$\langle \hat{\mu}(D)u, v\rangle := \int \langle D(x)u, v\rangle\, \mu(dx)$$

for all $D \in \mathrm{Irr}(G)$ $(u, v \in \mathscr{H}(D))$. Then $\hat{\mu}(D)$ is a bounded linear operator on $\mathscr{H}(D)$. (For information on Fourier transforms see [11] and [26].)

It is easy to show that for a centered probability measure μ on $\boldsymbol{R}$ the Fourier transform can be estimated in the following way:

$$|\hat{\mu}(t) - 1| = \left| \int (e^{itx} - 1 - itx)\, \mu(dx) \right| \leqslant \frac{t^2}{2} \int x^2\, \mu(dx) \quad \text{for all } t \in \boldsymbol{R}.$$

We prove a similar result in the case of the Heisenberg group. Recall that a probability measure μ on $\boldsymbol{H}$ is centered if $\int x_1\, \mu(dx) = \int x_2\, \mu(dx) = 0$.

LEMMA 1. *Let μ be a centered probability measure on $\boldsymbol{H}$. Then for all $D \in \mathrm{Irr}(\boldsymbol{H})$ and $u \in \mathscr{H}_0(D)$ there exists a constant $c(D, u)$ such that*

$$\|\hat{\mu}(D)u - u\| \leqslant c(D, u) \int |x|^2\, \mu(dx).$$

Proof. In the case of the one-dimensional representations $D_{\alpha,\beta}$, $\alpha, \beta \in \boldsymbol{R}$, we have simply

$$\begin{aligned}
|\hat{\mu}(D_{\alpha,\beta})-1| &= \left|\int (\exp[i(\alpha x_1+\beta x_2)]-1-i\alpha x_1-i\beta x_2)\,\mu(dx)\right| \\
&\leqslant \left|\int (\exp[i\alpha x_1]-1-i\alpha x_1)\exp[i\beta x_2]\,\mu(dx)\right| \\
&\quad + \left|\int (\exp[i\beta x_2]-1-i\beta x_2)\,\mu(dx)\right| \\
&\quad + \left|\int i\alpha x_1(\exp[i\beta x_2]-1)\,\mu(dx)\right| \\
&\leqslant \frac{\alpha^2}{2}\int x_1^2\,\mu(dx)+\frac{\beta^2}{2}\int x_2^2\,\mu(dx)+|\alpha\beta|\int |x_1 x_2|\,\mu(dx) \\
&\leqslant \frac{\alpha^2+\beta^2}{2}\int (x_1^2+x_2^2)\,\mu(dx) \leqslant \frac{\alpha^2+\beta^2}{2}\int |x|^2\,\mu(dx).
\end{aligned}$$

In the case of the infinite-dimensional representations $D_{\pm\lambda}$, $\lambda\in\boldsymbol{R}\setminus\{0\}$, we use a similar method. For $\lambda>0$ and $u\in\mathscr{S}(\boldsymbol{R})=\mathscr{H}_0(D_{\pm\lambda})$ we have the decomposition

$$\hat{\mu}(D_{\pm\lambda})u(s)-u(s)=I_1+I_2,$$

where

$$\begin{aligned}
I_1 &= \int \exp[i(\pm\lambda x_3\pm\lambda^{1/2}x_1 s+\lambda x_1 x_2/2)]\,(u(s+\lambda^{1/2}x_2)-u(s))\,\mu(dx), \\
I_2 &= \int (\exp[i(\pm\lambda x_3\pm\lambda^{1/2}x_1 s+\lambda x_1 x_2/2)]-1\mp i\lambda x_1 s)\,\mu(dx)\,u(s).
\end{aligned}$$

In the case of the first integral we use the Taylor formula

$$u(s+\lambda^{1/2}x_2)-u(s)=\lambda^{1/2}x_2 u'(s)+\lambda x_2^2\int_0^1 (1-t)\,u''(s+t\lambda^{1/2}x_2)\,dt.$$

Since $\int x_2\,\mu(dx)=0$, we have to deal only with the second term of the Taylor formula. We have

$$\begin{aligned}
\|I_1\| &= \Big\|\lambda x_2^2\int \exp[i(\pm\lambda x_3\pm\lambda^{1/2}x_1 s+\lambda x_1 x_2/2)] \\
&\qquad\times\int_0^1 (1-t)\,u''(s+t\lambda^{1/2}x_2)\,dt\,\mu(dx)\Big\| \leqslant \lambda\frac{\|u''\|}{2}\int |x|^2\,\mu(dx).
\end{aligned}$$

For the second integral of the decomposition of $\hat{\mu}(D_{\pm\lambda})u(s)-u(s)$ we have

$$\begin{aligned}
&|\exp[i(\pm\lambda x_3\pm\lambda^{1/2}x_1 s+\lambda x_1 x_2/2)]-1\mp i\lambda x_1 s| \\
&\quad\leqslant |\exp[\pm i\lambda x_3]-1|+|\exp[i\lambda x_1 x_2/2]-1|+|\exp[\pm i\lambda^{1/2}x_1 s]-1\mp i\lambda x_1 s| \\
&\quad\leqslant \lambda(|x_3|+|x_1 x_2|/2+x_1^2 s^2/2)\leqslant c\lambda|x|^2(1+s^2)
\end{aligned}$$

using property (2) of homogeneous norms.

Summarizing we have

$$\|\hat{\mu}(D_{\pm\lambda})u-u\|\leqslant c\lambda(\|u\|+\|s^2 u\|+\|u''\|)\int |x|^2\,\mu(dx)\qquad\text{for all }\lambda\in\boldsymbol{R}\setminus\{0\}.$$

Hence we obtain the assertion. ■

7. The accompanying Poisson system of a triangular system. The *accompanying Poisson system* of a triangular system $(\mu_{nk})_{k=1,\ldots,k_n;n\geqslant 1}$ is defined by

$$\nu_{nk} := \exp(\mu_{nk} - \varepsilon_e), \qquad k = 1, \ldots, k_n;\ n \geqslant 1.$$

For a commutative triangular system $(\mu_{nk})_{k=1,\ldots,k_n;n\geqslant 1}$ the row products of its accompanying Poisson system are the Poisson measures

$$\exp\Big(\sum_{k=1}^{k_n} (\mu_{nk} - \varepsilon_e)\Big), \qquad n \geqslant 1.$$

The *accompanying sequence of Poisson semigroups* of a system $(\mu_{nk})_{k=1,\ldots,k_n;n\geqslant 1}$ is defined by

$$S_n := (\nu_t^{(n)})_{t\geqslant 0}, \qquad \nu_t^{(n)} := \exp\Big(t \sum_{k=1}^{k_n} (\mu_{nk} - \varepsilon_e)\Big) \quad \text{for } n \geqslant 1,\ t \geqslant 0.$$

Applying Theorem 5.1 one can simply obtain necessary and sufficient conditions for the convergence of the accompanying sequence of Poisson semigroups to a Gaussian one.

PROPOSITION 7.1. *Let $(\mu_{nk})_{k=1,\ldots,k_n;n\geqslant 1}$ be a commutative triangular system on a Lie group with the accompanying sequence of Poisson semigroups $(S_n)_{n\geqslant 1}$. The following statements are equivalent:*

(i) *$S_n \to S$, where $S = (\nu_t)_{t\geqslant 0}$ is the Gaussian semigroup with the generating functional*

$$A(f) = \sum_{i=1}^{m} a_i (X_i f)(e) + \tfrac{1}{2} \sum_{i,j=1}^{m} a_{ij} (X_i X_j f)(e).$$

(ii) (a) $\lim_{n\to\infty} \sum_{k=1}^{k_n} \mu_{nk}(G\backslash B) = 0$ *for all* $B \in \mathscr{B}(G)$ *with* $e \notin B^-$;

(b) $\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{G^\times} \zeta_i(x)\zeta_j(x)\,\mu_{nk}(dx) = a_{ij}$ *for all* $1 \leqslant i, j \leqslant m$;

(c) $\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{G^\times} \zeta_i(x)\,\mu_{nk}(dx) = a_i$ *for all* $1 \leqslant i \leqslant m$.

Proof. The Poisson semigroup $S_n = (\nu_t^{(n)})_{t\geqslant 0}$ has the canonical decomposition $(b_i^{(n)}, 0, \zeta_n)_{1\leqslant i\leqslant m}$, where

$$b_i^{(n)} := \sum_{k=1}^{k_n} \zeta_i(x)\,\mu_{nk}(dx), \qquad \zeta_n := \sum_{k=1}^{k_n} \mu_{nk}.$$

The Gaussian semigroup $S := (\nu_t)_{t\geqslant 0}$ has the canonical decomposition $(a_i, a_{ij}, 0)_{1\leqslant i,j\leqslant m}$. Theorem 5.1 yields the assertions. ∎

Remark 4. In the case of a stratified Lie group, conditions in (ii) can be replaced by

(a) $\lim_{n\to\infty} \sum_{k=1}^{k_n} \mu_{nk}\{|x| > \varepsilon\} = 0$ for every $\varepsilon > 0$;

(b) $\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x|<1} x_i x_j\,\mu_{nk}(dx) = a_{ij}$ for all $1 \leqslant i, j \leqslant m$;

(c) $\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x|<1} x_i\,\mu_{nk}(dx) = a_i$ for all $1 \leqslant i \leqslant m$.

In order to obtain necessary and sufficient conditions for the convergence of triangular system to a Gaussian measure we have to ensure that the convergence of the triangular system implies the convergence of the accompanying sequence of Poisson semigroups. We shall use the following statement due to Siebert [26], Proposition 8.1.

PROPOSITION 7.2. *Let $(\mu_{nk})_{k=1,\dots,k_n;n\geqslant 1}$ be a commutative and infinitesimal system of probability measures on a locally compact group G. Suppose that*

$$\limsup_{n\to\infty}\sum_{k=1}^{k_n}\|\hat{\mu}_{nk}(D)u-u\|<\infty \qquad \text{for all } D\in\operatorname{Irr}(G),\ u\in\mathscr{H}_0(D).$$

Then the system $(\mu_{nk})_{k=1,\dots,k_n;n\geqslant 1}$ is convergent if and only if the accompanying Poisson system is convergent, and in the affirmative case their limits coincide.

8. The Lindeberg–Feller theorem on the Heisenberg group. First we prove a convergence theorem for a symmetric triangular system in the case of bounded (homogeneous) moments of second order.

THEOREM 8.1. *Let $(\mu_{nk})_{k=1,\dots,k_n;n\geqslant 1}$ be a commutative system of symmetric probability measures on the Heisenberg group $\boldsymbol{H}$ which satisfies the condition*

$$\sup_{n\geqslant 1}\sum_{k=1}^{k_n}\int|x|^2\,\mu_{nk}(dx)<\infty. \tag{3}$$

Then the following statements are equivalent:

(i) (a) $(\mu_{nk})_{k=1,\dots,k_n;n\geqslant 1}$ *is infinitesimal;*

(b) $\mu_{n1}*\dots*\mu_{nk_n}\to\nu$ *as* $n\to\infty$, *where* $\nu=\nu_1$, $(\nu_t)_{t\geqslant 0}$ *is the (symmetric) Gaussian semigroup with the generating functional*

$$A(f)=\tfrac{1}{2}\sum_{i,j=1}^{3}a_{ij}(X_iX_jf)(e).$$

(ii) (a) $\lim_{n\to\infty}\sum_{k=1}^{k_n}\mu_{nk}\{|x|>\varepsilon\}=0$ *for every* $\varepsilon>0$;

(b) $\lim_{n\to\infty}\sum_{k=1}^{k_n}\int_{|x|<1}x_ix_j\,\mu_{nk}(dx)=a_{ij}$ *for all* $1\leqslant i,j\leqslant 3$.

Proof. (i)$\Rightarrow$(ii). Condition (3) together with Lemma 1 implies

$$\sup_{n\geqslant 1}\sum_{k=1}^{k_n}\|\hat{\mu}_{nk}(D)u-u\|<\infty \qquad \text{for all } D\in\operatorname{Irr}(\boldsymbol{H}) \text{ and } u\in\mathscr{H}_0(D).$$

Using Proposition 7.2 we conclude that (a) and (b) of (i) imply

$$\exp\Big(\sum_{k=1}^{k_n}(\mu_{nk}-\varepsilon_e)\Big)\to\nu.$$

Since the Poisson measure $\nu_t^{(n)}=\exp\big(t\sum_{k=1}^{k_n}(\mu_{nk}-\varepsilon_e)\big)$ is symmetric for all $n\geqslant 1$, $t\geqslant 0$, the convolution operator $T_t^{(n)}:=T_{\nu_t^{(n)}}$ corresponding to the measure $\nu_t^{(n)}$ is a selfadjoint positive semidefinite contraction operator on the (complex) Hilbert space $L^2(H)$. Since $S_n = (\nu_t^{(n)})_{t \geqslant 0}$ is a convolution semigroup, $(T_t^{(n)})_{t \geqslant 0}$ is a (strongly continuous) semigroup of operators on $L^2(H)$. In view of [23], Section 141, there exist spectral resolutions $(E_\varrho^{(n)})_{\varrho \in [0,1]}$ such that

$$T_t^{(n)} = \int_0^1 \varrho^t \, dE_\varrho^{(n)} \quad \text{for every } t \geqslant 0, n \in \boldsymbol{N}.$$

Similarly, the convolution operator $T_t := T_{\nu_t}$ corresponding to the measure ν_t is a selfadjoint positive semidefinite contraction operator on $L^2(H)$ and admits a spectral decomposition

$$T_t = \int_0^1 \varrho^t \, dE_\varrho \quad \text{for every } t \geqslant 0.$$

Now assumption (i) (b) yields $T_1^{(n)} \to T_1$ in the strong operator topology (cf. Theorem 1.5.5 in [12]). By Lemma 6.2.21 in [12] we conclude that $T_t^{(n)} \to T_t$ and $\nu_t^{(n)} \to \nu_t$ for all $t \geqslant 0$. Using Proposition 6.1 in [26] we obtain $S_n \to S$ and Proposition 7.1 becomes applicable.

(ii) $\Rightarrow$ (i). Condition (ii) (a) implies that the system $(\mu_{nk})_{k=1,\ldots,k_n; n \geqslant 1}$ is infinitesimal. By Proposition 7.1 we have $S_n \to S$, which together with Proposition 7.2 implies (i) (b). ∎

By the method of symmetrization, Theorem 8.1 can be generalized for normal and centered systems.

If $\mu \in \mathscr{P}(G)$, the *adjoint measure* $\tilde{\mu}$ is defined by $\tilde{\mu}(f) := \mu(f^*)$ for every continuous function $f\colon G \to \boldsymbol{C}$ with compact support, where $f^*\colon G \to \boldsymbol{C}$ is defined by $f^*(x) := f(x^{-1})$ for all $x \in G$. A measure $\mu \in \mathscr{P}(G)$ is said to be *normal* if $\mu * \tilde{\mu} = \tilde{\mu} * \mu$.

A triangular system $(\mu_{nk})_{k=1,\ldots,k_n; n \geqslant 1}$ is called *normal* if for all $n \geqslant 1$, $1 \leqslant j, k \leqslant k_n$ the equality

$$\mu_{nj} * \tilde{\mu}_{nk} = \tilde{\mu}_{nk} * \mu_{nj}$$

holds. Particularly, $\mu_{nj} * \tilde{\mu}_{nj} = \tilde{\mu}_{nj} * \mu_{nj}$, i.e., the measure μ_{nj} is normal for all $n \geqslant 1$, $1 \leqslant j \leqslant k_n$.

The system $(\mu_{nk})_{k=1,\ldots,k_n; n \geqslant 1}$ is called *centered* if for all $n \geqslant 1$, $1 \leqslant k \leqslant k_n$ the measure μ_{nk} is centered.

A convolution semigroup $(\nu_t)_{t \geqslant 0}$ is said to be *normal* if for all $t \geqslant 0$ the measure ν_t is normal, i.e., $\nu_t * \tilde{\nu}_t = \tilde{\nu}_t * \nu_t$. A Gaussian semigroup $(\nu_t)_{t \geqslant 0}$ on a nilpotent Lie group with the generating functional

$$A(f) = \sum_{i=1}^m a_i (X_i f)(e) + \tfrac{1}{2} \sum_{i,j=1}^m a_{ij} (X_i X_j f)(e)$$

is *normal* if $a_i = 0$ for $d_i = 1$, i.e., ν_t is centered for all $t \geqslant 0$ (cf. the representation of Gaussian semigroups by the help of Wiener processes due to Roynette [24]).

THEOREM 8.2. *Let $(\mu_{nk})_{k=1,\ldots,k_n;n\geqslant 1}$ be a commutative, normal and centered system of probability measures on the Heisenberg group* $\boldsymbol{H}$ *which satisfies the condition*

$$\sup_{n\geqslant 1}\sum_{k=1}^{k_n}\int |x|^2\,\mu_{nk}(dx) < \infty.$$

Then the following statements are equivalent:

(i) (a) $(\mu_{nk})_{k=1,\ldots,k_n;n\geqslant 1}$ *is infinitesimal;*

(b) $\mu_{n1}*\ldots*\mu_{nk_n}\to\nu$ *as* $n\to\infty$, *where* $\nu=\nu_1$, $(\nu_t)_{t\geqslant 0}$ *is the (normal) Gaussian semigroup with the generating functional*

$$A(f) = a_3(X_3 f)(e)+\tfrac{1}{2}\sum_{i,j=1}^{3} a_{ij}(X_i X_j f)(e).$$

(ii) (a) $\lim_{n\to\infty}\sum_{k=1}^{k_n}\mu_{nk}\{|x|>\varepsilon\}=0$ *for every* $\varepsilon>0$;

(b) $\lim_{n\to\infty}\sum_{k=1}^{k_n}\int_{|x|<1} x_i x_j\,\mu_{nk}(dx)=a_{ij}$ *for all* $1\leqslant i,j\leqslant 3$;

(c) $\lim_{n\to\infty}\sum_{k=1}^{k_n}\int_{|x|<1} x_i\,\mu_{nk}(dx)=a_i$ *for all* $1\leqslant i\leqslant 3$, *where* $a_1=a_2=0$.

Proof. (i)$\Rightarrow$(ii). Let us consider the accompanying sequence of Poisson semigroups $S_n=(\nu_t^{(n)})_{t\geqslant 0}$, $n\geqslant 1$. Since $(\mu_{nk})_{k=1,\ldots,k_n;n\geqslant 1}$ is commutative and normal, we have

$$\nu_t^{(n)}*\tilde{\nu}_t^{(n)}=\tilde{\nu}_t^{(n)}*\nu_t^{(n)}\qquad\text{for all } n\geqslant 1,\ t\geqslant 0.$$

Therefore $\pi_t^{(n)}:=\nu_t^{(n)}*\tilde{\nu}_t^{(n)}$, $t\geqslant 0$, is a symmetric Poisson semigroup for all $n\geqslant 1$ see [7]).

Since $S=(\nu_t)_{t\geqslant 0}$ is a normal Gaussian semigroup, $\pi_t:=\nu_t*\tilde{\nu}_t$, $t\geqslant 0$, is a symmetric Gaussian semigroup.

As in the proof of Theorem 8.1 we have $\nu_1^{(n)}\to\nu_1$, which implies $\pi_1^{(n)}\to\pi_1$, and using the symmetry of $\pi_t^{(n)}$ and π_t we can conclude that $(\pi_t^{(n)})_{t\geqslant 0}\to(\pi_t)_{t\geqslant 0}$ as $n\to\infty$. Clearly, the Lévy measure of the Poisson semigroup $(\pi_t^{(n)})_{t\geqslant 0}$ is $\sum_{k=1}^{k_n}(\mu_{nk}+\tilde{\mu}_{nk})$ for $n\geqslant 1$. Thus by Theorem 5.1 we obtain

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}(\mu_{nk}\{|x|>\varepsilon\}+\tilde{\mu}_{nk}\{|x|>\varepsilon\})=0\qquad\text{for every } \varepsilon>0.$$

Obviously, this implies

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}\mu_{nk}\{|x|>\varepsilon\}=0\qquad\text{for every } \varepsilon>0,$$

which together with Proposition 9.2 in [26] yields that any of the limit points of the accompanying sequence $S_n=(\nu_t^{(n)})_{t\geqslant 0}$ of Poisson semigroups is a Gaussian semigroup or a degenerated one. Now $\nu_1^{(n)}\to\nu$ implies that for each strictly monotone sequence $(n_k)_{k\geqslant 1}$ in N there exists a subsequence $(n_{k_l})_{l\geqslant 1}$ of $(n_k)_{k\geqslant 1}$ and a continuous convolution semigroup $(\mu_t)_{t\geqslant 1}$ such that $\mu_1=\nu$ and $\nu_t^{(n_{k_l})}\to\mu_t$

for all $t \geqslant 0$ (see [15]). We have already known that $(\mu_t)_{t \geqslant 0}$ can be only a Gaussian semigroup. Since a Gaussian measure on a simply connected nilpotent Lie group can be uniquely embedded into a Gaussian semigroup (see [2] in the case of 2-step nilpotent Lie groups and [21] for the general case), we conclude that $\mu_t = \nu_t$ for all $t \geqslant 0$. Consequently, for all fixed $t \geqslant 0$ any subsequence $(\nu_t^{(n_k)})_{k \geqslant 1}$ of the sequence $(\nu_t^{(n)})_{n \geqslant 1}$ has a convergent subsequence and the limit is always equal to ν_t. This proves $S_n \to S$ and Proposition 7.1 becomes applicable.

(ii) $\Rightarrow$ (i) follows from 7.1 ∎

If we suppose the convergence of variances of the first two coordinates and the Lindeberg condition on the third coordinate, then we obtain the usual form of the Lindeberg–Feller theorem.

COROLLARY 1. *Let $(\mu_{nk})_{k=1,\ldots,k_n; n \geqslant 1}$ be a commutative, normal and centered system of probability measures on $\boldsymbol{H}$ which satisfies the conditions*

$$\sup_{n \geqslant 1} \sum_{k=1}^{k_n} \int |x_3| \, \mu_{nk}(dx) < \infty, \tag{4}$$

$$\lim_{n \to \infty} \sum_{k=1}^{k_n} \int_{|x| \geqslant 1} |x_3| \, \mu_{nk}(dx) = 0, \tag{5}$$

$$\lim_{n \to \infty} \sum_{k=1}^{k_n} \int x_3 \, \mu_{nk}(dx) = 0, \tag{6}$$

$$\lim_{n \to \infty} \sum_{k=1}^{k_n} \int x_i x_j \, \mu_{nk}(dx) = a_{ij} \quad \text{for all } i, j = 1, 2. \tag{7}$$

Then the following statements are equivalent:

(i) (a) $\lim_{n \to \infty} \max_{1 \leqslant k \leqslant k_n} \int |x|^2 \, \mu_{nk}(dx) = 0$;

(b) $\mu_{n1} * \ldots * \mu_{nk_n} \to \nu$ *as* $n \to \infty$, *where* $\nu = \nu_1$, $(\nu_t)_{t \geqslant 0}$ *is the (symmetric) Gaussian semigroup with the generating functional*

$$A(f) = \tfrac{1}{2} \sum_{i,j=1}^{2} a_{ij} (X_i X_j f)(e).$$

(ii) (a) $\lim_{n \to \infty} \sum_{k=1}^{k_n} \mu_{nk} \{|x| > \varepsilon\} = 0$ *for every* $\varepsilon > 0$;

(b) $\lim_{n \to \infty} \sum_{k=1}^{k_n} \int_{|x| < 1} x_i x_j \, \mu_{nk}(dx) = a_{ij}$ *for all* $i, j = 1, 2$.

(iii) $\lim_{n \to \infty} \sum_{k=1}^{k_n} \int_{|x| \geqslant \varepsilon} |x|^2 \, \mu_{nk}(dx) = 0$ *for every* $\varepsilon > 0$.

Proof. (i) $\Rightarrow$ (ii). Condition (i) (a) implies that the system $(\mu_{nk})_{k=1,\ldots,k_n; n \geqslant 1}$ is infinitesimal, since

$$\mu_{nk} \{x\colon |x| \geqslant \varepsilon\} \leqslant \varepsilon^{-2} \int |x|^2 \, \mu_{nk}(dx) \tag{8}$$

for arbitrary $\varepsilon > 0$. From assumptions (7) and (4) we get (3). Applying Theorem 8.2 we obtain (i) $\Rightarrow$ (ii).

(ii) ⇒ (iii). By (ii) (b) and (7) the equality

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x|\geqslant 1} x_i^2 \, \mu_{nk}(dx) = 0$$

holds for $i = 1, 2, 3$. This together with (5) implies

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x|\geqslant 1} |x|^2 \, \mu_{nk}(dx) = 0.$$

Now, from (ii) (a) we infer that

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{\varepsilon_1 \leqslant |x| \leqslant \varepsilon_2} |x|^2 \, \mu_{nk}(dx) = 0 \tag{9}$$

for arbitrary $0 < \varepsilon_1 < \varepsilon_2$. Hence we obtain (iii).

(iii) ⇒ (i). For any $\varepsilon > 0$ we have

$$\int |x|^2 \, \mu_{nk}(dx) \leqslant \varepsilon^2 + \int_{|x|\geqslant\varepsilon} |x|^2 \, \mu_{nk}(dx) \leqslant \varepsilon^2 + \sum_{k=1}^{k_n} \int_{|x|\geqslant\varepsilon} |x|^2 \, \mu_{nk}(dx).$$

Thus we obtain (i) (a). To prove (i) (b) we shall show that conditions (ii) of Theorem 8.2 are satisfied. Estimation (8) implies that (ii) (a) of Theorem 8.2 holds. Thus we have also (9). Clearly, (iii) implies

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x|\geqslant\varepsilon} x_i^2 \, \mu_{nk}(dx) = 0 \quad \text{for } i = 1, 2.$$

Hence

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x|\geqslant\varepsilon} x_i x_j \, \mu_{nk}(dx) = 0 \quad \text{for } i, j = 1, 2,$$

which together with assumption (7) gives (ii) (b) of Theorem 8.2 for $i, j = 1, 2$. Using estimate (2), for $0 < \varepsilon < 1$ we have

$$\int_{|x|<\varepsilon} x_3^2 \, \mu_{nk}(dx) \leqslant c \int_{|x|<\varepsilon} |x|^4 \, \mu_{nk}(dx) \leqslant c\varepsilon^2 \int |x|^2 \, \mu_{nk}(dx).$$

Further we have

$$\int_{\varepsilon\leqslant|x|<1} x_3^2 \, \mu_{nk}(dx) \leqslant c\mu_{nk}(x\colon |x| \geqslant \varepsilon)$$

and, consequently,

$$\limsup_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x|<1} x_3^2 \, \mu_{nk}(dx) \leqslant c\varepsilon^2 \sup_{n\geqslant 1} \sum_{k=1}^{k_n} \int |x|^2 \, \mu_{nk}(dx).$$

Since $0 < \varepsilon < 1$ is arbitrary, we conclude (ii) (b) of Theorem 8.2 for $i = j = 3$. Using the same arguments we can obtain (ii) (b) of Theorem 8.2 for the remaining cases. Clearly, we have

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x|\geq 1} |x_i| \mu_{nk}(dx) = 0 \quad \text{for } i = 1, 2, 3,$$

which together with the assumption that the system is centered and assumption (6) implies (ii) (c) of Theorem 8.2. ∎

Remark 5. Conditions (i) (a) and (iii) are the classical Feller and Lindeberg conditions, respectively. Assumption (4) is needed in order to have bounded (homogeneous) moments of second order. Assumption (5) is in fact the Lindeberg condition for the third coordinate.

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