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DOMAINS OF ATTRACTION OF STABLE MEASURES ON THE HEISENBERG GROUP

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Abstract. Let H_d be the $(2d+1)$-dimensional Heisenberg group and $(\mu_t)_{t\geq 0}$ be a continuous convolution semigroup of probability measures on H_d. Let moreover μ_1 be full. A probability measure ν is said to *belong to the domain of attraction* of μ_1 if there exists a sequence $(\sigma_n)_n$ of automorphisms of H_d such that $\sigma_n \nu^n \xrightarrow[n\to\infty]{} \mu_1$ weakly. We prove some simple necessary and sufficient conditions on ν for the existence of such automorphisms if $(\mu_t)_{t\geq 0}$ has no Gaussian component. Furthermore, the domain of normal attraction of a Gaussian measure on H_d is considered.

1. Introduction. Let H_d be the $(2d+1)$-dimensional Heisenberg group. Suppose that μ and ν are probability distributions on H_d and that μ is full, i.e., not concentrated on a proper closed connected subgroup of H_d. We say that ν *belongs to the domain of attraction* of μ if there exists a sequence of automorphisms $(\sigma_n)_n \subset \mathrm{Aut}(H_d)$ such that

$$(*) \qquad \sigma_n \nu^n \xrightarrow[n\to\infty]{} \mu,$$

where ν^n denotes the n-th convolution power of ν and the convergence is the weak convergence. We say that μ is *stable* if there exists a continuous convolution semigroup $(\mu_t)_{t\geq 0}$ of probability measures on H_d with $\mu_1 = \mu$ and a continuous one-parameter group $(\tau_t)_{t>0} \subset \mathrm{Aut}(H_d)$ such that $\tau_s \mu_t = \mu_{st}$ for all $s, t > 0$. It is a result of [20] that μ is stable if and only if μ has a non-empty domain of attraction.

We are interested therefore in obtaining necessary and sufficient conditions for ν to belong to the domain of attraction of a full measure μ. In [26] we have investigated the case of $\mathscr{D}$-domains of attraction where the norming automorphisms τ_n are only allowed to belong to the group $\mathscr{D} \subset \mathrm{Aut}(G)$ of dilations and G is an arbitrary stratified Lie group. This is due to the fact that we do not know enough about the structure of the automorphisms of an arbitrary stratified Lie group. If $G = H_d$, the structure of an automorphism $\tau \in \mathrm{Aut}(H_d)$ and also of a one-parameter group $(\tau_t)_{t>0} \subset \mathrm{Aut}(H_d)$ is well known (see [1]). Roughly speaking, every automorphism τ of $\boldsymbol{H}_d$ is the composition of a kind of *block-diagonal* automorphism $\psi_{F,s}$ with an inner automorphism of $\boldsymbol{H}_d$, where

$$\psi_{F,s} = \begin{pmatrix} & & 0 \\ & sF & \vdots \\ & & 0 \\ 0 \dots 0 & & \pm s^2 \end{pmatrix},$$

$s > 0$ and F is a symplectic mapping of $\boldsymbol{R}^{2d}$. We define $\mathscr{B}$ as the closed subgroup of $\operatorname{Aut}(\boldsymbol{H}_d)$ of automorphisms without an inner part and consider only $\mathscr{B}$-stable continuous convolution semigroups and $\mathscr{B}$-domains of attraction, where the one-parameter group $(\tau_t)_{t>0}$ and the norming automorphisms σ_n are only allowed to belong to $\mathscr{B}$.

In the classical situation on $\boldsymbol{R}$ or $\boldsymbol{R}^d$, several descriptions of domains of attraction are known; see, e.g., [2], [13]–[16], [18], [9], [10], [7], and [24]. Especially, Meerschaert [15] proved on $\boldsymbol{R}^d$ necessary and sufficient conditions on a measure ν belonging to the domain of attraction of a full nonnormal measure μ, and Jurek [9] has described the domain of normal attraction of such a measure. If μ is a full Gaussian measure on $\boldsymbol{R}^d$, Jurek has shown that ν belongs to the domain of normal attraction of μ if and only if ν has a finite second moment and the same covariance matrix as μ.

This paper is organised as follows: In Section 2 we introduce the structure of the Heisenberg group $\boldsymbol{H}_d$ and repeat some important notions about semigroups. In Section 3 we recall the definition of fullness of a measure and of various domains of attraction. In view of the convergence criteria for discrete semigroups in Section 3 of [26] (see also [22]) we need to pass over from (*) to a functional limit, i.e., the convergence of the discrete semigroups $(\sigma_n \nu^{[nt]})_{t \geq 0}$ to a continuous convolution semigroup $(\mu_t)_{t \geq 0}$ with $\mu_1 = \mu$. Due to a result of Nobel ([20], Theorem 6) for full measures the limit convolution semigroup is stable, and therefore uniquely determined by μ_1; hence such a transition is possible.

In Section 4 we will investigate the automorphisms and one-parameter groups of automorphisms of $\boldsymbol{H}_d$. We define a pseudo-inner product on $\boldsymbol{H}_d$ and an automorphism norm for automorphisms in $\mathscr{B}$ which behave like the usual inner product and the operator norm on $\boldsymbol{R}^d$. Furthermore, we show that there are some very important connections between stable convolution semigroups on $\boldsymbol{H}_d$ and on $\boldsymbol{R}^{2d} \cong \boldsymbol{H}_d/[\boldsymbol{H}_d, \boldsymbol{H}_d]$, resp. $\boldsymbol{R} \cong [\boldsymbol{H}_d, \boldsymbol{H}_d]$.

In Section 5 we will state and prove the main results of this paper: A description of the domain of (normal) attraction of a full measure μ without Gaussian component. The domain of normal attraction of a full Gaussian measure is also concerned. With the use of our pseudo-inner product we will give a necessary condition on ν belonging to the domain of normal attraction of such a measure. Unfortunately, this condition is only sufficient if we make the further assumption of the existence of the second homogeneous moment of v. In contrast to the above-mentioned vector space case on $\boldsymbol{R}^d$, the existence of the second homogeneous moment of v is not necessary on $\boldsymbol{H}_d$ as shown in an explicit example. We think this example is very surprising.

2. Notation and preliminaries. Let $\psi(p, q)$ denote the usual Euclidean inner product on $\boldsymbol{R}^d$ and let

$$\sigma\big((p, q), (p', q')\big) \stackrel{\text{def}}{=} \psi(p, q') - \psi(q, p')$$

be the usual symplectic form on $\boldsymbol{R}^{2d}$. Let

$$\mathfrak{h}_d \stackrel{\text{def}}{=} \boldsymbol{R}^d \times \boldsymbol{R}^d \times \boldsymbol{R},$$

furnished with the bracket product

$$[(p, q, t), (p', q', t')] \stackrel{\text{def}}{=} \big(0, 0, \sigma((p, q), (p', q'))\big),$$

be a realisation of the Heisenberg Lie algebra. Then $(\mathfrak{h}_d, [\cdot, \cdot])$ is a step 2 stratified Lie algebra. Using the Campbell–Hausdorff formula, we define on $\boldsymbol{R}^{2d+1}$ the multiplication

$$x \circ y \stackrel{\text{def}}{=} x + y + \tfrac{1}{2}[x, y], \qquad x, y \in \boldsymbol{R}^{2d+1}. \tag{1}$$

Then $(\boldsymbol{R}^{2d+1}, \circ)$ is called the $(2d+1)$-*dimensional Heisenberg group*, denoted by $\boldsymbol{H}_d$, which is a stratified Lie group of step 2. We set

$$\boldsymbol{H}_d^{\times} \stackrel{\text{def}}{=} \boldsymbol{H}_d \backslash \{e\},$$

where e denotes the neutral element of $\boldsymbol{H}_d$. With this definition, the exponential mapping $\exp\colon \mathfrak{h}_d \to \boldsymbol{H}_d$ is just the identity. Let

$$\{X_i \stackrel{\text{def}}{=} e_i\colon\ i = 1, \ldots, 2d+1\}$$

be the natural basis of $\mathfrak{h}_d$. Then the functions

$$\xi_i\colon \boldsymbol{H}_d \to \boldsymbol{R}, \qquad \xi_i(x) \stackrel{\text{def}}{=} x_i$$

define a system of global coordinates on $\boldsymbol{H}_d$ with $\xi_i(\exp X_j) = \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker symbol.

Let

$$\begin{aligned} P_1\colon \boldsymbol{H}_d \to \boldsymbol{R}^{2d}, \qquad & P_1(p, q, t) \stackrel{\text{def}}{=} (p, q), \\ P_2\colon \boldsymbol{H}_d \to \boldsymbol{R}, \qquad & P_2(p, q, t) \stackrel{\text{def}}{=} t, \end{aligned} \tag{2}$$

denote the *projections* on the steps of $\mathfrak{h}_d$. Thereby P_1 is a group homomorphism from $\boldsymbol{H}_d$ onto (the vector space) $\boldsymbol{R}^{2d}$, whereas in general we have $P_2(x \circ y) \neq P_2(x) + P_2(y)$. In the following we use the notation $x = (\bar{x}, x') \in \boldsymbol{H}_d$, where $P_1(x) = \bar{x}$ and $P_2(x) = x'$. For $t > 0$ we denote by $\delta_t\colon \boldsymbol{H}_d \to \boldsymbol{H}_d$ the dilation given by

$$\delta_t x \stackrel{\text{def}}{=} \delta_t(\bar{x}, x') \stackrel{\text{def}}{=} (t\bar{x}, t^2 x'),$$

and for $x \in \boldsymbol{H}_d$ we define

$$|x| \stackrel{\text{def}}{=} (\|\bar{x}\|_2^2 + |x'|)^{1/2}.$$

Then $|\cdot|: \boldsymbol{H}_d \to \boldsymbol{R}_+$ is a homogeneous norm on $\boldsymbol{H}_d$, i.e., a continuous mapping with $|\delta_t x| = t|x|$, $|x^{-1}| = |x|$ and $|x| = 0 \Leftrightarrow x = e$. It is well known that any two homogeneous norms $|\cdot|_1$, $|\cdot|_2$ on $\boldsymbol{H}_d$ are equivalent, that is there exist constants $C_1, C_2 > 0$ such that

$$C_1 |x|_1 \leqslant |x|_2 \leqslant C_2 |x|_1 \quad \text{for all } x \in \boldsymbol{H}_d.$$

By $C^b(\boldsymbol{H}_d)$ we denote the space of bounded continuous complex-valued functions on $\boldsymbol{H}_d$ equipped with the supremum norm $\|\cdot\|_\infty$. Let $M_+^b(\boldsymbol{H}_d)$ be the set of bounded positive Radon measures on $\boldsymbol{H}_d$, and $M^1(\boldsymbol{H}_d)$ the set of probability measures on $\boldsymbol{H}_d$ which, furnished with the convolution product and the weak topology $\sigma(M^1(\boldsymbol{H}_d), C^b(\boldsymbol{H}_d))$, is a topological semigroup. The point measure in $x \in \boldsymbol{H}_d$ is denoted by ε_x. We use the notation

$$\langle \mu, f \rangle \stackrel{\text{def}}{=} \int_{\boldsymbol{H}_d} f d\mu \quad \text{for } \mu \in M_+^b(\boldsymbol{H}_d) \text{ and } f \in C^b(\boldsymbol{H}_d).$$

Denote by $\mathscr{M}(\boldsymbol{H}_d^\times)$ the class of all σ-finite Radon measures on $\boldsymbol{H}_d^\times$ which are finite on sets bounded away from the neutral element e. Let ∂B denote the topological boundary of a set $B \subset \boldsymbol{H}_d$. For $\nu_n, \nu \in \mathscr{M}(\boldsymbol{H}_d^\times)$ we will write $\nu_n \xrightarrow[n\to\infty]{} \nu$ if and only if $\nu_n(B) \xrightarrow[n\to\infty]{} \nu(B)$ for all Borel sets B bounded away from the neutral element such that $\nu(\partial B) = 0$. We will call this convergence the *convergence in* $\mathscr{M}(\boldsymbol{H}_d^\times)$.

Let $\mathscr{D}(\boldsymbol{H}_d)$ be the space of all C^∞-functions with compact support on $\boldsymbol{H}_d$, and let $\operatorname{supp} f$ denote the support of a function f. In our situation $\mathscr{E}(\boldsymbol{H}_d)$, the space of *regular functions* on $\boldsymbol{H}_d$, is the space of all bounded C^∞-functions on $\boldsymbol{H}_d$. We regard every element $X \in \mathfrak{h}_d$ as a (left invariant) differential operator on $\boldsymbol{H}_d$: for $f \in \mathscr{D}(\boldsymbol{H}_d)$ we define

$$(Xf)(x) \stackrel{\text{def}}{=} \lim_{t\to 0} \frac{1}{t}\big(f(x \exp(tX)) - f(x)\big).$$

A family $(\mu_t)_{t \geqslant 0} \subset M^1(\boldsymbol{H}_d)$ is said to be a *continuous convolution semigroup* (abbreviated by c.c.s.) if

$$\mu_s * \mu_t = \mu_{s+t} \text{ for all } s, t \geqslant 0 \quad \text{and} \quad \lim_{t \downarrow 0} \mu_t = \varepsilon_e.$$

Its generating distribution A is defined by

$$\langle A, f \rangle \stackrel{\text{def}}{=} \frac{d^+}{dt}\bigg|_{t=0} \langle \mu_t, f \rangle = \lim_{t \downarrow 0} \frac{1}{t}(\langle \mu_t, f \rangle - f(e))$$

for all $f \in D(A) \stackrel{\text{def}}{=} \{f \in C^b(\boldsymbol{H}_d) \,|\, \langle A, f \rangle \text{ exists}\}$. We have $\mathscr{E}(\boldsymbol{H}_d) \subset D(A)$ and A admits on $\mathscr{E}(\boldsymbol{H}_d)$ the unique decomposition (the *Lévy–Khinchin formula*)

$$(3)\quad \langle A,f\rangle = \sum_{i=1}^{2d+1} p_i(X_i f)(e) + \sum_{i,j=1}^{2d+1} a_{i,j}(X_i X_j f)(e) + \int_{H_d^\times} \Big[f(x)-f(e)-\sum_{i=1}^{2d+1} \zeta_i(x)(X_i f)(e)\Big]\, d\eta(x),$$

where $p_1, \ldots, p_{2d+1}$ are real numbers, $(a_{i,j})_{1\leqslant i,j\leqslant 2d+1}$ is a real symmetric positive semidefinite matrix, η is a Lévy measure on $\boldsymbol{H}_d$, i.e., a positive σ-finite Radon measure on $\boldsymbol{H}_d^\times$, with

$$\int_{H_d^\times} \min\Big(1, \sum_{i=1}^{2d+1} x_i^2\Big)\, d\eta(x) < \infty,$$

and $\{\zeta_1, \ldots, \zeta_{2d+1}\}$ is a system of local coordinates of the first kind in $\mathscr{D}(\boldsymbol{H}_d)$ adapted to the basis $\{X_1, \ldots, X_{2d+1}\}$. It follows from 4.1.9 Lemma in [5] that we can suppose without loss of generality that $\zeta_i(x) = x_i$ for all $x \in \{x \in \boldsymbol{H}_d\colon |x| \leqslant 1\}$ and $i = 1, \ldots, 2d+1$. Since every c.c.s. $(\mu_t)_{t\geqslant 0}$ on $\boldsymbol{H}_d$ is uniquely determined by the restriction of its generating distribution A on $\mathscr{E}(\boldsymbol{H}_d)$, we shall write $A = [(p_i), (a_{i,j}), \eta]$. It is well known that a c.c.s. whose generating distribution takes the form $A = [(p_i), (a_{i,j}), 0]$ is called a *Gaussian semigroup.* We will call a c.c.s. with generating distribution $A = [(p_i), 0, \eta]$ a *c.c.s. without Gaussian component.*

3. Full measures and $\mathscr{B}$-domains of attraction. In view of the characterization theorems of [26], Section 3, we have to require a weak form of a *functional limit theorem*, i.e., the convergence of the discrete convolution semigroups $\sigma_n \nu^{[nt]} \underset{n\to\infty}{\longrightarrow} \mu_t$ (for all $t > 0$), to obtain necessary conditions on ν and the sequence $(\sigma_n)_n$ in $\sigma_n \nu^n \underset{n\to\infty}{\longrightarrow} \mu_1$. As in [26], Section 4, we need the notion of fullness of a measure and a convergence of types theorem to pass from $\sigma_n \nu^n \underset{n\to\infty}{\longrightarrow} \mu_1$ to a functional limit. Due to [1], [19] and [4] we have

DEFINITION 3.1. A measure $\mu \in M^1(\boldsymbol{H}_d)$ is called *full* if $P_1(\mu)$ is not concentrated on a proper subspace of $\boldsymbol{R}^{2d}$.

With this definition the following *convergence of types theorem* holds:

PROPOSITION 3.2 ([1], [4]). *Let $\mu_n, \mu, \lambda \in M^1(\boldsymbol{H}_d)$ and $\sigma_n \in \mathrm{Aut}(\boldsymbol{H}_d)$ be a sequence of automorphisms. Suppose that $\mu_n \underset{n\to\infty}{\longrightarrow} \mu$ and $\sigma_n \mu_n \underset{n\to\infty}{\longrightarrow} \lambda$. If μ and λ are full, then the set $\{\sigma_n | n \in \boldsymbol{N}\}$ is relatively compact in $\mathrm{Aut}(\boldsymbol{H}_d)$ and for every accumulation point σ we have $\sigma\mu = \lambda$.*

Let now $\mathscr{B} \subset \mathrm{Aut}(\boldsymbol{H}_d)$ be a closed subgroup of automorphisms and $(\tau_t)_{t>0} \subset \mathscr{B}$ be a continuous one-parameter group of automorphisms. We will specify $\mathscr{B}$ in Section 4. Now we recall some specializations of known definitions on domains of attraction and stability of c.c.s. on $\boldsymbol{H}_d$.

DEFINITION 3.3. A c.c.s. $(\mu_t)_{t\geqslant 0} \subset M^1(\boldsymbol{H}_d)$ is called *$\mathscr{B}$-stable* if there exists a continuous one-parameter group $(\tau_t)_{t>0} \subset \mathscr{B}$ such that $\tau_s \mu_t = \mu_{st}$ for all $s, t > 0$.

DEFINITION 3.4. Let μ, $\nu \in M^1(\boldsymbol{H}_d)$. Then ν is said to *belong to the $\mathscr{B}$-domain of attraction of* μ, denoted by $\nu \in \mathrm{DOA}_{\mathscr{B}}(\mu)$, if there exists a sequence $(\sigma_n)_n \subset \mathscr{B}$ such that

$$\sigma_n \nu^n \underset{n\to\infty}{\longrightarrow} \mu.$$

If μ is embeddable in a $(\tau_t)_{t>0}$-stable c.c.s. $(\mu_t)_{t\geqslant 0} \subset M^1(G)$ (i.e., $\mu_1 = \mu$), we say that ν *belongs to the domain of normal attraction of* $\mu = \mu_1$, and write $\nu \in \mathrm{DONA}(\mu, (\tau_t)_{t>0})$, if

$$\tau_{1/n} \nu^n \underset{n\to\infty}{\longrightarrow} \mu$$

holds.

Taking into consideration that in general we do not know if a c.c.s. $(\mu_t)_{t\geqslant 0}$ on $\boldsymbol{H}_d$ is uniquely determined by $\mu = \mu_1$, we have to distinguish between attraction to a semigroup and attraction to a single measure. Therefore we have

DEFINITION 3.5. Let $(\mu_t)_{t\geqslant 0} \subset M^1(\boldsymbol{H}_d)$ be a c.c.s. Then $\nu \in M^1(\boldsymbol{H}_d)$ is said to *belong to the $\mathscr{B}$-domain of attraction of* $(\mu_t)_{t\geqslant 0}$, denoted by $\nu \in \mathrm{DOA}_{\mathscr{B}}((\mu_t)_{t\geqslant 0})$, if there exists a sequence $(\sigma_n)_n \subset \mathscr{B}$ such that

$$\sigma_n \nu^{[nt]} \underset{n\to\infty}{\longrightarrow} \mu_t \quad \text{for all } t > 0.$$

If additionally the c.c.s. $(\mu_t)_{t\geqslant 0}$ is $(\tau_t)_{t>0}$-stable, we say that $\nu \in M^1(\boldsymbol{H}_d)$ *is in the domain of normal attraction of* $(\mu_t)_{t\geqslant 0}$, and write $\nu \in \mathrm{DONA}((\mu_t)_{t\geqslant 0}, (\tau_t)_{t>0})$, if

$$\tau_{1/n} \nu^{[nt]} \underset{n\to\infty}{\longrightarrow} \mu_t \quad \text{for all } t > 0.$$

Clearly, we have

$$\mathrm{DOA}_{\mathscr{B}}((\mu_t)_{t\geqslant 0}) \subset \mathrm{DOA}_{\mathscr{B}}(\mu_1)$$

and

$$\mathrm{DONA}((\mu_t)_{t\geqslant 0}, (\tau_t)_{t>0}) \subset \mathrm{DONA}(\mu_1, (\tau_t)_{t>0}).$$

To prove necessary conditions about measures ν in the domain of attraction of μ_1 we need (in view of the characterization theorems of [26], Section 3) the opposite inclusions, i.e., the passage from $\sigma_n \nu^n \underset{n\to\infty}{\longrightarrow} \mu_1$ to a functional limit $\sigma_n \nu^{[nt]} \underset{n\to\infty}{\longrightarrow} \mu_t$ for all $t > 0$. In fact, we have:

PROPOSITION 3.6 (see [20], Theorem 6). *Let $(\mu_t)_{t\geqslant 0}$ be a c.c.s. If μ_1 is full, we have*

$$\mathrm{DOA}_{\mathscr{B}}(\mu_1) = \mathrm{DOA}_{\mathscr{B}}((\mu_t)_{t\geqslant 0}).$$

In the case of normal attraction we have without further assumptions:

PROPOSITION 3.7 (see [12], Proposition 4). *Let $(\tau_t)_{t>0} \subset \mathrm{Aut}(\boldsymbol{H}_d)$ and $(\mu_t)_{t\geqslant 0}$ be a $(\tau_t)_{t\geqslant 0}$-stable c.c.s. Then we have*

$$\mathrm{DONA}(\mu_1, (\tau_t)_{t>0}) = \mathrm{DONA}((\mu_t)_{t\geqslant 0}, (\tau_t)_{t>0}).$$

4. Automorphisms of the Heisenberg group. In this section we define a special subgroup $\mathscr{B} \subset \mathrm{Aut}(\boldsymbol{H}_d)$ of automorphisms of $\boldsymbol{H}_d$ and obtain some very useful relations between stable c.c.s. on $\boldsymbol{H}_d$ and their images under the projections P_1 and P_2 on $\boldsymbol{R}^{2d}$, resp. $\boldsymbol{R}$.

Furthermore, we introduce the notion of a pseudo-inner product on $\boldsymbol{H}_d$ and an automorphism norm for automorphisms in $\mathscr{B}$ which behave like the usual inner product and the operator norm on $\boldsymbol{R}^d$. Recall from (2) the definition of the projections P_1 and P_2.

DEFINITION 4.1. For $x, y \in \boldsymbol{H}_d$ let us define by

$$\langle x, y\rangle \stackrel{\text{def}}{=} \psi(\bar{x}, \bar{y}) + \sqrt{|x' y'|} \tag{4}$$

the *pseudo-inner product* on $\boldsymbol{H}_d$, where $\psi(\bar{x}, \bar{y}) = \sum_{i=1}^{2d} x_i y_i$ is the usual inner product on $\boldsymbol{R}^{2d}$.

The notion of pseudo-inner product is justified by

LEMMA 4.2. *We have* $\langle x, x\rangle = |x|^2$, $\langle \delta_t x, y\rangle = \langle x, \delta_t y\rangle = t\langle x, y\rangle$, $|\langle x, y\rangle| \leqslant |x|\cdot|y|$ *for all* $x, y \in \boldsymbol{H}_d$ *and* $t > 0$.

Proof. The first two assertions are obvious. One needs to show the last assertion only for $|x| = |y| = 1$. We have

$$|\langle x, y\rangle| \leqslant \sum_{i=1}^{2d} \sqrt{x_i^2 y_i^2} + \sqrt{|x' y'|} \leqslant \tfrac{1}{2}\Big(\sum_{i=1}^{2d} (x_i^2 + y_i^2) + |x'| + |y'|\Big)$$
$$= \tfrac{1}{2}(|x|^2 + |y|^2) = 1. \;\blacksquare$$

Following [1], (1.2) Proposition, every automorphism $\tau \in \mathrm{Aut}(\boldsymbol{H}_d)$ has the unique decomposition

$$\tau = \mathrm{inn}(v) \circ \alpha_F \circ \delta_s, \tag{5}$$

where $\mathrm{inn}(v)$ denotes for $v \in \boldsymbol{R}^{2d}$ the inner automorphism

$$\mathrm{inn}(v)(x) \stackrel{\text{def}}{=} (\bar{x}, x' + \sigma(v, \bar{x})) = (v, 0) \circ x \circ (v, 0)^{-1}.$$

Let $\mathrm{Sp}(\boldsymbol{R}^{2d})$ be the set of all symplectic mappings on $\boldsymbol{R}^{2d}$ (with respect to the symplectic form σ), and $\widetilde{\mathrm{Sp}}(\boldsymbol{R}^{2d})$ be the set of all skew-symplectic mappings. We set

$$\mathrm{S}(\boldsymbol{R}^{2d}) \stackrel{\text{def}}{=} \mathrm{Sp}(\boldsymbol{R}^{2d}) \cup \widetilde{\mathrm{Sp}}(\boldsymbol{R}^{2d})$$

and for $F \in \mathrm{S}(\boldsymbol{R}^{2d})$ we define

$$\alpha_F(x) = \begin{cases} (F\bar{x}, x') & \text{if } F \in \mathrm{Sp}(\boldsymbol{R}^{2d}), \\ (F\bar{x}, -x') & \text{if } F \in \widetilde{\mathrm{Sp}}(\boldsymbol{R}^{2d}) \end{cases}$$

for $x \in \boldsymbol{H}_d$. In the following we will only use automorphisms τ without an inner part, i.e., $v = 0$ in (5). We use the abbreviated notation

$$\psi_{F,s} \stackrel{\text{def}}{=} \alpha_F \circ \delta_s$$

and define

$$\mathscr{B} \stackrel{\text{def}}{=} \{\psi_{F,s} \mid F \in S(\boldsymbol{R}^{2d}),\ s > 0\} \tag{6}$$

to be the closed subgroup of admissible automorphisms. From the definition we get

$$\delta_t \circ \psi_{F,s} = \psi_{F,s} \circ \delta_t \tag{7}$$

for all $\psi_{F,s} \in \mathscr{B}$ and $t > 0$; hence the automorphisms in $\mathscr{B}$ commute with dilations. Let $\mathrm{sp}(\boldsymbol{R}^{2d})$ denote the Lie algebra of $\mathrm{Sp}(\boldsymbol{R}^{2d})$ and for $M \in \mathrm{sp}(\boldsymbol{R}^{2d})$ let $\mathrm{spec}(M)$ be the set of all complex eigenvalues of M counted by multiplicity. It follows from [1], (2.6) Corollary, that the contracting one-parameter subgroups in $\mathscr{B}$ are given by

$$\sigma_{M,m}(t) \stackrel{\text{def}}{=} \psi_{t^M, t^m}, \tag{8}$$

where $m > 0$ and $M \in \mathrm{sp}(\boldsymbol{R}^{2d})$ with $\mathrm{Re}\,\lambda > -m$ for all $\lambda \in \mathrm{spec}(M)$. The pair (M, m) will be called the *exponent of the one-parameter group* $(\sigma_{M,m}(t))_{t>0}$. From the definition of the projections P_1 and P_2 and the automorphisms in $\mathscr{B}$ we obtain the following very useful identities:

$$P_1 \circ \psi_{F,s} = (sF) \circ P_1, \quad P_2 \circ \psi_{F,s} = \pm s^2 P_2. \tag{9}$$

Remark 4.3. An easy computation shows that in general the identities (9) are not valid if one replaces $\psi_{F,s}$ by an arbitrary automorphism with inner part.

For linear mappings T on a normed vector space the operator norm $\|T\|$ is very useful. We will now define an analogous quantity for automorphisms in $\mathscr{B}$ and we will show that some of the usual estimates are valid for our automorphism norm. Since the set

$$\Sigma \stackrel{\text{def}}{=} \{x \in \boldsymbol{H}_d\colon |x| = 1\}$$

is compact and $|\tau \circ \delta_t x| = t|\tau x|$ for all $\tau \in \mathscr{B}$, $t > 0$ and $x \in \boldsymbol{H}_d$, the definition

$$|\tau| \stackrel{\text{def}}{=} \sup_{x \neq 0} \frac{|\tau x|}{|x|} = \sup_{x \in \Sigma} |\tau x| \tag{10}$$

makes sense for all $\tau \in \mathscr{B}$. We will call $|\tau|$ the *automorphism norm* of $\tau \in \mathscr{B}$. This notion is justified by

LEMMA 4.4. *For $\tau_1, \tau_2, \tau \in \mathscr{B}$, $x \in \boldsymbol{H}_d$ and $r > 0$ we have:*

(a) $|\tau x| \leqslant |\tau| \cdot |x|$, $|\tau_1 \tau_2| \leqslant |\tau_1| \cdot |\tau_2|$, $|\delta_r \tau| = |\tau \delta_r| = r|\tau|$, $|\delta_r| = r$.

(b) *Let $(\tau_n)_n \subset \mathscr{B}$ be a sequence of automorphisms. Then the following assertions are equivalent:*

(1) $\tau_n x \underset{n\to\infty}{\longrightarrow} e$ *for all* $x \in \boldsymbol{H}_d$, *i.e.* $(\tau_n)_n$ *is contracting;*

(2) $|\tau_n| \underset{n\to\infty}{\longrightarrow} 0$.

Proof. (a) follows directly from the definition.

(b) (1) $\Rightarrow$ (2). Since every τ_n is continuous, for an arbitrary $x_0 \in \Sigma$ there exists an open neighbourhood $B_{x_0} \subset \Sigma$ such that $\tau_n x \underset{n\to\infty}{\longrightarrow} e$ uniformly for all $x \in B_{x_0}$. Using the compactness of Σ we conclude that $\tau_n x \underset{n\to\infty}{\longrightarrow} e$ uniformly on Σ, and hence $|\tau_n| \underset{n\to\infty}{\longrightarrow} 0$.

(2) $\Rightarrow$ (1). This follows from Lemma 4.4 (a). ∎

Recall from Definition 3.1 that we call a measure $\mu \in M^1(\boldsymbol{H}_d)$ full if $P_1(\mu)$ is not concentrated on a proper subspace of $\boldsymbol{R}^{2d}$. We will now prove some useful relations between c.c.s. on $\boldsymbol{H}_d$ and their projections under P_1 and P_2 on $P_1(\boldsymbol{H}_d) = \boldsymbol{R}^{2d}$ and on $P_2(\boldsymbol{H}_d) = \boldsymbol{R}$.

PROPOSITON 4.5. *Let $(\mu_t)_{t\geq 0} \subset M^1(\boldsymbol{H}_d)$ be a c.c.s. and $(\sigma_{M,m}(t))_{t>0} \subset \mathscr{B}$ be a continuous one-parameter group of automorphisms. We have*:

(i) *If μ_t is full, then $P_1(\mu_t)$ is full on $\boldsymbol{R}^{2d}$, i.e., not concentrated on a proper subspace of $\boldsymbol{R}^{2d}$.*

(ii) *$(\mu_t)_{t\geq 0}$ is $(\sigma_{M,m}(t))_{t>0}$-stable, then $(P_1(\mu_t))_{t\geq 0}$ is $(t^{m+M})_{t>0}$-stable.*

(iii) *Let $(\mu_t)_{t\geq 0}$ be a $(\sigma_{M,m}(t))_{t>0}$-stable c.c.s. without Gaussian component and generating distribution $A = [(a_i), 0, \eta]$. Then*:

(a) *The generating distribution $P_1(A)$, defined by $\langle P_1(A), f\rangle \overset{\text{def}}{=} \langle A, f\circ P_1\rangle$ for $f \in \mathscr{D}(\boldsymbol{R}^{2d})$, generates the $(t^{m+M})_{t>0}$-stable c.c.s. $(P_1(\mu_t))_{t\geq 0}$ on $\boldsymbol{R}^{2d}$ without Gaussian component. The Lévy measure of $(P_1(\mu_t))_{t\geq 0}$ is given by*

$$f \mapsto \int_{\boldsymbol{H}_d^\times} f\circ P_1\, d\eta$$

for $f \in \mathscr{D}(\boldsymbol{R}^{2d})$ with $0 \notin \operatorname{supp} f$. We denote this Lévy measure by $P_1(\eta)$.

(b) *The generating distribution $P_2(A)$, defined by $\langle P_2(A), f\rangle = \langle A, f\circ P_2\rangle$ for $f \in \mathscr{D}(\boldsymbol{R})$, generates a $(t^{2m})_{t>0}$-stable c.c.s. $(\lambda_t)_{t\geq 0}$ on $\boldsymbol{R}$ without Gaussian component. The Lévy measure of $(\lambda_t)_{t\geq 0}$ is given by*

$$f \mapsto \int_{\boldsymbol{H}_d^\times} f\circ P_2\, d\eta$$

for $f \in \mathscr{D}(\boldsymbol{R})$ with $0 \notin \operatorname{supp} f$. We denote this Lévy measure by $P_2(\eta)$.

(iv) *Let $(\mu_t)_{t\geq 0}$ be a $(\sigma_{M,m}(t))_{t>0}$-stable Gaussian semigroup with generating distribution A. Then*

(a) *$(P_1(\mu_t))_{t\geq 0}$ is a $(t^{m+M})_{t>0}$-stable Gaussian semigroup on $\boldsymbol{R}^{2d}$ with generating distribution $P_1(A)$.*

(b) *$P_2(A)$ is the generating distribution of a $(t^{2m})_{t>0}$-stable Gaussian semigroup $(\lambda_t)_{t\geq 0}$ on $\boldsymbol{R}$.*

Proof. (i) is Definition 3.1. (ii) follows from (8) and (9).

(iii) (a) Using (8), (9) and $\sigma_{M,m}(t)(A) = tA$ for all $t > 0$ we conclude that $P_1(A)$ is the generating distribution of a $(t^{m+M})_{t>0}$-stable c.c.s. without Gaussian component. For all regular functions $f \in \mathscr{E}(\boldsymbol{R}^{2d})$ we have

$$\langle P_1(A), f\rangle = \langle A, f\circ P_1\rangle = \lim_{t\downarrow 0}\frac{1}{t}\langle \mu_t - \varepsilon_e, f\circ P_1\rangle = \lim_{t\downarrow 0}\frac{1}{t}\langle P_1(\mu_t) - \varepsilon_0, f\rangle,$$

and hence $P_1(A)$ is the generating distribution of $(P_1(\mu_t))_{t\geq 0}$. The Lévy measure of $(P_1(\mu_t))_{t\geq 0}$ is given by

$$\lim_{t\downarrow 0}\frac{1}{t}\langle P_1(\mu_t), f\rangle = \lim_{t\downarrow 0}\frac{1}{t}\langle \mu_t, f\circ P_1\rangle = \langle \eta, f\circ P_1\rangle$$

for $f\in C^b(\boldsymbol{R}^{2d})$ with $0\notin \operatorname{supp} f$.

(b) The proof of (b) and (iv) is similar. ∎

Remark 4.6. (a) Note that in Proposition 4.5 (iii), (iv) the equality $P_1(A)=0$ or $P_2(A)=0$ is possible.

(b) It follows from the definition of the group law of $\boldsymbol{H}_d$ that in general

$$P_2(\mu_t) * P_2(\mu_s) \neq P_2(\mu_t * \mu_s),$$

i.e., $(P_2(\mu_t))_{t\geq 0}$ is not a convolution semigroup. Hence one cannot speak of the Lévy measure of the family $(P_2(\mu_t))_{t\geq 0}$. Note that for $f\in\mathscr{E}(\boldsymbol{R})$ we have

$$\frac{d^+}{dt}\langle P_2(\mu_t), f\rangle|_{t=0} = \frac{d^+}{dt}\langle \lambda_t, f\rangle|_{t=0}.$$

5. $\mathscr{B}$-domains of attraction. In this section we will obtain necessary and sufficient conditions for attraction to a $\mathscr{B}$-stable c.c.s. Our results extend those of Meerschaert [15] and Jurek [9] on $\boldsymbol{R}^d$, who characterize the domains of attraction in terms of convergence towards the Lévy measure of the c.c.s. In contrast to [26] we will not prove the assertion directly on $\boldsymbol{H}_d$, but we will use Proposition 4.5 and the known results on $\boldsymbol{R}^{2d}$, resp. $\boldsymbol{R}$, in our proof. Therefore we will call our method of proof the *projection method*. Furthermore, we use the *convergence criteria for discrete semigroups* stated and proved in [26].

Our first result is the description of the $\mathscr{B}$-domain of attraction of a $\mathscr{B}$-stable full c.c.s. without Gaussian component.

THEOREM 5.1. *Let $(\mu_t)_{t\geq 0}\subset M^1(\boldsymbol{H}_d)$ be a full $(\sigma_{M,m}(t))_{t>0}$-stable c.c.s. without Gaussian component and generating distribution $A=[(a_i), 0, \eta]$. For $\nu\in M^1(\boldsymbol{H}_d)$ the condition*

(i) $\nu\in \mathrm{DOA}_{\mathscr{B}}(\mu_1)$

implies

(ii) *there exists a sequence $(\tau_n)_n\subset\mathscr{B}$ with $|\tau_n|\xrightarrow[n\to\infty]{}0$ and*

$$n(\tau_n\nu)\xrightarrow[n\to\infty]{}\eta \quad \text{in } \mathscr{M}(\boldsymbol{H}_d^{\times}).$$

If additionally ν and μ_1 are symmetric, we have (ii) ⇒ (i).

Proof. In view of Proposition 3.6 we can apply Theorem 3.1 of [26].

(i) ⇒ (ii). Because of Theorem 3.1 (2) (c) in [26] we have only to show that $|\tau_n|\xrightarrow[n\to\infty]{}0$ holds. From Proposition 1 of [20] we conclude that $\tau_n\nu\xrightarrow[n\to\infty]{}\varepsilon_e$. Since μ_1 is full, ν must be full, and hence $(\tau_n)_n$ is contracting; therefore Lemma 4.4 (b) yields the assertion.

(ii) ⇒ (i). Let μ_1, ν be symmetric and $\tau_n = \psi_{F_n, s_n} \in \mathscr{B}$ with $|\tau_n| \xrightarrow[n\to\infty]{} 0$. From the definition of $|\tau_n|$ we easily obtain

(11) $$\|s_n F_n\| \xrightarrow[n\to\infty]{} 0 \quad \text{and} \quad s_n \xrightarrow[n\to\infty]{} 0.$$

Using (9) it follows from $n(\tau_n \nu) \xrightarrow[n\to\infty]{} \eta$ that

(12) $$n\big((s_n F_n) \circ P_1(\nu)\big) \xrightarrow[n\to\infty]{} P_1(\eta) \quad \text{in } \mathscr{M}(\boldsymbol{R}^{2d} \setminus \{0\}).$$

An application of Proposition 4.5 yields that $P_1(A)$ is the generating distribution of a full $(t^{m+M})_{t>0}$-stable symmetric c.c.s. $\big(P_1(\mu_t)\big)_{t\geq 0}$ without Gaussian component and Lévy measure $P_1(\eta)$ on $\boldsymbol{R}^{2d}$. With the use of the Theorem of [15] it follows from (11) and (12) that

$$P_1(\nu) \in \mathrm{DOA}\big(P_1(\mu_1)\big),$$

even $(s_n F_n) P_1(\nu)^n \xrightarrow[n\to\infty]{} P_1(\mu_1)$, because $(\mu_t)_{t\geq 0}$ is symmetric. It is well known that this is equivalent to

$$(s_n F_n)\big(P_1(\nu)\big)^{[nt]} \xrightarrow[n\to\infty]{} P_1(\mu_t) \quad \text{for all } t > 0.$$

Using Theorem 3.1 of [26] we conclude that

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} n \int_{\|\bar{x}\|_2 < \varepsilon} x_i^2 \, d\big((s_n F_n) \circ P_1(\nu)\big)(\bar{x}) = 0$$

for $i = 1, \ldots, 2d$. Because of

$$\int_{|x|<\varepsilon} \xi_i(x)^2 \, d(\tau_n \nu)(x) \leq \int_{\|\bar{x}\|<\varepsilon} x_i^2 \, d\big((s_n A_n) \circ P_1(\nu)\big)(\bar{x})$$

we obtain

(13) $$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} n \int_{|x|<\varepsilon} \xi_i(x)^2 \, d(\tau_n \nu)(x) = 0$$

for $i = 1, \ldots, 2d$. In view of Theorem 3.1 in [26] it remains to show (13) for $i = 2d+1$. For $a > 0$ let $\theta_a\colon \boldsymbol{R} \to \boldsymbol{R}$, $\theta_a(x) \stackrel{\text{def}}{=} ax$, denote the homothetical transformation on $\boldsymbol{R}$. From (9) and $n(\tau_n \nu) \xrightarrow[n\to\infty]{} \eta$ we obtain

(14) $$n\big(\theta_{s_n^2} P_2(\nu)\big) \xrightarrow[n\to\infty]{} P_2(\eta) \quad \text{in } \mathscr{M}(\boldsymbol{R} \setminus \{0\}).$$

By Proposition 4.5 we see that $P_2(A)$ is the generating distribution of a $(t^{2m})_{t>0}$-stable c.c.s. $(\lambda_t)_{t\geq 0}$ without Gaussian component on $\boldsymbol{R}$ and Lévy measure $P_2(\eta)$. Since $(\mu_t)_{t\geq 0}$ is symmetric, it follows that $(\lambda_t)_{t\geq 0}$ is symmetric. Now the case that $(\lambda_t)_{t\geq 0}$ is a point measure, i.e., $P_2(\eta) = 0$, is possible, so we need to consider two cases.

First case. $P_2(\eta) = 0$. Then we have $n\big(\theta_{s_n^2} P_2(\nu)\big) \xrightarrow[n\to\infty]{} 0$ in $\mathscr{M}(\boldsymbol{R} \setminus \{0\})$. Hence

(15) $$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} n \int_{|x'|<\varepsilon} x'^2 \, d\big(\theta_{s_n^2} P_2(\nu)\big)(x') = 0.$$

Second case. $P_2(\eta) \neq 0$. Then $(\lambda_t)_{t\geq 0}$ is a full c.c.s. on $\boldsymbol{R}$ without Gaussian component. With the use of Theorem 2 in [14] or the Theorem in [15] we infer from (14) that $P_2(v) \in \mathrm{DOA}(\lambda_1)$, even $\theta_{s_n^2} P_2(v)^n \underset{n\to\infty}{\longrightarrow} \lambda_1$, since $(\lambda_t)_{t\geq 0}$ and v are symmetric. Using Theorem 3.1 of [26] we conclude that (15) holds. Because of

$$n \int_{|x|<\varepsilon} x_{2d+1}^2 \, d(\tau_n v)(x) \leqslant n \int_{|x_{2d+1}|<\varepsilon^2} x_{2d+1}^2 \, d(\tau_n v)(x) \leqslant n \int_{|x'|<\varepsilon} x'^2 \, d\big(\theta_{s_n^2} P_2(v)\big)(x'),$$

we get (13) for $i = 2d+1$ from (15). This completes the proof. ∎

As in [26] we can prove as a corollary to Theorem 5.1 a description of the domain of normal attraction of a full $(\sigma_{M,m}(t))_{t>0}$-stable c.c.s. $(\mu_t)_{t\geq 0}$ without Gaussian component. Hence we need a desintegration formula of the Lévy measure of $(\mu_t)_{t\geq 0}$ and, consequently, a compact cross-section $\Sigma \subset \boldsymbol{H}_d^\times$ such that the mapping

$$\psi\colon \boldsymbol{R}_+^* \times \Sigma \to \boldsymbol{H}_d^\times, \qquad (t, u) \mapsto \sigma_{M,m}(t)\, u,$$

is a homeomorphism. In contrast to Lemma 5.6 of [26], in general we cannot realize Σ as the unit sphere with respect to the usual homogeneous norm $|\cdot|$, but only with respect to a special homogeneous norm $|\cdot|_i$ which will depend on the exponent (M, m).

First we need some information about the exponent (M, m) of the one-parameter group of a full $(\sigma_{M,m}(t))_{t>0}$-stable c.c.s. without Gaussian component.

PROPOSITION 5.2. *Let $(\mu_t)_{t\geq 0} \subset M^1(\boldsymbol{H}_d)$ be a full $(\sigma_{M,m}(t))_{t>0}$-stable c.c.s. without Gaussian component. Then necessarily $m > \frac{1}{2}$ and $\operatorname{Re}\lambda > -m+\frac{1}{2}$ for all $\lambda \in \operatorname{spec}(M)$.*

Proof. According to Theorem 1 of [1] we have $m \geqslant \frac{1}{2}$ and $\operatorname{Re}\lambda \geqslant -m+\frac{1}{2}$ for all $\lambda \in \operatorname{spec}(M)$ and $mI_{2d}+M \in \mathrm{GL}(\boldsymbol{R}^{2d})$, where I_{2d} denotes the identity in $\mathrm{GL}(\boldsymbol{R}^{2d})$. Let us first suppose that $m = \frac{1}{2}$. Because of $M \in \mathrm{sp}(\boldsymbol{R}^{2d})$ it follows that

$$\operatorname{trace}(M) \overset{\text{def}}{=} \sum_{\lambda\in\operatorname{spec}(M)} \lambda = 0.$$

Consequently, $\min_{\lambda\in\operatorname{spec}(M)} \operatorname{Re}\lambda \leqslant 0$ and $\max_{\lambda\in\operatorname{spec}(M)} \operatorname{Re}\lambda \geqslant 0$. As $m = \frac{1}{2}$, we get $\operatorname{Re}\lambda \geqslant 0$ for all $\lambda \in \operatorname{spec}(M)$, and hence $\operatorname{Re}\lambda = 0$ for all $\lambda \in \operatorname{spec}(M)$. In view of Proposition 4.5, $\big(P_1(\mu_t)\big)_{t\geq 0}$ is a full $(t^{m+M})_{t>0}$-stable c.c.s. on $\boldsymbol{R}^{2d}$ without Gaussian component, and then $m+\operatorname{Re}\lambda = \frac{1}{2}$ is a contradiction to [28]. Let us now suppose $m > \frac{1}{2}$ and $\operatorname{Re}\lambda = -m+\frac{1}{2}$ for some $\lambda \in \operatorname{spec}(M)$. This is again a contradiction to [28]. ∎

The *desintegration formula* is given by:

PROPOSITION 5.3. *Let $(\mu_t)_{t\geq 0} \subset M^1(\boldsymbol{H}_d)$ be a full $(\sigma_{M,m}(t))_{t>0}$-stable c.c.s. without Gaussian component and Lévy measure η. Then there exists a homogeneous norm $|\cdot|_i = |\cdot|_{i,(M,m)}$ on $\boldsymbol{H}_d$ and a finite Borel measure χ on*

$\Sigma \stackrel{\text{def}}{=} \{x \in \boldsymbol{H}_d;\ |x|_i = 1\}$ *such that:*

$$\eta(B) = \int_{\Sigma} \int_0^{\infty} 1_B\big(\sigma_{M,m}(t)u\big) t^{-2}\, dt\, d\chi(u) \tag{16}$$

for all Borel subsets $B \subset \boldsymbol{H}_d^{\times}$.

Proof. We use some of the ideas of Jurek [11]. For $x \in \boldsymbol{H}_d$ we define

$$|x|_i \stackrel{\text{def}}{=} \int_0^1 |\sigma_{M,m}(t)x| \frac{dt}{t}.$$

It follows from Proposition 5.2 and [6], p. 136, that there exists a constant $K > 0$ such that $|\sigma_{M,m}(t)x| \leqslant K|x|\, t^{1/2}$ for all $x \in \boldsymbol{H}_d$ and $t > 0$. Hence $|x|_i$ is well defined. An easy computation shows that $|\cdot|_i$ is a homogeneous norm on $\boldsymbol{H}_d$, so it is equivalent to $|\cdot|$. Furthermore, the mapping

$$\psi\colon\ \boldsymbol{R}_+^* \times \Sigma \to \boldsymbol{H}_d \backslash \{0\}, \qquad (t, u) \mapsto \sigma_{M,m}(t)u,$$

is one-to-one, onto and continuous. By an argument analogous to that in the proof of Corollary 2 in [11] it follows that ψ is a homeomorphism. Now one can derive (16) as in the proof of Lemma 5.6 in [26]. ■

Remark 5.4. With the use of the desintegration formula (16) one easily gets:

$$\eta\big(\partial\{\sigma_{M,m}(t)u \mid u \in B,\ t \geqslant a\}\big) = 0 \Leftrightarrow \chi(\partial B) = 0$$

for all Borel subsets $B \subset \Sigma$ and all $a > 0$.

As a corollary to Theorem 5.1, by using Proposition 5.3 now we can prove the following description of the domain of normal attraction.

COROLLARY 5.5. *Let* $(\mu_t)_{t \geqslant 0} \subset M^1(\boldsymbol{H}_d)$ *be a full* $\big(\sigma_{M,m}(t)\big)_{t>0}$*-stable c.c.s. without Gaussian component and let its Lévy measure* η *be decomposed as in* (16). *For a measure* $\nu \in M^1(\boldsymbol{H}_d)$ *the condition*

(i) $\nu \in \mathrm{DONA}\big(\mu_1, (\sigma_{M,m}(t))_{t>0}\big)$

implies

(ii) $\lim_{t\to\infty} t\nu\{\sigma_{M,m}(s)u \mid u \in B,\ s \geqslant t\} = \chi(B)$ *for all Borel sets* $B \subset \Sigma$ *with* $\chi(\partial B) = 0$.

If additionally ν *and* μ_1 *are symmetric, we also have* (ii) $\Rightarrow$ (i).

Proof. The proof is identical to the proof of Corollary 5.11 in [26] if one replaces the dilation δ_t by $\sigma_{M,m}(t)$. Observe also Remark 5.4. ■

Remark 5.6. Of course, the exponent (M, m) of a full $\big(\sigma_{M,m}(t)\big)_{t>0}$-stable c.c.s. $(\mu_t)_{t \geqslant 0}$ is not uniquely determined. But our definition of the domain of normal attraction does not depend on the choice of the exponent (M, m) as seen by the following general argument:

Let G be a simply-connected nilpotent Lie group, $(\tau_t)_{t>0} \subset \mathrm{Aut}(G)$ a continuous one-parameter group, and $(\mu_t)_{t \geqslant 0}$ be a full $(\tau_t)_{t>0}$-stable c.c.s.

Suppose that there exists another one-parameter group $(\sigma_t)_{t>0} \subset \mathrm{Aut}(G)$ such that $(\mu_t)_{t\geqslant 0}$ is also $(\sigma_t)_{t>0}$-stable. Then by a simple computation we infer that

$$\{\sigma_t \tau_{1/t}\}_{t>0} \subset \mathcal{I}_{\mathcal{B}}(\mu_1) \stackrel{\text{def}}{=} \{\tau \in \mathcal{B} \mid \tau\mu_1 = \mu_1\},$$

where $\mathcal{I}(\mu_1)$ is the invariance group of μ_1. Since μ_1 is full, it follows from [20], 2.3, that $\mathcal{I}(\mu_1)$ is compact. For $\nu \in \mathrm{DONA}(\mu_1, (\tau_t)_{t>0})$ we conclude that

$$\sigma_{1/n} \nu^n = (\sigma_{1/n} \tau_n) \tau_{1/n} \nu^n \underset{n\to\infty}{\longrightarrow} \mu_1$$

by the compactness of $\mathcal{I}(\mu_1)$. Hence $\nu \in \mathrm{DONA}(\mu_1, (\sigma_t)_{t>0})$.

Now we will investigate the domain of normal attraction of a full $(\sigma_{M,m}(t))_{t>0}$-stable Gaussian semigroup $(\mu_t)_{t\geqslant 0}$. First we show as in Proposition 5.2 that in this case the exponent (M, m) has a special structure and it will follow that only certain Gaussian semigroups on $\boldsymbol{H}_d$ are $\mathcal{B}$-stable.

PROPOSITION 5.7. *Let $(\mu_t)_{t\geqslant 0} \subset M^1(\boldsymbol{H}_d)$ be a full $(\sigma_{M,m}(t))_{t>0}$-stable Gaussian semigroup. Then the exponent (M, m) necessarily satisfies $m = \frac{1}{2}$ and* $\mathrm{Re}\,\lambda = 0$ *for all* $\lambda \in \mathrm{spec}(M)$.

Proof. As in the proof of Proposition 5.2 we have $m \geqslant \frac{1}{2}$ and $\mathrm{Re}\,\lambda \geqslant -m+\frac{1}{2}$ for all $\lambda \in \mathrm{spec}(M)$. From Proposition 4.5 we conclude that $(P_1(\mu_t))_{t>0}$ is a full $(t^{m+M})_{t>0}$-stable Gaussian semigroup on $\boldsymbol{R}^{2d}$. Theorem 4 of [28] gives $\mathrm{Re}\,\lambda = -m+\frac{1}{2}$ for all $\lambda \in \mathrm{spec}(M)$. Since $\mathrm{trace}(M) = 0$, we get $m = \frac{1}{2}$, and hence $\mathrm{Re}\,\lambda = 0$ for all $\lambda \in \mathrm{spec}(M)$. ∎

In view of the Lévy–Khinchin formula (3) the generating distribution A of an arbitrary Gaussian semigroup $(\mu_t)_{t\geqslant 0} \subset M^1(\boldsymbol{H}_d)$ takes the form

$$A = \sum_{i=1}^{2d+1} p_i X_i + \sum_{i,j=1}^{2d+1} \tilde{a}_{i,j} X_i X_j,$$

where p_i $(i = 1, \ldots, 2d+1)$ are real numbers and $(\tilde{a}_{i,j})$ is a real symmetric positive semidefinite matrix. If $(\mu_t)_{t\geqslant 0}$ is full and $(\sigma_{M,m}(t))_{t>0}$-stable, we conclude from the equality $\sigma_{M,m}(t)\,A = tA$ for all $t > 0$ and Proposition 5.7 that

$$A = aX_{2d+1} + \sum_{i,j=1}^{2d} a_{i,j} X_i X_j, \tag{17}$$

where $a \in \boldsymbol{R}$ and $C \stackrel{\text{def}}{=} (a_{i,j})_{1\leqslant i,j\leqslant 2d}$ is a real symmetric positive semidefinite matrix. It follows easily from Proposition 4.5 that C is the covariance matrix of the full $(t^{m+M})_{t>0}$-stable Gaussian semigroup $(P_1(\mu_t))_{t\geqslant 0}$ on $\boldsymbol{R}^{2d}$. So we infer from Theorem 5.1 in [27] that $CM = MC$ and C is positive definite. Therefore we have shown:

PROPOSITION 5.8. *Let $(\mu_t)_{t\geqslant 0} \subset M^1(\boldsymbol{H}_d)$ be a full $(\sigma_{M,m}(t))_{t>0}$-stable Gaussian semigroup. Then its generating distribution A is of the form* (17) *with a real symmetric positive definite matrix $C = (a_{i,j})_{1\leqslant i,j\leqslant 2d}$ which satisfies $MC = CM$. We will call C the covariance matrix of the c.c.s. $(\mu_t)_{t\geqslant 0}$.*

We are now in a position to prove a particular nice description of the domain of normal attraction of a full $\mathscr{B}$-stable Gaussian measure on $\boldsymbol{H}_d$. We will use our projection method, i.e., Proposition 4.5, and the pseudo-inner product (4) on $\boldsymbol{H}_d$. Our result is in part analogous to a theorem of Jurek (see [10], Theorem 4.1) on $\boldsymbol{R}^d$.

THEOREM 5.9. *Let* $(\mu_t)_{t\geqslant 0}\subset M^1(\boldsymbol{H}_d)$ *be a full* $(\sigma_{M,m}(t))_{t>0}$*-stable Gaussian semigroup with covariance matrix* C (cf. Proposition 5.8). *Then for* $\nu\in M^1(\boldsymbol{H}_d)$ *the condition*

(i) $\nu\in \mathrm{DONA}\big(\mu_1, (\sigma_{M,m}(t))_{t>0}\big)$

implies

(ii) $\langle (C\bar{y}, 0), (\bar{y}, 0)\rangle = \int_{\boldsymbol{H}_d}\langle x, (\bar{y}, 0)\rangle^2\, d\nu(x)$ *for all* $\bar{y}\in\boldsymbol{R}^{2d}$.

If moreover μ_1 *and* ν *are symmetric and additionally* ν *has a finite second homogeneous moment, i.e.,* $\int_{\boldsymbol{H}_d}|x|^2\, d\nu(x)<\infty$, *we also have* (ii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii). According to Propositions 4.5 and 5.8 we have

$$P_1(\nu)\in \mathrm{DONA}\big(P_1(\mu_1), (t^{m+M})_{t>0}\big),$$

and $P_1(\mu_1)$ is a full $(t^{m+M})_{t>0}$-stable Gaussian measure on $\boldsymbol{R}^{2d}$ with covariance matrix C. Hence from Theorem 4.1 of [10] it follows that

$$\psi(C\bar{y}, \bar{y}) = \int_{\boldsymbol{R}^{2d}}\psi(\bar{x}, \bar{y})^2\, dP_1(\nu)(\bar{x}) \qquad \text{for all } \bar{y}\in\boldsymbol{R}^{2d}.$$

Since $\int_{\boldsymbol{R}^{2d}}\psi(\bar{x}, \bar{y})^2\, dP_1(\nu)(\bar{x}) = \int_{\boldsymbol{H}_d}\langle x, (\bar{y}, 0)\rangle^2\, d\nu(x)$ for all $y\in\boldsymbol{R}^{2d}$, the definition (4) yields the assertion.

(ii) $\Rightarrow$ (i). Let μ_1 and ν be symmetric and $\int_{\boldsymbol{H}_d}|x|^2\, d\nu(x)<\infty$. Theorem 5.9 (ii) is equivalent to

$$\int_{\boldsymbol{H}_d} x_i x_j\, d\nu(x) = a_{i,j} \qquad \text{for all } i, j = 1, \ldots, 2d.$$

Hence the well-known central limit theorem on $\boldsymbol{H}_d$ (see [23], [21] and [26], Remark 3.5 (4)) implies

$$\delta_{1/\sqrt{n}}\nu^n \xrightarrow[n\to\infty]{} \mu_1. \tag{18}$$

Since $(\mu_t)_{t\geqslant 0}$ is a full $(\sigma_{M,m}(t))_{t>0}$-stable Gaussian semigroup on $\boldsymbol{H}_d$, it follows from (17) that $(\mu_t)_{t\geqslant 0}$ is also $(\delta_{1/\sqrt{t}})_{t>0}$-stable. Of course, we have

$$\mu_1\in \mathrm{DONA}\big(\mu_1, (\delta_{1/\sqrt{t}})_{t>0}\big) \quad \text{and} \quad \mu_1\in \mathrm{DONA}\big(\mu_1, (\sigma_{M,m}(t))_{t>0}\big),$$

so

$$\delta_{1/\sqrt{n}}\mu_1^n \xrightarrow[n\to\infty]{} \mu_1 \quad \text{and} \quad \sigma_{M,m}(1/n)\,\mu_1^n \xrightarrow[n\to\infty]{} \mu_1.$$

By the convergence of types theorem (Proposition 3.2) we see that $\{\sigma_{M,m}(1/n)\,\delta_{\sqrt{n}}\,|\, n\in\boldsymbol{N}\}$ is relatively compact in $\mathrm{Aut}(\boldsymbol{H}_d)$, hence in $\mathscr{B}$, and all limit points τ fulfil $\tau\mu_1 = \mu_1$, i.e., $\tau \in \mathscr{I}(\mu_1)$. Finally, (18) and

$$\sigma_{M,m}(1/n)\,\nu^n = \big(\sigma_{M,m}(1/n)\,\delta_{\sqrt{n}}\big)\,\delta_{1/\sqrt{n}}\,\nu^n$$

yield the assertion. This completes the proof. ■

Theorem 5.9 is weaker than the corresponding result of Jurek ([10], Theorem 4.1) on $\boldsymbol{R}^d$ because we need to require the existence of the second homogeneous moment of ν to obtain a sufficient condition. In contrast to the vector space case on $\boldsymbol{R}^d$, in general the existence of the second homogeneous moment for measures ν in the domain of normal attraction of a full $\mathscr{B}$-stable Gaussian measure is not necessary. This is shown in the following example.

Example 5.10. Let $(\mu_t)_{t\geqslant 0} \subset M^1(\boldsymbol{H}_1)$ be the full Gaussian semigroup on $\boldsymbol{H}_1$ with generating distribution $A = X_1^2 + X_2^2$. Then $(\mu_t)_{t\geqslant 0}$ is $(\delta_{1/\sqrt{t}})_{t>0}$-stable. We will write the coordinates of $\boldsymbol{H}_1$ in the form $x = (x_1, x_2, t) \in \boldsymbol{H}_1$ with $P_1(x) = (x_1, x_2)$ and $P_2(x) = t$. Now let $\nu \in M^1(\boldsymbol{H}_1)$ be the measure with density

$$(x_1, x_2, t) \mapsto \frac{e}{4\pi}\exp\left[-\tfrac{1}{2}(x_1^2 + x_2^2)\right]\frac{\log|t|+1}{t^2(\log|t|)^2}\,1_{\{|t|\geqslant e\}}(t) \tag{19}$$

with respect to the Haar measure dx on $\boldsymbol{H}_1$. Then we have

$$\nu \in \mathrm{DONA}\big(\mu_1, (\delta_{1/\sqrt{t}})_{t>0}\big) \quad \text{but} \int_{\boldsymbol{H}_1} |x|^2\,d\nu(x) = \infty,$$

i.e., ν has no finite second homogeneous moment.

(α) Since μ_1 and ν are symmetric, we can use Corollary 3.4 of [26]. Hence

$$\delta_{1/\sqrt{t}}\,\nu^n \underset{n\to\infty}{\longrightarrow} \mu_1$$

is equivalent to

(i) $$n\int_{|x|<\varepsilon} x_i^2\,d(\delta_{1/\sqrt{n}}\nu)(x) \underset{n\to\infty}{\longrightarrow} 1 \quad \text{for } i = 1, 2,$$

(ii) $$n\int_{|x|<\varepsilon} x_i x_j\,d(\delta_{1/\sqrt{n}}\nu)(x) \underset{n\to\infty}{\longrightarrow} 0 \quad \text{for } i, j = 1, 2,\ i \neq j,$$

(iii) $$n\int_{|x|<\varepsilon} x_i t\,d(\delta_{1/\sqrt{n}}\nu)(x) \underset{n\to\infty}{\longrightarrow} 0 \quad \text{for } i = 1, 2,$$

(iv) $$n\int_{|x|<\varepsilon} t^2\,d(\delta_{1/\sqrt{n}}\nu)(x) \underset{n\to\infty}{\longrightarrow} 0,$$

(v) $$n(\delta_{1/\sqrt{n}}\nu)\{x\colon |x| \geqslant \varepsilon\} \underset{n\to\infty}{\longrightarrow} 0$$

for all $\varepsilon > 0$.

By an easy computation we obtain (i) and (ii). One takes into consideration that

$$\left(-\frac{1}{t\log t}\right)' = \frac{\log t + 1}{t^2(\log t)^2}.$$

(iii) follows by a simple symmetry argument. Using the definition (19) of the density of ν we get

$$n \int_{|x|<\varepsilon} t^2 \, d(\delta_{1/\sqrt{n}}\nu)(x) \leqslant \frac{e}{n} \int_{e}^{n\varepsilon^2} \frac{\log t + 1}{(\log t)^2} \, dt.$$

After a substitution and integration by parts we obtain

$$\operatorname{li}(x) \overset{\text{def}}{=} \int_{e}^{x} \frac{dt}{\log t} = O\left(\frac{x}{\log x}\right) \quad \text{as } x \to \infty.$$

Another integration by parts gives

$$\int_{e}^{x} \frac{\log t + 1}{(\log t)^2} \, dt = -\frac{x}{\log x} + e - \operatorname{li}(x) = O\left(\frac{x}{\log x}\right) \quad \text{as } x \to \infty;$$

therefore we obtain (iv).

From the relation

$$\{x \in \boldsymbol{H}_1;\ |x| \geqslant \sqrt{n}\varepsilon\} \subset \left\{x \in \boldsymbol{H}_1;\ \|(x_1, x_2)\| \geqslant \frac{\sqrt{n}\varepsilon}{2}\right\} \cup \left\{x \in \boldsymbol{H}_1;\ |t| \geqslant \frac{n\varepsilon^2}{4}\right\}$$

it follows that

$$(20) \quad n(\delta_{1/\sqrt{n}}\nu)\{x;\ |x| \geqslant \varepsilon\} \leqslant n\nu\left\{x;\ \|(x_1, x_2)\| \geqslant \frac{\sqrt{n}\varepsilon}{2}\right\} + n\nu\left\{x;\ |t| \geqslant \frac{n\varepsilon^2}{4}\right\}.$$

Using the well-known properties of Gaussian measures on $\boldsymbol{R}^2$ we conclude that the first term on the right-hand side of (20) converges against zero. For the second term we obtain

$$n\nu\left\{x;\ |t| \geqslant \frac{n\varepsilon^2}{4}\right\} = en \int_{n\varepsilon^2/4}^{\infty} \frac{\log t + 1}{t^2 (\log t)^2} \, dt = \frac{4e}{\varepsilon^2 \log(n\varepsilon^2/4)} \xrightarrow[n \to \infty]{} 0,$$

and hence (v) is valid.

(β) We have

$$\int_{\boldsymbol{H}_1} |x|^2 \, d\nu(x) = \int_{\boldsymbol{H}_1} \|(x_1, x_2)\|^2 \, d\nu(x) + \int_{\boldsymbol{H}_1} |t| \, d\nu(x).$$

Trivially, the first term on the right-hand side is finite, but for the second term we compute

$$\int_{\boldsymbol{H}_1} |t| \, d\nu(x) = e \int_{e}^{\infty} \frac{\log t + 1}{t (\log t)^2} \, dt = \infty,$$

i.e., ν has no second homogeneous moment.

Concluding remarks. In [25] we have shown that a measure ν in the domain of attraction of a full stable c.c.s. on $\boldsymbol{H}_d$ has certain moments. More precisely, let $(\mu_t)_{t\geqslant 0} \subset M^1(\boldsymbol{H}_d)$ be a full $(\sigma_{M,m}(t))_{t>0}$-stable c.c.s. and define

$$\delta_0 \stackrel{\text{def}}{=} \max\left(\{m+\operatorname{Re}\lambda \mid \lambda\in\operatorname{spec}(M)\} \cup \{m\}\right).$$

Then a measure ν in the $\mathscr{B}$-domain of attraction of μ_1 has all homogeneous moments of order less than $1/\delta_0$. This result is analogous to that of [8]. Details will appear elsewhere.

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