PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 14, Fasc. 2 (1993), pp. 347–358

ON THE ALMOST UNIFORM CONVERGENCE IN NONCOMMUTATIVE L_2-SPACES

BY

EWA HENSZ, RYSZARD JAJTE AND ADAM PASZKIEWICZ (ŁÓDŹ)

Abstract. We introduce a kind of convergence in L_2 over a von Neumann algebra and prove a few typical results being the analogues of classical pointwise theorems.

1. In [4] a notion of the almost sure convergence in L_2 over a von Neumann algebra has been introduced and several limit theorems have been proved (cf. [6]). The main goal of this paper* is to define another kind of convergence in the noncommutative L_2-space which coincides with the ordinary almost everywhere convergence in the case of a commutative von Neumann algebra $L_\infty(X, \mathscr{F}, \mu)$. We shall call our new convergence the *almost uniform convergence* in L_2. Moreover, we prove some typical limit theorems (an individual ergodic theorem, a martingale convergence theorem, a Rademacher–Menshov theorem) for this convergence.

Let us begin with some notation and definitions. Let M be a σ-finite von Neumann algebra with a faithful and normal state Φ. In our case, the GNS representation of (M, Φ) is faithful and normal so, without any loss of generality, we may assume that M acts in its GNS representation Hilbert space, say H, in a standard way. In particular, we have $H = L_2(M, \Phi)$ being the completion of M under the norm $x \mapsto \Phi(x^* x)^{1/2}$, and $\Phi(x) = (x\Omega, \Omega)$, $x \in M$, where Ω is a cyclic and separating vector in H (cf. [13]). The norm in H will be denoted by $\|\cdot\|$, the operator norm in M by $\|\cdot\|_\infty$. $\operatorname{Proj} M$ denotes the lattice of all orthogonal projections in M, $p^\perp = 1-p$ for $p \in \operatorname{Proj} M$. We always have $|x|^2 = x^* x$ for $x \in M$, and M^{sa} (or M^+) consists of all selfadjoint (or positive) operators from M.

A linear map α: $M \to M$ is said to be a *Schwarz map* if $\alpha(|x|^2) \geqslant |\alpha(x)|^2$ for $x \in M$. A Schwarz map satisfying the condition $\Phi(\alpha x) \leqslant \Phi(x)$ for $x \in M^+$ is called a *kernel*. A kernel α in M can always be extended in a unique way to a contraction β in H. Namely, we put $\beta(x\Omega) = \alpha(x)\Omega$ for $x \in M$, and then we

* Research supported by KBN grant 211529101.

extend the obtained contraction from $M\Omega$ to the whole H by continuity. In this case, we say that the contraction β in H is *induced by the kernel* α in M. The most important kernels are Φ-preserving $*$-endomorphisms of M and Φ-preserving conditional expectations. They induce in H isometries and orthogonal projections, respectively.

2. Let us recall (following Lance [7] and Sinai and Anshelevich [12]; cf. also Paszkiewicz [10]) that a sequence $(x_n) \subset M$ is said to be *almost uniformly convergent* to $x \in M$ ($x_n \to x$ a.u. in M) if for every $\varepsilon > 0$ there exists a $p \in \operatorname{Proj} M$ with $\Phi(p^\perp) < \varepsilon$ such that $\|(x_n - x)p\|_\infty \to 0$ as $n \to \infty$.

We start with the following definition:

2.1. Definition. A sequence (ξ_n) in H is said to be *almost uniformly convergent* in H to a $\xi \in H$ ($\xi_n \to \xi$ a.u. in H) if for every sequence $(y_n) \subset M$ with $\sum_{n=1}^{\infty} \|\xi_n - \xi - y_n\Omega\|^2 < \infty$ we have $y_n \to 0$ a.u. in M.

Clearly, by Egorov's theorem and Beppo Levi's theorem, the almost uniform convergence in $H = L_2(M, \Phi)$ coincides with the usual almost everywhere convergence in the case $M = L_\infty(X, \mathscr{F}, \mu)$ over a probability space $(X, \mathscr{F}, \mu)$.

Note that for a sequence (x_n) in M if $x_n\Omega$ is a.u. convergent in H to an $x\Omega \in H$, then x_n is almost uniformly convergent in M to x. The inverse implication is just an open question.

Finally, let us recall (see [4]) that a sequence $(\xi_n) \subset H$ is said to be *almost surely convergent to zero* ($\xi_n \to 0$ a.s.) if for every $\varepsilon > 0$ there exists a projection $p \in \operatorname{Proj} M$ with $\Phi(p^\perp) < \varepsilon$ and $\|\xi_n\|_p \to 0$ as $n \to \infty$. Here the modular $\|\cdot\|_p$ ($p \in \operatorname{Proj} M$) is defined as follows: for a $\xi \in H$ we have

$$\|\xi\|_p = \inf\Big\{\Big\|\sum_{k=1}^{\infty} x_k p\Big\|:\ \xi = \sum_{k=1}^{\infty} x_k\Omega \text{ in } H,\ (x_k) \subset M$$

$$\text{and } \sum_{k=1}^{\infty} x_k p \text{ converges in norm in } M\Big\}.$$

2.2. Theorem. *Let (ξ_n) be a sequence in H. If $\xi_n \to 0$ a.u. in H and $\|\xi_n\| \to 0$, then $\xi_n \to 0$ a.s.*

Proof. Let us choose operators $x_i^n \in M$ ($n, i = 1, 2, \ldots$) such that

$$\xi_n = \sum_{i=1}^{\infty} x_i^n \Omega \text{ in } H, \tag{1}$$

$$\sum_{n=1}^{\infty} \|\xi_n - x_1^n \Omega\|^2 < \infty, \tag{2}$$

$$\sum_{j=2}^{\infty} 4^j \|x_j^n \Omega\| \leqslant 2\,\|\xi_n - x_1^n \Omega\|, \quad n = 1, 2, \ldots \tag{3}$$

By assumption, there exists an increasing sequence $(k(n))$ of positive integers such that

$$\sum_{n=1}^{\infty} \|\xi_{k(n)}\|^2 < \infty. \tag{4}$$

Let us fix a bijection π of the set $N \times \{N \setminus \{1\}\}$ onto the set $N_0 = \{1, 3, 5, \ldots\}$ of all odd positive numbers. Now, we define a sequence $(y_n)_{n=1}^{\infty} \subset M$ putting $y_n = x_1^n$ for $n \notin \{k(m)\colon m \in N\}$, $y_{k(2m)} = x_1^{k(m)}$ $(m \in N)$ and $y_{k(\pi(i,j))} = 2^j x_j^i$ $(i, j \in N,\ j \neq 1)$. We get

$$\sum_{n=1}^{\infty} \|\xi_n - y_n \Omega\|^2 < \infty. \tag{5}$$

Indeed, by (2), we have

$$\sum_{n \notin \{k(m): m \in N\}} \|\xi_n - y_n \Omega\|^2 = \sum_{n \notin \{k(m)\}} \|\xi_n - x_1^n \Omega\|^2 < \infty.$$

By (4), we obtain

$$\sum_{m=1}^{\infty} \|\xi_{k(2m)} - y_{k(2m)} \Omega\|^2 \leqslant 2 \sum_{m=1}^{\infty} (\|\xi_{k(2m)}\|^2 + \|x_1^{k(m)} \Omega\|^2) < \infty$$

because, by (2) and (4), we have

$$\sum_{m=1}^{\infty} \|x_1^{k(m)} \Omega\|^2 \leqslant 2 \sum_{m=1}^{\infty} (\|x_1^{k(m)} \Omega - \xi_{k(m)}\|^2 + \|\xi_{k(m)}\|^2) < \infty.$$

Finally, by (3) and (2), we get

$$\sum_{\substack{i,j \in N \\ j \neq 1}} \|\xi_{k(\pi(i,j))} - y_{k(\pi(i,j))} \Omega\|^2 \leqslant 2 \sum_{\substack{i,j \in N \\ j \neq 1}} (\|\xi_{k(\pi(i,j))}\|^2 + 4^j \|x_j^i \Omega\|^2) < \infty,$$

which yields (5).

Then, by (5) and the almost uniform convergence of (ξ_n) in H, we get $y_n \to 0$ a.u. in M, so for $\varepsilon > 0$ there exists a projection $p \in \operatorname{Proj} M$ with $\Phi(p^{\perp}) < \varepsilon$ such that $\|y_n p\|_{\infty} \to 0$ as $n \to \infty$.

In particular, we obtain

$$\|x_1^n p\|_{\infty} \to 0 \quad \text{as } n \to \infty \tag{6}$$

and

$$\|y_{k(\pi(i,j))} p\|_{\infty} \to 0 \quad \text{as } \max(i, j) \to \infty,\ j \neq 1,$$

because $\pi(i, j) \to \infty$ as $\max(i, j) \to \infty$. Thus

$$\sup_{\substack{i,j \in N \\ j \neq 1}} \|y_{k(\pi(i,j))} p\|_{\infty} \leqslant C < \infty$$

and

$$\|y_{k(\pi(i,j))} p\|_{\infty} \to 0 \quad \text{as } n \to \infty,\ j \neq 1. \tag{7}$$

Then the series $\sum_{j=2}^{\infty} \|x_j^n p\|_\infty = \sum_{j=2}^{\infty} 2^{-j} \|y_{k(\pi(i,j))} p\|_\infty$ is uniformly convergent with respect to n.

But, by (1), we have

$$\|\xi_n\|_p \leqslant \sum_{j=1}^{\infty} \|x_j^n p\|_\infty,$$

so, by (6) and (7), we get $\|\xi_n\|_p \to 0$ as $n \to \infty$. The proof is completed. ∎

3. Now, we formulate some limit theorems for the almost uniform convergence in H. By Theorem 2.2 we can regard them as new stronger versions of some results from [4] and [6].

3.1. THEOREM (individual ergodic theorem). *Let β be a contraction in H generated by a kernel α in M. Then, for each $\xi \in H$, $\sigma_n(\xi) \to \tilde{\xi}$ a.u. in H, where $\sigma_n = n^{-1} \sum_{k=0}^{n-1} \beta^k$ and $\tilde{\xi} = \lim_{n\to\infty} \sigma_n(\xi)$ in H given by the mean ergodic theorem.*

3.2. THEOREM (martingale convergence theorem). *Let $(M_n)_{n=1}^{\infty}$ be a decreasing sequence of von Neumann subalgebras of M with conditional expectations $\boldsymbol{E}_n$. Let $\tilde{\boldsymbol{E}}_n$ denote the orthogonal projection in $H = L_2(M, \Phi)$ generated by $\boldsymbol{E}_n$, i.e., $\tilde{\boldsymbol{E}}_n(x\Omega) = \boldsymbol{E}_n(x)\Omega$ for $x \in M$. Then, for every $\xi \in H$, $\tilde{\boldsymbol{E}}_n \xi$ converges almost uniformly in H to $\tilde{\xi} = \tilde{\boldsymbol{E}}_\infty \xi$, where $\tilde{\boldsymbol{E}}_\infty = \bigwedge_{n=1}^{\infty} \tilde{\boldsymbol{E}}_n$.*

3.3. THEOREM (Rademacher–Menshov theorem). *Let $(\xi_n)_{n=1}^{\infty}$ be an orthogonal sequence in H such that*

$$\sum_{n=1}^{\infty} \log^2(n+1) \|\xi_n\|^2 < \infty. \tag{8}$$

Then

$$\sigma_n = \sum_{j=1}^{n} \xi_j \to \sigma \text{ a.u. in } H,$$

where σ is the sum of the series $\sum_{j=1}^{\infty} \xi_j$ in H.

4. For proofs of the above results we shall need some auxiliary results.

Both in the classical and the noncommutative theory, behind the proofs of the individual ergodic theorems or martingale convergence theorems there are always some "maximal inequalities". We use the following theorem of Goldstein ([3], cf. also [5]).

4.1. THEOREM (maximal inequality for ergodic averages). *Let $\alpha\colon M \to M$ be a normal positive map such that $\alpha \mathbf{1} \leqslant \mathbf{1}$ and $\Phi(\alpha x) \leqslant \Phi(x)$ for $x \in M^+$. Let $(x_n) \subset M^+$ and let (ε_n) be a sequence of positive numbers. We put $s_n = n^{-1} \sum_{k=0}^{n-1} \alpha^k$. Then there exists a projection $p \in \operatorname{Proj} M$ such that*

$$\|p s_n(x_k) p\|_\infty < 2\varepsilon_k \quad \textit{for } n, k = 1, 2, \ldots$$

and

$$\Phi(p^\perp) \leqslant 2 \sum_{n=1}^{\infty} \varepsilon_n^{-1} \Phi(x_n).$$

A simple consequence of this theorem is the following

4.2. PROPOSITION. *Let $(D_n) \subset M^+$ and $\sum_{k=1}^{\infty} \Phi(D_k) < \infty$. Then for each $\varepsilon > 0$ there exists a $p \in \operatorname{Proj} M$ with $\Phi(p^\perp) < \varepsilon$ such that $\|pD_n p\|_\infty \to 0$ as $n \to \infty$.*

Now, we formulate a lemma about some approximation possibilities. We omit its vexatious proof which is based only on "calculating mechanism" with its main tool – the triangle inequality.

4.3. LEMMA. *Let $(\xi_n)_{n=1}^{\infty}$ be an orthogonal sequence in H. Let $(m(n))_{n=1}^{\infty}$ and $(k(n))_{n=1}^{\infty}$ be two sequences of indices such that $m(j) > 2^{j+1}$ $(j = 1, 2, \ldots)$ and $k(n) = s$ when $2^s < n \leqslant 2^{s+1}$ $(s = 0, 1, 2, \ldots)$. Then there exists a sequence (ε_i) of positive numbers such that for all $(x_i) \subset M$ (with $x_n = 0$ when $\xi_n = 0$) the inequalities*

$$\|\xi_i - x_i \Omega\| \leqslant \varepsilon_i, \quad i = 1, 2, \ldots,$$

imply

$$\sum_{n=1}^{\infty} \|\sigma_n - s_n \Omega - \sigma_{m(k(n))} + s_{m(k(n))} \Omega\|^2 < \infty, \tag{9}$$

where

$$\sigma_n = \sum_{j=1}^{n} \xi_j, \quad s_n = \sum_{j=1}^{n} x_j, \quad n = 1, 2, \ldots,$$

and

$$\sum_{i,j=2^k+1}^{m(k)} |\Phi(x_i^* x_j)| \leqslant 2 \sum_{j=2^k+1}^{m(k)} \|\xi_j\|^2, \quad k = 0, 1, 2, \ldots \tag{10}$$

The next lemma is a slight modification of Lemma 4.2 in [4] (cf. also Lemma 5.2.2 in [6]).

4.4. LEMMA. *Let $J \subset N$ and $\# J = \mu$. Let $(\xi_i)_{i=1}^{\infty}$ be a sequence of elements in H such that $\xi_i = 0$ for $i \notin J$. Let (ε_i) be an arbitrary sequence of positive numbers. Then there exist operators $x_i \in M$, $i \in N$ (with $x_i = 0$ when $\xi_i = 0$) and $B \in M^+$ such that*

$$\|\xi_i - x_i \Omega\| < \varepsilon_i, \; i = 1, 2, \ldots, \qquad \Big|\sum_{i=1}^{n} x_i\Big|^2 \leqslant B, \; n = 1, 2, \ldots,$$

$$\Phi(B) \leqslant 2(1+\log \mu)^2 \sum_{i,j \in J} |(\xi_i, \xi_j)|.$$

5. Proof of Theorem 3.1. Let us put

$$H_1 = \{\eta \in H\colon \beta\eta = \eta\} \quad \text{and} \quad H_2 = \{(x - \alpha x)\Omega\colon x \in M\}^-.$$

Clearly, we have $H = H_1 \oplus H_2$. Let $\xi \in H$. Obviously, $\xi - \tilde{\xi} \in H_2$. Then there exists a sequence $(x_k) \subset M$ such that

$$\xi - \tilde{\xi} = \sum_{k=1}^{\infty} z_k \Omega, \tag{11}$$

where $z_k = x_k - \alpha x_k$ $(k = 1, 2, \ldots)$ and

(12) $$\|z_k \Omega\| \leqslant 2^{1-k} \|\xi - \tilde{\xi}\|, \quad k = 1, 2, \ldots$$

Setting $s_n = n^{-1} \sum_{j=0}^{n-1} \alpha^j$, $n = 1, 2, \ldots$, we have

(13) $$\|s_n(z_k)\|_\infty \leqslant \frac{2}{n} \|x_k\|_\infty \to 0, \quad n \to \infty.$$

Moreover,

(14) $$\sigma_n(\xi) - \tilde{\xi} = \sigma_n(\xi - \tilde{\xi}) = \sum_{k=1}^{\infty} \sigma_n(z_k \Omega) = \sum_{k=1}^{\infty} s_n(z_k) \Omega.$$

Let $(k(n))_{n=1}^{\infty}$ be a sequence of indices such that putting

$$\varrho_n = \sum_{k=k(n)+1}^{\infty} s_n(z_k) \Omega,$$

we obtain

(15) $$\sum_{n=1}^{\infty} \|\varrho_n\|^2 < \infty.$$

Setting

$$v_n = \sum_{k=1}^{k(n)} s_n(z_n), \quad n = 1, 2, \ldots,$$

by (14), we have $\sigma_n(\xi) - \tilde{\xi} = v_n \Omega + \varrho_n$, $n = 1, 2, \ldots$ Now, let $y_n \in M$ $(n = 1, 2, \ldots)$ and

(16) $$\sum_{n=1}^{\infty} \|\sigma_n(\xi) - \tilde{\xi} - y_n \Omega\|^2 < \infty.$$

The proof will be completed if we show that $y_n \to 0$ a.u. in M.

Putting $\delta_n = y_n \Omega - (\sigma_n(\xi) - \tilde{\xi})$, $n = 1, 2, \ldots$, we get $y_n \Omega = v_n \Omega + \tau_n$, where $\tau_n = \varrho_n + \delta_n$, $n = 1, 2, \ldots$

Then, by (15) and (16), we obtain

(17) $$\sum_{n=1}^{\infty} \|\tau_n\|^2 < \infty.$$

On the other hand, we can write $\tau_n = t_n \Omega$, $n = 1, 2, \ldots$, where $t_n = y_n - v_n$ $(n = 1, 2, \ldots)$. Clearly, by (17), $\sum_{k=1}^{\infty} \Phi(|t_k|^2) < \infty$.

Thus, for each $\varepsilon > 0$ there exists a sequence of positive numbers $\delta_k \to 0$ such that

$$\sum_{k=1}^{\infty} \delta_k^{-1} \Phi(|t_k|^2) < \varepsilon/4.$$

Let us put $\varepsilon_k = 2^{4-k} \varepsilon^{-1} \|\xi - \tilde{\xi}\|^2$ $(k = 1, 2, \ldots)$. Then, by (12), we get

(18) $$\sum_{k=1}^{\infty} \varepsilon_k^{-1} \Phi(|z_k|^2) < \varepsilon/4.$$

Now, applying Theorem 4.1, we can find a projection $p \in \operatorname{Proj} M$ such that

$$\Phi(p^{\perp}) < 2 \sum_{k=1}^{\infty} \Phi(\varepsilon_k^{-1} |z_k|^2 + \delta_k^{-1} |t_k|^2) < \varepsilon$$

and

(19) $$\|p s_n(|z_k|^2) p\|_{\infty} \leqslant \varepsilon_k, \quad n, k = 1, 2, \ldots,$$

(20) $$\|p |t_k|^2 p\|_{\infty} \leqslant \delta_k, \quad k = 1, 2, \ldots$$

But s_n are (because of α) Schwarz maps as well, so $\|s_n(z_k) p\|_{\infty} \leqslant \|p s_n(|z_k|^2) p\|_{\infty}^{1/2}$, $n, k = 1, 2, \ldots$, and thus, by (19), we get

$$\sum_{k=1}^{\infty} \|s_n(z_k) p\|_{\infty} \leqslant \text{Const} \sum_{k=1}^{\infty} 2^{-k/2} < \infty.$$

Then the above series is uniformly convergent (with respect to n) and, by (13), we obtain

$$\|v_n p\|_{\infty} = \Big\| \sum_{k=1}^{k(n)} s_n(z_k) p \Big\|_{\infty} \to 0 \quad \text{as } n \to \infty.$$

Hence, by (20), we have $\|y_n p\|_{\infty} \leqslant \|v_n p\|_{\infty} + \|t_n p\|_{\infty} \to 0$ as $n \to \infty$ which, with $\Phi(p^{\perp}) < \varepsilon$, means that $y_n \to 0$ a.u. in M. The proof is completed. ∎

6. Proof of Theorem 3.2. The method of proof goes back to Neveu [9]. It was adapted to the context of von Neumann algebras by Dang-Ngoc [2]. We shall refer to [5]. By Lemma 3.1.5 of [5], p. 62, we find a sequence $0 = a_0 < a_1 < \ldots < a_n < 1$, $a_n \to 1$, and a sequence (n_r) of positive integers such that

$$\alpha = \sum_{r=1}^{\infty} (a_r - a_{r-1}) E_r$$

is a kernel on M and

(21) $$\sum_{r=1}^{\infty} \Big\| n_r^{-1} \sum_{j=0}^{n_r - 1} \alpha^j - E_r \Big\| \leqslant 1.$$

Denote by β the contraction in H generated by α, i.e. $\beta(x\Omega) = \alpha(x)\Omega$ for $x \in M$. Let

$$\sigma_n = n^{-1} \sum_{j=0}^{n-1} \beta^j \quad (n = 1, 2, \ldots).$$

Fix an arbitrary element ξ in H, and let $\bar{\xi} = \lim_{n\to\infty} \sigma_n(\xi)$ in H (by the mean ergodic theorem). Then, by (21), we also have $\tilde{E}_r \xi \to \bar{\xi}$, so

$$\bar{\xi} = \left(\bigwedge_{r=1}^{\infty} \tilde{E}_r\right) \xi.$$

In particular, $\tilde{E}_r \bar{\xi} = \bar{\xi}$ for $r = 1, 2, \ldots$

The continuation of our reasoning is very similar to that in the proof of Theorem 3.1, so we shall keep the earlier notation and only sketch the proof.

We have the decomposition $H = H_1 \oplus H_2$ and we can find, in the same manner, a sequence $(x_k) \subset M$ such that (11)–(14) hold.

Moreover, we get

$$\tilde{E}_r \xi - \bar{\xi} = \tilde{E}_r(\xi - \bar{\xi}) = \sum_{k=1}^{\infty} \tilde{E}_r(z_k \Omega)$$
$$= \sum_{k=1}^{\infty} \left(\tilde{E}_r(z_k) - n_r^{-1} \sum_{j=0}^{n_r - 1} \alpha^j(z_k)\right) \Omega + n_r^{-1} \sum_{j=0}^{n_r - 1} \beta^j(\xi) - \bar{\xi} \equiv \eta_r + \sigma_{n_r}(\xi) - \bar{\xi}.$$

By (21) and (12) we have the estimation

$$\sum_{r=1}^{\infty} \|\eta_r\| \leqslant \sum_{r=1}^{\infty} \sum_{k=1}^{\infty} \left\|\tilde{E}_r - n_r^{-1} \sum_{j=0}^{n_r - 1} \alpha^j\right\| \|z_k \Omega\| \leqslant \sum_{k=1}^{\infty} \|z_k \Omega\| \leqslant 2\|\xi - \bar{\xi}\| < \infty. \tag{22}$$

Let $(k(r))_{r=1}^{\infty}$ be a sequence of indices such that putting this time, for $r = 1, 2, \ldots,$

$$\bar{\varrho}_r = \sum_{k=k(r)+1}^{\infty} s_{n_r}(z_k), \tag{23}$$

we also obtain $\sum_{r=1}^{\infty} \|\bar{\varrho}_r\|^2 < \infty$, whereas setting

$$\bar{v}_r = \sum_{k=1}^{k(r)} s_{n_r}(z_k)$$

we get $\sigma_{n_r}(\xi) - \bar{\xi} = \bar{v}_r \Omega + \bar{\varrho}_r$. Let $y_r \in M$ $(r = 1, 2, \ldots)$ and

$$\sum_{r=1}^{\infty} \|\tilde{E}_r \xi - \bar{\xi} - y_r \Omega\|^2 < \infty. \tag{24}$$

To conclude the proof it is enough to show that $y_r \to 0$ a.u. in M.

Putting $\bar{\delta}_r = y_r \Omega - (\tilde{E}_r \xi - \bar{\xi})$, we get $y_r \Omega = \bar{v}_r \Omega + \bar{\tau}_r$, where $\bar{\tau}_r = \bar{\delta}_r + \eta_r + \bar{\varrho}_r$, and by (21), (24) and (23) we obtain

$$\sum_{r=1}^{\infty} \|\bar{\tau}_r\|^2 < \infty. \tag{25}$$

But, writing $\bar{\tau}_r = (y_r - \bar{v}_r)\Omega = \bar{t}_r \Omega$, where $\bar{t}_r \in M$, we have by (25)

$$\sum_{r=1}^{\infty} \Phi(|\bar{t}_r|^2) < \infty. \tag{26}$$

Now, for given $\varepsilon > 0$ and $\varepsilon_k = 2^{4-k}\varepsilon^{-1}\|\xi - \bar{\xi}\|$ $(k = 1, 2, \ldots)$, using (18) and (26) and applying Theorem 4.1, we can find a projection $p \in \operatorname{Proj} M$ with $\Phi(p^\perp) < \varepsilon$ and such that

$$\|ps_n(|z_k|^2)p\|_\infty \leqslant \varepsilon_k, \; n, k = 1, 2, \ldots, \qquad \|p|\bar{t}_k|^2 p\|_\infty \to 0, \; k \to \infty.$$

Since the series $\sum_{k=1}^\infty \|s_n(z_k)p\|_\infty$ is also uniformly convergent, by (13), we obtain

$$\|\bar{v}_r p\|_\infty = \Big\|\sum_{k=1}^{k(r)} s_{n_r}(z_k)p\Big\|_\infty \to 0 \quad \text{as } r \to \infty.$$

Thus, we get $\|\bar{y}_r p\|_\infty \leqslant \|\bar{v}_r p\|_\infty + \|\bar{t}_t p\|_\infty \to 0$ as $n \to \infty$, which completes the proof. ■

7. Proof of Theorem 3.3. Clearly, $\sum_{j=1}^\infty \xi_j$ is convergent in H. Exactly like in the classical case ([11], [8], cf. [1]) we have

$$\sum_{k=0}^\infty \|\sigma_{2^k} - \sigma\|^2 = \sum_{k=0}^\infty \sum_{j=2^k+1}^\infty \|\xi_j\|^2 < \infty. \tag{27}$$

Then there exists a sequence of indices $(m(k))_{k=1}^\infty$ with $m(k) > 2^{k+1}$, $k = 1, 2, \ldots$, such that

$$\sum_{k=1}^\infty 2^k \|\sigma_{m(k)} - \sigma\|^2 < \infty. \tag{28}$$

Define the sequence $(k(n))_{n=1}^\infty$ of indices by putting $k(n) = s$ for $2^s < n \leqslant 2^{s+1}$ $(n = 1, 2, \ldots)$. Applying Lemma 4.3 we can find a sequence (ε_i) such that for all $(x_i) \subset M$ (with $x_i = 0$ when $\xi_i = 0$) the inequalities $\|\xi_i - x_i\Omega\| < \varepsilon_i$ $(i = 1, 2, \ldots)$ imply (9) and (10).

Next, we use Lemma 4.4 by taking the sequence (ε_i) just found and as J the set $I_k = \{2^k+1, \ldots, 2^{k+1}\}$. Then there exist operators $x_i \in M$ $(i = 1, 2, \ldots)$ and $D_k \in M^+$ $(k = 0, 1, 2, \ldots)$ such that $\|\xi_i - x_i\Omega\| < \varepsilon_i$ for all i and

$$|s_n - s_{2^k}|^2 \leqslant D_k \quad \text{for } 2^k < n \leqslant 2^{k+1}, \; k = 0, 1, 2, \ldots, \tag{29}$$

where

$$s_n = \sum_{j=1}^n x_j \quad (n = 1, 2, \ldots)$$

and

$$\Phi(D_k) \leqslant 2(k+1)^2 \sum_{j=2^k+1}^{2^{k+1}} \|\xi_i\|^2. \tag{30}$$

Obviously, we get immediately

$$\sum_{n=1}^\infty \|\sigma_n - s_n\Omega - \sigma_{m(k(n))} + s_{m(k(n))}\Omega\|^2 < \infty \tag{31}$$

and

$$\sum_{i,j=2^k+1}^{m(k)} |\Phi(x_i^* x_j)| \leqslant 2 \sum_{j=2^k+1}^{m(k)} \|\xi_j\|^2, \quad k = 0, 1, 2, \ldots \tag{32}$$

By (30) and the assumption (8) we obtain

$$\sum_{k=0}^{\infty} \Phi(D_k) < \infty. \tag{33}$$

Let us put for $k = 0, 1, 2, \ldots$

$$E_k = |s_{m(k)} - s_{2^k}|^2 = \Big|\sum_{j=2^k+1}^{m(k)} x_j\Big|^2. \tag{34}$$

Then, by (32), we have

$$\Phi(E_k) = \sum_{i,j=2^k+1}^{m(k)} \Phi(x_i^* x_j) \leqslant 2 \sum_{j=2^k+1}^{\infty} \|\xi_j\|^2.$$

Thus, by (27), we get

$$\sum_{k=0}^{\infty} \Phi(E_k) < \infty. \tag{35}$$

Now, we take an arbitrary sequence $(y_n) \subset M$ such that

$$\sum_{n=1}^{\infty} \|(\sigma_n - \sigma) - y_n \Omega\|^2 < \infty. \tag{36}$$

The proof will be completed if we show that $y_n \to 0$ a.u. in M.
Let us put for $n = 1, 2, \ldots$

$$F_n = |y_n + (s_{m(k(n))} - s_n)|^2. \tag{37}$$

Then we have

$$\begin{aligned}
\sum_n \Phi(F_n) &= \sum_{k=0}^{\infty} \sum_{n=2^k+1}^{2^{k+1}} \|y_n + (s_{m(k(n))} - s_n)\Omega\|^2 \\
&\leqslant 3 \sum_{k=0}^{\infty} \sum_{n=2^k+1}^{2^{k+1}} \big(\|y_n \Omega - (\sigma_n - \sigma)\|^2 + \|\sigma_{m(k(n))} - \sigma\|^2 \\
&\quad + \|\sigma_n - s_n \Omega - \sigma_{m(k(n))} + s_{m(k(n))}\Omega\|^2\big) \\
&\equiv 3 \sum_{k=0}^{\infty} (\alpha_k + \beta_k + \gamma_k).
\end{aligned}$$

But, by (36),

$$\sum_{k=0}^{\infty} \alpha_k = \sum_n \|y_n \Omega - (\sigma_n - \sigma)\|^2 < \infty,$$

further, by (28),

$$\sum_{k=0}^{\infty} \beta_k = \sum_{k=0}^{\infty} \sum_{n=2^k+1}^{2^{k+1}} \|\sigma_{m(k(n))} - \sigma\|^2 = \sum_{k=0}^{\infty} 2^k \|\sigma_{m(k)} - \sigma\|^2 < \infty,$$

and, at last, by (31),

$$\sum_{k=0}^{\infty} \gamma_k = \sum_{n} \|\sigma_n - s_n \Omega - \sigma_{m(k(n))} + s_{m(k(n))} \Omega\|^2 < \infty.$$

Thus, we obtain

$$\sum_{n=1}^{\infty} \Phi(F_n) < \infty. \tag{38}$$

Applying Proposition 4.2 to the sequence $(D_1, E_1, F_1, D_2, E_2, F_2, \ldots)$, by (33), (35) and (38) for each $\varepsilon > 0$ we can find a $p \in \operatorname{Proj} M$ with $\Phi(p^{\perp}) < \varepsilon$ such that

$$\|pD_k p\|_{\infty} \to 0, \quad \|pE_k p\|_{\infty} \to 0, \quad \|pF_n p\|_{\infty} \to 0, \quad k \to \infty. \tag{39}$$

To conclude the proof we remark that, by (37), (34) and (29), we have

$$\begin{aligned} \|y_n p\|_{\infty} &\leqslant \|(y_n + s_{m(k(n))} - s_n) p\|_{\infty} \\ &\quad + \|(s_{m(k(n))} - s_{2^{k(n)}}) p\|_{\infty} + \|(s_n - s_{2^{k(n)}}) p\|_{\infty} \\ &= \|pF_n p\|_{\infty}^{1/2} + \|pE_{k(n)} p\|_{\infty}^{1/2} + \|pD_{k(n)} p\|_{\infty}^{1/2}, \end{aligned}$$

which, with (39) and $\Phi(p^{\perp}) < \varepsilon$, gives $y_n \to 0$, a.u. in M and completes the proof. ∎

REFERENCES

[1] G. Alexits, *Convergence Problems of Orthogonal Series*, Pergamon Press, New York–Oxford–Paris 1961.

[2] N. Dang-Ngoc, *Pointwise convergence of martingales in von Neumann algebras*, Israel J. Math. 34 (1979), pp. 273–280.

[3] M. S. Goldstein, *Theorems in almost everywhere convergence in von Neumann algebras* (in Russian), J. Operator Theory 6 (1981), pp. 233–311.

[4] E. Hensz and R. Jajte, *Pointwise convergence theorems in L_2 over a von Neumann algebra*, Math. Z. 193 (1986), pp. 413–429.

[5] R. Jajte, *Strong limit theorems in noncommutative probability*, Lecture Notes in Math. 1110, Springer, Berlin–Heidelberg–New York 1985.

[6] – *Strong limit theorems in noncommutative L_2-spaces*, ibidem 1477, Springer, Berlin–Heidelberg–New York 1991.

[7] E. C. Lance, *Ergodic theorem for convex sets and operator algebras*, Invent. Math. 37 (1976), pp. 201–214.

[8] D. Menchoff, *Sur les séries de fonctions orthogonales*, Fund. Math. 4 (1923), pp. 82–105.

[9] J. Neveu, *Deux remarques sur la théorie des martingales*, Z. Wahrsch. verw. Gebiete (1964), pp. 122–127.

[10] A. Paszkiewicz, *Convergence in W*-algebras*, J. Funct. Anal. 69 (1986), pp. 143–154.

[11] H. Rademacher, *Einige Sätze über Reihen von allgemeinen Orthogonalfunktionen*, Math. Ann. 87 (1922), pp. 112–138.
[12] Y. G. Sinai and V. V. Anshelevich, *Some problems of non-commutative ergodic theory*, Rússian Math. Surveys 31 (1976), pp. 156–174.
[13] M. Takesaki, *Theory of Operator Algebras. I*, Springer, Berlin–Heidelberg–New York 1979.

Institute of Mathematics, University of Łódź
ul. S. Banacha 22, 90-238 Łódź, Poland

Received on 5.10.1993