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INTERSECTIONS AND SHIFT FUNCTIONS OF STRONG MARKOV RANDOM CLOSED SETS

BY

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Abstract. If M and M_1 are independent subsets of the positive half-line, then the function $\chi(t) = \boldsymbol{P}\{M \cap (M_1+t) = \emptyset\}$ is said to be a *shift function* of M with respect to M_1. In the paper both sets M and M_1 are supposed to be strong Markov (or regenerative). It is shown that the shift function is a harmonic function with respect to the kernel determined by the transition probabilities of the corresponding semi-linear forward recurrence processes. Conditions for the uniqueness of such a harmonic function with given boundary values are presented.

1. Introduction. A *random closed set* is a random element with values in the family of all closed subsets of a given space (see [6], [15]). This immediately suggests the study of typical set-theoretic operations (union, Minkowski addition, intersection, convex hull, etc.) in relation to distribution of random closed sets. Whereas unions, convex hulls and Minkowski sums of random closed sets have been already systematically investigated (cf. [1], [18], [22]), the study of other operations (intersection and some morphological operations [3], [19]) causes difficulties. Mostly, it is impossible to calculate the distribution of the result in terms of the distributions of the components.

In this paper* we shall consider *intersections* of random closed subsets of the positive half-line. Even in this (relatively simple) case it is impossible to express the distribution of the intersection of two independent random sets M and M_1 by the distributions of M and M_1. In general, it is very difficult to evaluate even the probability that two independent identically distributed random sets have a non-empty intersection. However, sometimes it is important to know such probabilities, since they can, e.g., be interpreted as simultaneous failure times of several devices.

The probability $\boldsymbol{P}\{M \cap M_1 \neq \emptyset\}$ is, evidently, equal to the probability that the set $M \oplus \check{M}_1 = \{x-y\colon x \in M,\ y \in M_1\}$ contains the origin. For a deter-

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ministic set F, it is known that its central symmetrization $F \oplus \check{F}$ contains a certain neighborhood of the origin if F has positive Lebesgue measure. A random analogue of this fact should deal with the distribution of the set $M \oplus \check{M}_1$. The simplest characteristic of a random set distribution is its covering probability, i.e., the probability that a point belongs to the set in question. In our case the event $\{t \in M \oplus \check{M}_1\}$ is equivalent to the existence of two points at the distance t: one from M and the other one from M_1. Thus

$$\boldsymbol{P}\{t \in M \oplus \check{M}_1\} = \boldsymbol{P}\{M \cap M_1 + t \neq \emptyset\}.$$

In the present paper these probabilities are derived for so-called *strong Markov* random sets. These sets arise as levels (homecomings) of strong Markov random processes (see [4], [10], [11]). The starting point of this research was Feller's work [2], where so-called *recurrent events* were investigated. Extending this notion to the continuous case, Kingman [7] introduced the *regenerative phenomenon* and examined its structure in different cases (see [8], [10]). Kendall [5] considered regenerative phenomena from the point of view of random sets theory.

A series of restrictions, inherent in the theory of regenerative phenomena, is dropped in the theory of *strong Markov sets*. This theory is in turn imbedded in the general theory of regenerative systems (see [13]). Krylov and Jushkevich [11] introduced the notion of Markov random sets, which later was extensively investigated under the name of regenerative sets or strong Markov sets (see, e.g., [4], [17], [12]–[14], [21]). We use here the term "strong Markov set" to stress the *strong* Makov property of the corresponding sets.

The paper is organized as follows. Section 2 presents some notation and definitions. In Section 3 it is shown that the function $\chi(t) = \boldsymbol{P}\{M \cap M_1 + t \neq \emptyset\}$ (the so-called *shift function*) is a harmonic function with respect to the kernel given by transition probabilities of the forward semilinear process generated by M. Section 4 presents uniqueness conditions of such a harmonic function with given boundary values. If the corresponding integral equation admits many solutions, then the shift function can be found as the pointwise limit of unique solutions of appropriately modified equations. Shift functions for truncated random sets are considered in Section 5. In Section 6 a method of computation of the distribution of the random set $M \cap M_1$ is proposed.

2. Definition of strong Markov random sets. Let us recall several definitions and notation. Some of them are inspired by the theory of random closed sets [15]; others come from the theory of regenerative phenomena and Markov sets (see [13], [17]).

Let $\boldsymbol{R}_+$ be the set of non-negative real numbers, and let $\bar{\boldsymbol{R}}_+ = \boldsymbol{R}_+ \cup \{\infty\}$ be the compactified half-line. Furthermore, $\mathscr{B}$ (resp. $\mathscr{F}$) denotes the family of all Borel (resp. closed) subsets of $\bar{\boldsymbol{R}}_+$.

Let σ_f be the σ-algebra generated by the families $\{F \in \mathscr{F}\colon F \cap K \neq \emptyset\}$, where K runs through the class $\mathscr{K}$ of compacts in $\boldsymbol{R}_+$ (see [15]). A probability measure $\boldsymbol{P}$ on σ_f determines the distribution of the corresponding *random closed set* M. It follows from the Choquet theorem [15] that $\boldsymbol{P}$ is uniquely determined by the corresponding *capacity functional* $T(K) = \boldsymbol{P}\{M \cap K \neq \emptyset\}$, where K runs through $\mathscr{K}$.

For every F from $\mathscr{F}$ and $t \geqslant 0$ let us define the first point of F after t as

$$z_t^+(F) = \inf\{s \geqslant t\colon\ s \in F\},$$

and the forward semilinear process as $x_t^+(F) = z_t^+(F) - t$. If F is a closed set, then the values of either z_t^+ or x_t^+ for all $t \geqslant 0$ determine F. Furthermore, we use the notation $F|_t = F \cap [0, t]$, $F|^t = F \cap [t, \infty)$ for truncations and $F + t = \{t + x\colon\ x \in F\}$ for a shift of F.

A point x from F is said to *belong* to F' if x is an isolated point of F or x is a limit of a strong decreasing sequence of points of F. Thus, F' is the set of isolated or right-limit points of F.

A measurable map M of a complete probability space $(\Omega, \boldsymbol{F}, \boldsymbol{P})$ into $(\mathscr{F}, \sigma_f)$ is said to be a *random closed subset* of $\boldsymbol{R}_+$. Let $\boldsymbol{F}_t$, $t \geqslant 0$, be the $\boldsymbol{P}$-completion of the minimal σ-algebra, generated by the truncated random set $M|_t$. In other words, $\boldsymbol{F}_t$ is the minimal σ-algebra which contains all sets $\{\omega \in \Omega\colon\ M|_t(\omega) \cap K \neq \emptyset\}$ for K running through $\mathscr{K}$.

DEFINITION 2.1 (see [17]). A random closed subset M of $\boldsymbol{R}_+$, such that $0 \in M$ a.s., is said to be *strong Markov* if for every $(\boldsymbol{F}_t)$-stopping time τ belonging to M' a.s. on $\{\tau < \infty\}$ and for every K_1, K_2 from $\mathscr{K}$ the following conditions are valid:

(A1) The events $\{M|^\tau - \tau \cap K_1 \neq \emptyset\}$ and $\{M|_\tau \cap K_2 \neq \emptyset\}$ are independent under the condition $\{\tau < \infty\}$.

(A2) $\boldsymbol{P}\{M|^\tau - \tau \cap K_1 \neq \emptyset \mid \tau < \infty\} = \boldsymbol{P}\{M \cap K_1 \neq \emptyset\}$.

It is easy to verify that M is a strong Markov set if and only if the corresponding set M' is a regenerative set considered in [4] and [13].

An important example of a strong Markov set is the closure of a level set of a right-continuous strong Markov process (see [10], [4]). On the other hand, M is a strong Markov set if and only if M coincides with the closure of the range $\{\zeta_t\colon\ t \geqslant 0\}$ of a process ζ_t with independent increments and increasing trajectories (*subordinator*); see [12], [13]. The distribution of $\{\zeta_t\colon\ t \geqslant 0\}$ is determined by the corresponding *cumulant*

$$k(\theta) = -t^{-1} \log \boldsymbol{E} \exp\{-\theta\zeta_t\} = \varepsilon\theta + \int_0^\infty (1 - e^{-\theta x})\,\mu(dx) + \lambda, \tag{2.1}$$

where $\varepsilon \geqslant 0$ is a *shift coefficient*, μ is a measure on $(0, \infty]$ called the *Lévy measure*, and $\lambda = \mu(\{\infty\}) \geqslant 0$. The Lévy measure μ satisfies the condition

$$\int_0^\infty \min(x, 1)\,\mu(dx) < \infty.$$

(Hereafter integrals from 0 to ∞ are supposed to be taken over the domain $(0, \infty)$.) Sometimes $k(\theta)$ is said to be also the *cumulant* of M.

The following classification of strong Markov sets follows [9] and [10]. It is based on the properties of the main parameters ε, μ, λ of the cumulant (2.1).

1. Standard and light sets: classification based on the value of ε.

• If $\varepsilon > 0$, then M is said to be *standard*. Then the function $p(t) = \mathbf{P}\{t \in M\}$, $t \geqslant 0$, satisfies the condition $p(t) \to 1$ for $t \to 0$. This function p is called the *p-function* of M. It determines uniquely the distribution of a standard strong Markov set through the corresponding cumulant, namely

$$\int_0^\infty p(t) e^{-\theta t} dt = [k(\theta)]^{-1}.$$

• If $\varepsilon = 0$, then M is a so-called *light* set, i.e., $p(t) = 0$ almost everywhere with respect to the Lebesgue measure. In this case the Lebesgue measure of M is equal to zero with probability one. A typical example is the set of zeros for the Wiener process.

2. Recurrent and transient sets: classification based on the value of $\mu(\mathbf{R}_+)$.

• If $\mu(\mathbf{R}_+)$ is finite, then the set M is called *recurrent*. If, additionally, $\varepsilon > 0$, then M is the union of non-overlapping exponentially distributed segments. In other words, M is given by an alternating renewal process with exponentially distributed 1-phase. If $\varepsilon = 0$, then M is a renewal point process and M is said to be *discrete*.

• If $\mu(\mathbf{R}_+) = \infty$, then the set M is called *transient*.

3. Unbounded and bounded sets: classification based on the value of λ.

• If $\lambda = 0$, then the random set M is *unbounded*, i.e., $\sup M = \infty$ almost surely.

• If $\lambda > 0$, then $\sup M < \infty$ a.s. In this case M is said to be *bounded*.

Kingman [10] proved that $\{x_t^+ : t \geqslant 0\}$ is a strong Markov homogeneous process and its transition probabilities are uniquely determined by the family of probability distributions

$$P_t(A) = \mathbf{P}\{x_t^+ \in A\}, \quad t \geqslant 0, \ A \in \mathscr{B}. \tag{2.2}$$

Let us put $P_t(\{\infty\}) = \pi_t$. It follows from [10] that

$$\int_0^\infty \int_0^\infty P_t(dy) e^{-\theta t - \alpha y} dt = \frac{k(\theta) - k(\alpha)}{(\theta - \alpha) k(\theta)}, \tag{2.3}$$

and

$$\int_0^\infty \pi_t e^{-\theta t} = \frac{\lambda}{\theta k(\theta)}. \tag{2.4}$$

3. Main integral equations for shift functions. Let us formulate our main definition.

DEFINITION 3.1. Let M, M_1 be independent strong Markov sets. The function

$$\chi(t) = \boldsymbol{P}\{M \cap (M_1+t) = \emptyset\}, \qquad t \geqslant 0,$$

is called the *shift function* of the random closed set M_1 with respect to M. The dual function $\chi_1(t) = \boldsymbol{P}\{M_1 \cap (M+t) = \emptyset\}$ is defined similarly, If M and M_1 are identically distributed, then χ is called the *shift function of* M.

It is easy to show that the event $\{M \cap (M_1+t) = \emptyset\}$ belongs to the basic σ-algebra σ_f, so that the definition is correct. Evidently, $\chi(0) = \chi_1(0) = 0$.

Let us give another interpretation of the shift function. For closed sets F and F_1, their Minkowski sum is denoted by

$$F \oplus F_1 = \{s+t\colon\ s \in F,\ t \in F_1\}.$$

Furthermore, $\check{F} = \{-t\colon\ t \in F\}$.

LEMMA 3.2. *For every* $t \in \boldsymbol{R}_+$ *we have*

$$\chi(t) = \boldsymbol{P}\{M \cap (M_1+t) = \emptyset\} = \boldsymbol{P}\{t \notin M \oplus \check{M}_1\}.$$

By the way, Lemma 3.2 yields the Borel measurability of χ.

THEOREM 3.3. *The shift functions* χ *and* χ_1 *satisfy the following system of integral equations*:

$$\chi(t) = \int_0^\infty \chi_1(u)\, P_t(du) + P_t(\{\infty\}), \tag{3.1}$$

$$\chi_1(t) = \int_0^\infty \chi(u)\, G_t(du) + G_t(\{\infty\}),$$

where $P_t(\cdot)$ *and* $G_t(\cdot)$ *are distributions of the random variables* $x_t^+(M)$ *and* $x_t^+(M_1)$, *respectively.*

Proof. For every non-negative t we put $D_t = \inf\{s > t\colon\ s \in M\}$. This random variable is an $(\boldsymbol{F}_t)$-stopping time and the values of D_t lie in $M' \cup \{\infty\}$ almost surely. Hence (A1) and (A2) are valid for $\tau = D_t$. Furthermore, $D_t = z_t^+$ on $\{x_t^+ > 0\}$, whence for every compact K and a non-negative t

$$\boldsymbol{P}\{M|^{z_t^+} - z_t^+ \cap K \neq \emptyset | x_t^+\} = \boldsymbol{P}\{M \cap K \neq \emptyset\}$$

a.s. on $\{0 < x_t^+ < \infty\}$, and also

$$\begin{aligned}\chi(t) &= \boldsymbol{E}[\boldsymbol{P}\{M \cap (M_1+t) = \emptyset | x_t^+\}] \\ &= \boldsymbol{E}[1_{0<x_t^+<\infty} \boldsymbol{P}\{M \cap (M_1+t) = \emptyset | x_t^+\}] + \boldsymbol{P}\{x_t^+ = \infty\}\end{aligned}$$

$$= \boldsymbol{E}[1_{0<x_t^+<\infty} \boldsymbol{P}\{(M|^{z_t^+} - z_t^+) \cap (M_1 - x_t^+) = \emptyset | x_t^+\}] + \boldsymbol{P}\{x_t^+ = \infty\}$$

$$= \int_0^\infty P_t(du) \boldsymbol{P}\{M \cap (M_1 - u) = \emptyset\} + \boldsymbol{P}\{x_t^+ = \infty\}.$$

The second equation of (3.1) is derived similarly. ∎

COROLLARY 3.4. *The shift function of M satisfies the equation*

$$\chi(t) = \int_0^\infty \chi(u) P_t(du) + \pi_t. \tag{3.2}$$

Thus, the shift function χ is a *harmonic function* on the extended half-line $\bar{\boldsymbol{R}}_+$ with respect to the kernel $P_t(\cdot)$, satisfying the boundary conditions $\chi(0) = 0$ and $\chi(\infty) = 1$. A solution of (3.2) or (3.1) always exists if the kernels $P_t(\cdot)$ and $G_t(\cdot)$ correspond to strong Markov sets.

If otherwise is not stated, we consider *identically distributed* random sets M and M_1.

4. Uniqueness of the solutions of the main integral equation. In general, the equation (3.2) may have infinitely many solutions. For example, if $M = \{t \geqslant 0: w_t = 0\}$ is the set of zeros for the Wiener process w_t, then (3.2) takes the form

$$\chi(t) = \int_0^\infty \frac{\sqrt{t}}{\pi\sqrt{u}(u+t)} \chi(u)\, du.$$

It is easily seen that each constant function is a solution of this equation.

In the sequel, conditions for uniqueness of solutions of (3.2) will be given. It will be also shown that in the case of non-uniqueness the integral equation (3.2) can be modified to have a unique solution and the corresponding shift function is equal to the pointwise limit of solutions of these modified equations.

THEOREM 4.1. *Let M be a strong Markov set satisfying one of the following conditions*:

(B1) *M is standard and bounded, i.e., $\varepsilon > 0$ and $\lambda > 0$ in* (2.1).

(B2) *M is discrete and bounded, i.e., $\varepsilon = 0$, $\lambda > 0$ and $\mu(\boldsymbol{R}_+) < \infty$.*

(B3) *M is standard, i.e., $\varepsilon > 0$, and also $\int_0^\infty x\mu(dx) < \infty$.*

Then (3.2) *admits a unique bounded Borel solution.*

We begin with the following lemma:

LEMMA 4.2. *Let M be a bounded strong Markov set. Then the function $\pi_t = P_t(\{\infty\})$ is non-decreasing and strictly positive for each $t > 0$. If M is a.s. discrete, then also $\pi_{0+} = \lim_{t \downarrow 0} \pi_t > 0$.*

Proof. The relation $\pi_{0+} > 0$ for a discrete M and the monotonicity of π_t are obvious.

Let $\pi_{t_1} = 0$ for some $t_1 > 0$ and $t = \alpha t_1$, where $1/2 < \alpha < 1$. It follows from the monotonicity that $\pi_t = 0$, whence the stopping time $D_t = \inf\{s > t\colon s \in M\}$ is almost surely finite. By (A2), $M|^{D_t} - D_t$ and M have the same distribution. Thus,

$$0 = \pi_t = \boldsymbol{P}\{x_t^+(M|^{D_t} - D_t) = \infty\} = \boldsymbol{P}\{x_{t+D_t}^+(M|_t) = \infty\}$$
$$\geqslant \boldsymbol{P}\{x_{2t}^+(M) = \infty\} = \pi_{2t} = \pi_{(2\alpha)t_1}.$$

Similarly, $\pi_{(2\alpha)^n t_1} = 0$ for all $n \geqslant 1$. Hence π_t vanishes for all t, contrary to the conjecture $\sup M < \infty$ a.s. ∎

Proof of Theorem 4.1. Let $\gamma(t)$ be the difference between two bounded Borel solutions of (3.2). Then

$$\gamma(t) = \int_0^\infty \gamma(u) P_t(du) \leqslant \sup_{0<s<\infty} |\gamma(s)| \sup_{0<t<\infty} \big(1 - p(t) - \pi_t\big).$$

If M satisfies (B1), then $p(t) \to 1$ as $t \downarrow 0$, and

$$\sup_{0<t<\infty} \big(1 - p(t) - \pi_t\big) < 1 \tag{4.1}$$

by Lemma 4.2. If M satisfies (B2), then (4.1) again follows from Lemma 4.2. Furthermore, (B3) yields $\lim_{t\to\infty} p(t) > 0$, $\lim_{t\downarrow 0} p(t) = 1$, and also $p(t) > 0$ for all $t > 0$. Moreover, in this case the function p is continuous (see [7]). Therefore, (4.1) is also valid. Thus, in any case,

$$\gamma(t) \leqslant \sup_{0<s<\infty} |\gamma(s)|\, \theta$$

for $0 < \theta < 1$, whence $\gamma(t)$ is identically equal to zero. ∎

Now consider the case where (3.2) admits many solutions. The further study is based on the following lemma:

LEMMA 4.3. *For every random compact set M, the shift function χ of M can be found by*

$$\chi(t) = \lim_{a\downarrow 0} \chi_a(t), \qquad 0 < t < \infty, \tag{4.2}$$

where χ_a is the shift function of the set $M(a) = M \oplus [0, a]$.

Proof. Let $T(K) = \boldsymbol{P}\{\tilde{M} \cap K \neq \emptyset\}$ be the capacity functional of the random closed set $\tilde{M} = M \oplus \check{M}_1$, where M_1 is an independent copy of M. It follows from Lemma 3.2 that $\chi(t) = T(\{t\})$. It is known [15], [20] that T is upper semicontinuous on $\mathscr{K}$. Hence

$$\chi(t) = \lim_{a\downarrow 0} (1 - \boldsymbol{P}\{[t-a, t+a] \cap \tilde{M} \neq \emptyset\}).$$

It is easy to show that

$$\{[t-a, t+a] \cap M \neq \emptyset\} = \{M(a) \cap (M_1(a) + t) \neq \emptyset\},$$

whence (4.2) easily follows. It should be noted also that the convergence in (4.2) is monotone. ■

The value of $\chi_a(t)$ is the probability that M and M_1+t have two points at a distance less than a. Roughly speaking, χ_a is the shift function in the case where time can be measured with the error a.

THEOREM 4.4. *The shift function of a bounded strong Markov set M can be found by* (4.2), *where $\chi_a(t)$, $t>0$, is the unique bounded Borel solution of the integral equation*

$$\chi_a(t) = \int_{(a,\infty)} \chi_a(v-a)\,P_{t-a}(dv)+\pi_{t-a}, \qquad t \geqslant a, \tag{4.3}$$

and $\chi_a(t)=0$ for $t<a$.

Proof. Let $x_t^- = t-\sup\{s \leqslant t\colon s\in M\}$ be the backward semilinear process associated with M. For every $t, a \geqslant 0$ and $A \in \mathscr{B}$ define

$$P_t^a(A) = \boldsymbol{P}\{x_t^+\in A,\ x_t^- > a\} \qquad \text{and} \qquad \pi_t^a = P_t^a(\{\infty\}).$$

Then

$$\boldsymbol{P}\{M(a)\cap(M_1(a)+t)=\varnothing\} = \boldsymbol{P}\{M(a)\cap(M_1(a)+t)=\varnothing,\ x_t^- > a\}$$

and also $P_t^a(A) = P_{t-a}(A-a)$, $\pi_t^a = \pi_{t-a}$. Here M_1 is an independent copy of M.

Similarly to the proof of Theorem 3.3, we obtain (4.3) for the function

$$\chi_a(t) = \boldsymbol{P}\{M(a)\cap(M_1(a)+t)=\varnothing\}.$$

If $\gamma(t)$ is the difference between two bounded Borel solutions of (4.3) and $\Gamma = \sup_{0<s<\infty}|\gamma(s)|$, then

$$\Gamma \leqslant \Gamma\left(1-\boldsymbol{P}\{x_t^- \leqslant a\}-\pi_t^a\right) = \Gamma\varkappa(t),$$

where $\varkappa(t) = 1-T([t-a,\ t])-\pi_t^a$ and T is the capacity functional of M.

Lemma 4.2 yields $\sup_{t>a+\varepsilon}\varkappa(t)<1$ for each $\varepsilon>0$. Consider a sequence $t_n\downarrow a$ as $n\to\infty$. It follows from [13] that a strong Markov set is either a.s. perfect (contains no isolated points) or a.s. discrete (contains only isolated points). If M is a.s. perfect, then

$$\varkappa(t_n) \leqslant \boldsymbol{P}\{[t_n-a,\ t_n]\cap M=\varnothing\}\to 0 \qquad \text{as } n\to\infty.$$

If M is a.s. discrete, then $\varkappa(t_n)\leqslant 1-\pi_{0+}<1$ by Lemma 4.2. Thus, $\sup_{0<t<\infty}\varkappa(t)<1$, since $\varkappa(t)=0$ for $t<a$. Therefore, $\Gamma=0$, which implies that $\gamma(t)$ vanishes. ■

COROLLARY 4.5. *The shift function of the bounded strong Markov set M is the minimum positive solution of the equation* (3.2).

Proof. Let us show that each positive solution of (3.2) is greater than the solution of (4.3). Let $\gamma(t) = \chi(t)-\chi_a(t)$. Hence

$$\gamma(t) - \int_{(a,\infty)} \gamma(v-a)\,P_{t-a}(dv) = \varepsilon(t),$$

where

$$\varepsilon(t) = \int_{(a,\infty)} \chi(v)\, P_t(dv) - \int_{(a,\infty)} \chi(v-a)\, P_{t-a}(dv) + \pi_t - \pi_t^a.$$

It is easy to show that $\varepsilon(t)$ is a non-negative function. Then $\gamma(t) \geqslant 0$, since, for every $t \geqslant 0$,

$$\Big| \int_{(a,\infty)} \gamma(v-a)\, P_{t-a}(dv) \Big| \leqslant |\gamma(t)|. \quad ■$$

Let us suppose that the strong Markov set M is *unbounded*, that is $\lambda = 0$ in (2.1). It follows from Theorem 4.1 that the function $\chi(t)$ is identically equal to zero if $\varepsilon > 0$ and $\int_0^\infty x\mu(dx)$ is finite. If $\varepsilon = 0$ and $\mu(\boldsymbol{R}_+) < \infty$, then M is discrete and χ depends on atoms of the distribution $\mu(\cdot)/\mu(\boldsymbol{R}_+)$. If the latter is absolutely continuous, then $\chi(t) = 1$ for all $t > 0$.

EXAMPLE 4.6. If $\varepsilon = 0$, μ is concentrated at $\{1\}$ and $\lambda = 0$, then M is the (non-random) set of all non-negative integers (M is also strong Markov in this case), and $\chi(t)$ is equal to 0 if t is integer and to 1 otherwise.

In the sequel, a general method for the evaluation of shift functions of unbounded sets is proposed. For this, M will be replaced by its truncation.

THEOREM 4.7. *Let M be an unbounded strong Markov set with cumulant $k(\theta)$, and let χ be the shift function of M. Then χ is given by*

$$\chi(t) = \lim_{\lambda \downarrow 0} \lim_{a \downarrow 0} \chi_a^\lambda(t), \tag{4.4}$$

where $a, \lambda > 0$ and χ_a^λ is the unique bounded solution of the integral equation

$$\chi_a^\lambda(t) = \int_{(a,\infty)} \chi_a^\lambda(v-a)\, P_{t-a}(dv;\, \lambda) + \pi_{t-a}(\lambda). \tag{4.5}$$

The measure $P_t(\cdot;\, \lambda)$ and the function $\pi_t(\lambda)$ are determined by their Laplace transforms

$$\int_0^\infty P_t(A;\, \lambda)\, e^{-\theta t}\, dt = \frac{k(\theta)}{k(\theta)+\lambda} \int_0^\infty P_t(A)\, e^{-\theta t}\, dt, \qquad A \in \mathscr{B}, \tag{4.6}$$

$$\int_0^\infty \pi_t(\lambda)\, e^{-\theta t}\, dt = \frac{\lambda}{\theta\big(k(\theta)+\lambda\big)}. \tag{4.7}$$

Proof. Let ζ_t be a subordinator of M and let ξ be a random variable uniformly distributed on [0, 1] and independent of ζ_t. Put $\tau(\lambda) = \lambda^{-1} \log(\lambda/\xi)$. For every $\lambda > 0$ define $M^{(\lambda)}$ to be the closure of the range of the following subordinator:

$$\zeta_t^{\tau(\lambda)} = \begin{cases} \zeta_t, & t < \tau(\lambda), \\ +\infty, & \text{otherwise}. \end{cases} \tag{4.8}$$

Let χ^λ be the shift function of the random set $M^{(\lambda)}$. It is easy to show that $\chi(t) = \lim_{\lambda\downarrow 0}\chi^\lambda(t)$. The function χ^λ can be found by means of Theorem 4.4. Therefore, (4.5) follows from (4.3) and the equations (4.6) and (4.7) follow from (4.8), (2.3) and (2.4), since the cumulant of $\zeta_t^{\tau(\lambda)}$ is $k(\theta)+\lambda$. ∎

Remark. If the unbounded strong Markov set M is either standard or discrete, then we can put $a = 0$ in (4.5).

COROLLARY 4.8. *Let M be a strong Markov set and let $\chi_a(\cdot)$ be the shift function of the set $M(a) = M \oplus [0, a]$. Then $\chi_a(\cdot)$ is the maximum solution of* (4.3).

Proof. Let $\gamma(t)$ be the difference $\chi_a^\lambda(t) - x(t)$, where $x(t)$ is an arbitrary solution of (4.3). Then

$$\gamma(t) - \int_{(a,\infty)} \gamma(v-a)\, P_{t-a}(dv;\,\lambda) = \varepsilon(t),$$

where

$$\varepsilon(t) = \int_{(a,\infty)} x(v-a)\, P_{t-a}(dv;\,\lambda) - \int_{(a,\infty)} x(v-a)\, P_{t-a}(dv) + \pi_{t-a}(\lambda).$$

If $\xi = \sup M^{(\lambda)}$, then

$$P_t(A;\,\lambda) - P_a(A) = -\boldsymbol{P}\{x_t^+(M) \in A,\ \xi < t\}, \qquad A \in \mathscr{B},$$

and

$$\varepsilon(t) = -\int_{(a,\infty)} x(v-a)\, \boldsymbol{P}\{x_{t-a}^+ \in du,\ \xi < t-a\} + \pi_{t-a}(\lambda)$$
$$\geqslant -\boldsymbol{P}\{\xi < t-a\} + \pi_{t-a}(\lambda) = 0.$$

Thus, $\varepsilon(t)$ is positive, whence $\gamma(t) \geqslant 0$ for all positive t. ∎

In the course of evaluations, inverting the Laplace transforms (4.6) and (4.7) for different $\lambda > 0$ may cause difficulties. However, (4.5) can be modified to avoid this.

Let us consider the function

$$\bar{\gamma}_a^\lambda(\theta) = \theta \int_0^\infty e^{-\theta t} \chi_a^\lambda(t)\, dt.$$

The evaluation of the Laplace transform of both sides of (4.5) yields the following result:

THEOREM 4.9. *Let M be an unbounded strong Markov set. Assume that the Lévy measure μ has the density $f(\cdot)$, and that there exists a measure $\nu(\cdot)$ on $\boldsymbol{R}_+$ such that*

$$\int_0^\infty e^{-vu}\, \nu(dv) = f(u), \qquad u > 0.$$

Then, for every a, $\lambda \geqslant 0$, *the function* $\bar{\gamma}_a^\lambda(\theta)$ *is the unique bounded Borel solution of the equation*

$$(4.9) \qquad \bar{\gamma}_a^\lambda(\theta) = \frac{\theta e^{-\theta a}}{k(\theta)+\lambda} \int_0^\infty \frac{\nu(du)}{u(u+\theta)} e^{-ua} \bar{\gamma}_a^\lambda(u) + e^{-\theta a} \frac{\lambda}{k(\theta)+\lambda},$$

where $k(\theta)$ *is the cumulant of* M.

Assume that M and M_1 have different distributions. The following theorem can be obtained similarly to Theorem 4.1.

THEOREM 4.10. *If* M *or* M_1 *satisfies one of the conditions* (B1)–(B3), *then the system* (3.1) *has the unique bounded Borel solution* χ, χ_1.

If (3.1) has many solutions, then the shift function can be found in the same way as in Theorem 4.9.

Let us consider two examples of non-trivial shift functions.

EXAMPLE 4.11. Let the cumulant of M be equal to

$$(4.10) \qquad k(\theta) = \theta + \frac{\theta}{\theta+1} + \lambda,$$

i.e., in (2.1) we have $\mu(dx) = e^{-x}dx$, $\varepsilon = 1$ and $\lambda > 0$. In this case, M is an alternating bounded renewal process with both exponentially distributed phases. If the measure ν is concentrated at the point $\{1\}$ and $\nu(\{1\}) = 1$, then $\int_0^\infty e^{-xv}\nu(dv) = e^{-x}$ (the density of μ). It follows from Theorem 4.1 that in this case (4.9) has the unique solution (for $a = 0$) given by

$$\bar{\gamma}_0^\lambda(\theta) = \frac{\lambda}{\theta^2+(2+\lambda)\theta+\lambda}\left[\theta\frac{\lambda+2}{\lambda+1}+1\right].$$

For example, if $\lambda = 1$, then $\bar{\gamma}_0^1(\theta) = (3\theta/2+1)/(\theta^2+3\theta+1)$, whence

$$\chi_0^1(t) = 1-0.5e^{-0.38t}-0.5e^{-2.65t}.$$

EXAMPLE 4.12. Assume that M has the cumulant (4.10), and let M_1 be the set of zeros of the Wiener process. Then the Laplace transform of χ is given by

$$\int_0^\infty e^{-\theta t}\chi(t)\,dt = \theta^{-1}(\theta C(\lambda)+\lambda(\theta+1))/(\theta^2+(2+\lambda)\theta+\lambda),$$

where

$$C(\lambda) = \frac{\lambda\int_0^\infty \theta^{-1/2}(\theta^2+(2+\lambda)\theta+\lambda)^{-1}\,d\theta}{1-\int_0^\infty \sqrt{\theta}(\theta+1)^{-1}(\theta^2+(2+\lambda)\theta+\lambda)^{-1}\,d\theta}.$$

5. Truncated shift functions. Further we shall consider shift functions of truncated strong Markov sets. Recall that $M|_t$ means $M \cap [0, t]$ for all $t > 0$.

DEFINITION 5.1. The *truncated* shift function of a strong Markov set M with respect to M_1 is defined by

$$\chi(q; u) = \boldsymbol{P}\{M|_{q+u} \cap (M_1|_q + u) = \varnothing\}.$$

The function $\chi_1(q; u) = \boldsymbol{P}\{(M|_q + u) \cap M_1|_{q+u} = \varnothing\}$ is defined similarly. If M and M_1 have the same distribution, then the function $\chi(q; u)$ is said to be the *truncated* shift function of M.

It is easy to show that $\lim_{q\to\infty} \chi(q; u) = \chi(u)$, where χ is the shift function of M (non-truncated) and

$$\chi(0; u) = 1 - \boldsymbol{P}\{u \in M\}, \qquad \chi(q; 0) = 0. \tag{5.1}$$

The definition of truncated shift functions yields the following result:

LEMMA 5.2. *For every non-negative t, s and u, we have*

$$\boldsymbol{P}\{M|_s \cap (M_1|_t + u) = \varnothing\} = \chi(\min\{t, s-u\}; u),$$
$$\boldsymbol{P}\{(M|_t + u) \cap M_1|_s = \varnothing\} = \chi_1(\min\{t, s-u\}; u).$$

The following theorem is an analogue of Theorems 3.3 and 4.1 for truncated shift functions.

THEOREM 5.3. *If M and M_1 are strong Markov sets, then the corresponding truncated shift functions satisfy the following system of integral equations:*

$$\begin{aligned} \chi(q; u) &= \int_{(0,q)} P_u(dv)\,\chi_1(q-v; v) + \tilde{P}_u^q - P_u(\{q\})\,g(q), \\ \chi_1(q; u) &= \int_{(0,q)} G_u(dv)\,\chi(q-v; v) + \tilde{G}_u^q - G_u(\{q\})\,p(q), \end{aligned} \tag{5.2}$$

where $P_u(A) = \boldsymbol{P}\{x_u^+(M) \in A\}$ and $G_u(A) = \boldsymbol{P}\{x_u^+(M_1) \in A\}$ for $A \in \mathscr{B}$. Furthermore,

$$\tilde{P}_u^q = P_u([q, \infty]),\ \tilde{G}_u^q = G_u([q, \infty]) \ \textit{and} \ g(q) = \boldsymbol{P}\{q \in M_1\},\ p(q) = \boldsymbol{P}\{q \in M\}.$$

If one of the sets M, M_1 is either standard or discrete and for this set $\mu(a, \infty) + \lambda > 0$ for every $a > 0$, where μ and λ are the elements of the corresponding cumulant, then (5.2) *has the unique bounded Borel solution satisfying* (5.1).

Proof. It follows from Definition 5.1 and Lemma 5.2 that

$$\begin{aligned} \chi(q; u) &= \boldsymbol{E}[1_{x_t^+(M)\in(0,q)} \boldsymbol{P}\{(M|^{z_u^+} - z_u^+)|_{q+u-z_u^+} \cap (M_1|_q + u - z_u^+) = \varnothing \mid x_u^+(M)\}] \\ &\qquad + P_u(\{q\})(1 - g(q)) + P_u((q, \infty]) \\ &= \boldsymbol{E}[1_{0<x_u^+(M)<q} \boldsymbol{P}\{(M|_{q-x_u^+} + x_u^+) \cap M_1|_q = \varnothing \mid x_u^+(M)\}] + \tilde{P}_u^q - P_u(\{q\})\,g(q), \end{aligned}$$

whence the first equation of the system (5.2) is true. The second one can be obtained similarly.

Let $\gamma(q; u)$ and $\gamma_1(q; u)$ be the differences between two Borel solutions of (5.2) satisfying (5.1). For each $q \geqslant 0$ and $s < q$ let us put

$$\Gamma(s) = \sup_{0 \leqslant u \leqslant s} |\gamma(s; u)|, \quad \bar{\Gamma}(s) = \sup_{0 \leqslant s \leqslant q} \Gamma(s)$$

(respectively, $\Gamma_1(s)$ and $\bar{\Gamma}_1(s)$ for γ_1). Without loss of generality suppose that M satisfies the condition of the second (uniqueness) part of Theorem 5.3, that is, $\tilde{P}_u^q > 0$ for all positive u and a. Then

$$|\gamma_1(q; u)| \leqslant \bar{\Gamma}(q)\, G_u\big((0, q)\big) \leqslant \bar{\Gamma}(q)$$

and

$$|\gamma(q; u)| \leqslant \int_{(0,q)} \Gamma_1(q-v)\, P_u(dv) \leqslant \bar{\Gamma}_1(q)\, \big(1-p(u)-\tilde{P}_u^q\big) \leqslant \bar{\Gamma}(q)\big(1-p(u)-\tilde{P}_u^q\big).$$

If M is standard, then pick a point $u_0 > 0$ such that $p(t) > 1/2$ for all $t \leqslant u_0$. Hence

$$\sup_{u_0 < u < q} (1-\tilde{P}_u^q) = 1 - \inf_{u_0 < u < q} \boldsymbol{P}\{[u, u+q] \cap M = \emptyset\}$$
$$\leqslant 1 - \boldsymbol{P}\{[u_0, 2q) \cap M = \emptyset\} = 1 - \tilde{P}_{u_0}^{2q} = \theta(q) < 1.$$

Thus

$$\bar{\theta}(s) = \sup_{0 < q \leqslant s} \theta(q) \leqslant 1 - \tilde{P}_{u_0}^{2s} < 1$$

and $\bar{\Gamma}(q) \leqslant \bar{\Gamma}(q) \max\big(1/2, \theta(q)\big)$, whence $\bar{\Gamma}(q) = 0$.

If M is a.s. discrete, then

$$\sup_{0 < u \leqslant q} (1-\tilde{P}_u^q) \leqslant 1 - \boldsymbol{P}\{(0, 2q) \cap M = \emptyset\} = \theta_1(q) < 1$$

and $\bar{\Gamma}(q) \leqslant \bar{\Gamma}(q)\,\theta_1(q)$, whence $\bar{\Gamma}(q) = 0$ for all q. ■

If the solution of (5.2) is not unique, then the methods familiar from Section 4 are applicable, i.e. truncated shift functions can be found as pointwise limits of unique solutions of modified integral equations.

6. One Kingman's problem. If M_1 and M_2 are strong Markov sets, then their intersection $M = M_1 \cap M_2$ is also a strong Markov set. If both M_1 and M_2 are standard and $p_i(t) = \boldsymbol{P}\{t \in M_i\}$, $i = 1, 2$, are the corresponding p-functions, then M is also standard and $p(t) = \boldsymbol{P}\{t \in M\} = p_1(t)\, p_2(t)$. In this case the distribution of the intersection is completely determined by its p-function. Thus, properties of intersections of standard sets can be investigated through products of p-functions (see [9]).

Unfortunately, this approach does not work in the case where at least one of the sets M_1, M_2 is light (see [9], where the problem of developing methods for this case was posed).

Since the p-function does not serve any longer as the main characteristic of strong Markov sets in the light case, we turn to another characteristic – the cumulant of the corresponding subordinator. It determines the distribution of a strong Markov set uniquely. Furthermore, (2.3) yields a formula which relates this cumulant to the capacity functional of a strong Markov set M on the family of segments:

$$\hat{\phi}(\theta, \alpha) = \int_0^\infty \int_0^\infty e^{-\theta t - \alpha s} \phi(t, s)\, dt\, ds = \frac{\theta k(\alpha) - \alpha k(\theta)}{(\theta - \alpha)\theta\alpha k(\theta)}, \tag{6.1}$$

where $\phi(t, s) = \boldsymbol{P}\{[t, t+s) \cap M = \varnothing\}$. Thus, the cumulant of the subordinator can be found from (6.1) if the corresponding capacity functional is known.

LEMMA 6.1. *Up to a constant we have*

$$k(\alpha) = \frac{\hat{\phi}(1, \alpha)}{(1-\alpha)\alpha} + \alpha, \qquad \alpha \neq 1.$$

The evaluation of $\phi(t, s)$ can be reduced to the evaluation of the corresponding truncated shift functions χ_1 and χ_2. This can be done by the technique developed in Section 5.

THEOREM 6.2. *If M_1 and M_2 are strong Markov sets, χ_1, χ_2 are the corresponding truncated shift functions, $M = M_1 \cap M_2$, then, for all positive t and s,*

$$\begin{aligned}\phi(t, s) = &\int_{[0,s)} P_t^2(du_2) \int_{[0,u_2)} P_t^1(du_1)\chi_1(s-u_2, u_2-u_1) \\ &+ \int_{[0,s)} P_t^1(du_1) \int_{[0,u_1)} P_t^2(du_2)\chi_2(s-u_1, u_1-u_2) + \Phi_t^s,\end{aligned}$$

where $P_t^i(A) = \boldsymbol{P}\{x_t^+(M_i) \in A\}$, $i = 1, 2$, $A \in \mathscr{B}$, and

$$\begin{aligned}\Phi_t^s &= \boldsymbol{P}\{x_t^+(M_1) \geqslant s \text{ or } x_t^+(M_2) \geqslant s\} \\ &= P_t^1([s, \infty]) + P_t^2([s, \infty]) - P_t^1([s, \infty])P_t^2([s, \infty]).\end{aligned}$$

The theorem follows from the strong Markov property of x_t^+, Lemma 5.2 and the relation

$$\begin{aligned}&\boldsymbol{P}\{M_1 \cap M_2 \cap [t, t+s) = \varnothing\} \\ &\qquad = \Phi_t^s + \int_{[0,s)} \int_{[0,s)} \boldsymbol{P}\{(M_1 - u_1) \cap (M_2 - u_2) \\ &\qquad\qquad \cap [-u_1 - u_2, s - u_1 - u_2) = \varnothing\} P_t^1(du_1) P_t^2(du_2). \quad \blacksquare\end{aligned}$$

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