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A FUNCTIONAL CENTRAL LIMIT THEOREM UNDER UNIFORM MIXING ON A LOCALLY COMPACT ABELIAN GROUP

BY

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Abstract. A central limit theorem and a corresponding functional central limit theorem are given under a uniform mixing condition for uniformly infinitesimal triangular arrays of random variables which take values in a locally compact second countable Abelian group. The limiting distribution in the central limit theorem is Gaussian and the limiting distribution in the functional central limit theorem is the distribution of a Gaussian process with independent increments and continuous sample paths — a Wiener-type process.

1. Introduction. Throughout this article G will denote a locally compact second countable Abelian group, and $\hat{G}$ will denote the dual group of G. Thus, $\hat{G}$ is the set of continuous homomorphisms of G into the unit circle group T of complex numbers of modulus one, with topology induced from the complex plane and the group operation of complex multiplication. We endow $\hat{G}$ with the topology of uniform convergence on compact subsets and the natural group operation induced by the operation on T. Then $\hat{G}$ is, like G, a locally compact second countable Abelian group. We shall denote by $\langle x, y\rangle$ the value of the homomorphism $y \in \hat{G}$ at the point $x \in G$. Choose and fix a local inner product g on $G \times \hat{G}$; that is, g is a function with the properties specified in Lemma 5.3 on p. 83 of Parthasarathy [5].

All random variables that we consider will be assumed to be Borel measurable and their distributions will therefore be Borel measures. The mode of convergence in the central limit theorem will be weak convergence of probability measures on G. The characteristic function of a probability measure μ on G is

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the complex-valued function $\hat{\mu}$ defined on $\hat{G}$ by

$$\hat{\mu}(y) := \int_G \langle x, y\rangle \mu(dx) \quad \text{for all } y \in \hat{G}.$$

It is well known that there is a one-one correspondence between probability measures on G and their characteristic functions on $\hat{G}$ and that a sequence $\{\mu_n\}$ of probability measures on G converges weakly to the probability measure μ on G as $n \to \infty$ if and only if $\hat{\mu}_n(y) \to \hat{\mu}(y)$ as $n \to \infty$ for each $y \in \hat{G}$.

The limiting distribution in the central limit theorem below is Gaussian.

DEFINITION 1. A continuous nonnegative function φ defined on $\hat{G}$ is called a *continuous nonnegative quadratic form on* $\hat{G}$ if it satisfies the equation

$$\varphi(y_1+y_2)+\varphi(y_1-y_2) = 2\varphi(y_1)+2\varphi(y_2) \quad \text{for all } y_1, y_2 \in \hat{G}.$$

A probability measure μ on G is called *Gaussian* if its characteristic function is of the form

$$\hat{\mu}(y) = \langle x_0, y\rangle \exp[-\tfrac{1}{2}\varphi(y)], \quad y \in \hat{G}, \tag{1}$$

where x_0 is a fixed point of G and φ is a continuous nonnegative quadratic form on $\hat{G}$. The Gaussian distribution is *symmetric* if x_0 is the identity of G. ∎

In fact, any function which has the form of the right-hand side of equation (1) is the characteristic function of some (Gaussian) probability measure on G. The above definition of Gaussian distributions is equivalent to the definition of Gaussian distributions in Parthasarathy [5], the equivalence also being proved there. Heyer [4] also considers other candidates for the description 'Gaussian distribution' and calls the distributions defined in Definition 1 *Gaussian distributions in the sense of Parthasarathy.*

For general facts about locally compact second countable Abelian groups we refer the reader to Hewitt and Ross [3] and Rudin [6], and for the theory of probability measures on such groups we recommend Heyer [4] and Parthasarathy [5].

The results in this paper concern a triangular array of G-valued random variables $\{X_{n,j}\colon j = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots\}$ defined on a probability space $(\Omega, \mathscr{F}, P)$. Given such an array, define the following sub-σ-fields of $\mathscr{F}$: for each positive integer n and positive integers a, b such that $a \leqslant b$, let

$$\mathscr{M}(n, a, b) := \sigma\{X_{n,j}\colon a \leqslant j \leqslant b\},$$

where $X_{n,j} = e$, the identity of G, whenever j is not an integer in the range $1 \leqslant j \leqslant k_n$. For positive integers n, k define

$$\psi_n(k) := \sup_{\substack{1 \leqslant s \leqslant \\ \leqslant s+k \leqslant k_n}} \sup\{|P(B|A)-P(B)|\colon A \in \mathscr{M}(n, 1, s),\ P(A) > 0,\ B \in \mathscr{M}(n, s+k, k_n)\},$$

where the supremum of the empty set is taken to be 0.

2. The central limit theorem

THEOREM 1. *Let $\{X_{n,j}: j = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots\}$ be a triangular array of G-valued random variables and suppose that the following conditions hold as $n \to \infty$ for every neighbourhood N of the identity in G and for every $y \in \hat{G}$:*

(i)
$$\sum_{j=1}^{k_n} P(X_{n,j} \notin N) \to 0,$$

(ii)
$$\sum_{j=1}^{k_n} |\mathrm{E} g(X_{n,j}, y)| \to 0,$$

(iii)
$$\mathrm{E}\big[\big(\sum_{j=1}^{k_n} g(X_{n,j}, y)\big)^2\big] \to \varphi(y),$$

where φ is a continuous nonnegative quadratic form on $\hat{G}$. Suppose also that

(iv)
$$\sup_n \psi_n(j) \to 0 \quad \text{as } j \to \infty.$$

Then, as $n \to \infty$, the distribution of the row sum $S_n := \sum_{j=1}^{k_n} X_{n,j}$ converges weakly to the Gaussian probability measure with characteristic function

$$y \to \exp\big[-\tfrac{1}{2}\varphi(y)\big], \quad y \in \hat{G}. \ ■$$

Remark 1. Assumption (i) is a typical uniform infinitesimality condition when the limiting distribution is Gaussian, and assumption (ii) ensures that the triangular array is asymptotically centred. Condition (iii) provides the limiting variance, and assumption (iv) is a uniform mixing condition, which controls the extent to which random variables in the same row of the triangular array can be dependent. ■

Proof of Theorem 1. Fix $y \in \hat{G}$. For each $n = 1, 2, \ldots$ and $j = 1, 2, \ldots, k_n$ define

$$U_{n,j} := g(X_{n,j}, y) - \mathrm{E} g(X_{n,j}, y).$$

Then

$$\begin{aligned}
&\big|\mathrm{E}\big[\big(\sum_{j=1}^{k_n} U_{n,j}\big)^2\big] - \varphi(y)\big| \\
&\quad \leqslant \big|\mathrm{E}\big[\big(\sum_{j=1}^{k_n} g(X_{n,j}, y)\big)^2\big] - \varphi(y)\big| + \big|\mathrm{E} \sum_{j=1}^{k_n} g(X_{n,j}, y)\big|^2 \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

so, by assumption (iii),

$$\sigma_n^2 := \mathrm{Var}\big(\sum_{j=1}^{k_n} U_{n,j}\big) \to \varphi(y) \quad \text{as } n \to \infty.$$

Choose $\varepsilon > 0$ and suppose y is such that $\varphi(y) > 0$. Then $\sigma_n > \frac{1}{2}\sqrt{\varphi(y)}$ for all sufficiently large n and, because of assumption (ii), we also have

$$|\mathrm{E} g(X_{n,j}, y)| < \frac{\varepsilon}{4}\sqrt{\varphi(y)}$$

for every j whenever n is sufficiently large. Choose a neighbourhood M of the identity in G such that

$$|g(x, y)| < \frac{\varepsilon}{4}\sqrt{\varphi(y)}$$

whenever $x \in M$. Then, using assumption (i),

$$\begin{aligned}\limsup_{n\to\infty} \sum_{j=1}^{k_n} P(|U_{n,j}| \geqslant \varepsilon\sigma_n) &\leqslant \limsup_{n\to\infty} \sum_{j=1}^{k_n} P\left(|U_{n,j}| \geqslant \frac{\varepsilon}{2}\sqrt{\varphi(y)}\right) \\ &\leqslant \limsup_{n\to\infty} \sum_{j=1}^{k_n} P\left(|g(X_{n,j}, y)| > \frac{\varepsilon}{4}\sqrt{\varphi(y)}\right) \\ &\leqslant \limsup_{n\to\infty} \sum_{j=1}^{k_n} P(X_{n,j} \notin M) = 0.\end{aligned}$$

Therefore

$$\limsup_{n\to\infty} \sigma_n^{-2} \sum_{j=1}^{k_n} \mathrm{E}\big(U_{n,j}^2 \cdot 1(|U_{n,j}| \geqslant \varepsilon\sigma_n)\big) \leqslant \frac{c^2}{\varphi(y)} \limsup_{n\to\infty} \sum_{j=1}^{k_n} P(|U_{n,j}| \geqslant \varepsilon\sigma_n) = 0,$$

where $c := 2\sup_{x\in G} |g(x, y)| < \infty$.

Now apply the main result of Utev [7] to the triangular array of real--valued random variables $\{U_{n,j}\colon j = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots\}$ to deduce that, if $\varphi(y) > 0$, the distribution of $\sigma_n^{-1}\sum_{j=1}^{k_n} U_{n,j}$ converges weakly to the standard normal distribution as $n \to \infty$; note that the uniform mixing condition on the $U_{n,j}$'s needed to apply Utev's result is implied by the corresponding assumption (iv) on the $X_{n,j}$'s. Consequently,

$$\mathrm{E}\Big[\exp\Big(i\big(\sigma_n^{-1} \sum_{j=1}^{k_n} U_{n,j}\big)t\Big)\Big] \to \exp[-\tfrac{1}{2}t^2]$$

uniformly in $t \in K$ as $n \to \infty$ for each compact subset K of the real line. In particular, for any real number t and any sequence $\{t_n\}$ of real numbers converging to t,

$$\mathrm{E}\Big[\exp\Big(i\big(\sigma_n^{-1} \sum_{j=1}^{k_n} U_{n,j}\big)t_n\Big)\Big] \to \exp[-\tfrac{1}{2}t^2] \quad \text{as } n \to \infty.$$

Let $t_n = \sigma_n$. Then $t_n \to \sqrt{\varphi(y)}$ and we obtain

$$\mathrm{E}\Big[\exp\Big(i \sum_{j=1}^{k_n} U_{n,j}\Big)\Big] \to \exp[-\tfrac{1}{2}\varphi(y)] \quad \text{as } n \to \infty. \tag{2}$$

This holds for each $y \in \hat{G}$ such that $\varphi(y) > 0$.

If y is such that $\varphi(y) = 0$, then $\mathrm{Var}\big(\sum_{j=1}^{k_n} U_{n,j}\big) \to 0$, so

$$\sum_{j=1}^{k_n} U_{n,j} \xrightarrow{P} 0,$$

where $\xrightarrow{P}$ denotes convergence in probability, and

$$\mathrm{E}\left[\exp\left(i\sum_{j=1}^{k_n} U_{n,j}\right)\right] \to 1 = \exp\left[-\tfrac{1}{2}\varphi(y)\right] \quad \text{as } n\to\infty .$$

Hence (2) holds for all $y\in\hat{G}$.

Now choose a neighbourhood N of the identity in g such that $\langle x, y\rangle = \exp\big(ig(x, y)\big)$ for all $x\in N$. Then

$$\left|\mathrm{E}\langle S_n, y\rangle - \mathrm{E}\exp\left(i\sum_{j=1}^{k_n} U_{n,j}\right)\right| \leqslant \mathrm{E}\left|\left\langle \sum_{j=1}^{k_n} X_{n,j}, y\right\rangle - \exp\left(i\sum_{j=1}^{k_n} g(X_{n,j}, y)\right)\right|$$
$$+\left|\mathrm{E}\left[\left\{\exp\left(i\sum_{j=1}^{k_n} g(X_{n,j}, y)\right)\right\}\left\{1-\exp\left(-i\sum_{j=1}^{k_n}\mathrm{E}g(X_{n,j}, y)\right)\right\}\right]\right|$$
$$\leqslant 2\sum_{j=1}^{k_n} P(X_{n,j}\notin N) + \left|1-\exp\left(-i\sum_{j=1}^{k_n}\mathrm{E}g(X_{n,j}, y)\right)\right| \to 0 \quad \text{as } n\to\infty .$$

The proof of the theorem is completed by application of the Lévy–Cramér continuity theorem. ■

3. The functional central limit theorem. Denote by $\boldsymbol{D} := \boldsymbol{D}([0, 1], G)$ the Skorokhod space of G-valued càdlàg functions defined on the unit interval $[0, 1]$ in the real line. We endow $\boldsymbol{D}$ with the Skorokhod topology. If ϱ is a metric which gives the topology of G, then the Skorokhod topology can be defined on $\boldsymbol{D}$ in the same way as in Billingsley [1] or Parthasarathy [5] simply by replacing the metric on the real line by ϱ where appropriate. If $\mathscr{B}_{\boldsymbol{D}}$ denotes the σ-field of Borel subsets of $\boldsymbol{D}$ relative to the Skorokhod topology, then much of the theory of probability measures on $(\boldsymbol{D}, \mathscr{B}_{\boldsymbol{D}})$ and their weak convergence can be developed along the same lines as for the case when G is the real line.

Before stating the functional central limit theorem we need some further terminology.

DEFINITION 2. We shall call a stochastic process $\boldsymbol{S} := \{S(t)\colon t\in[0, 1]\}$ a *Wiener-type process on G* if $\boldsymbol{S}$ has independent increments, the sample paths of $\boldsymbol{S}$ are almost surely continuous, $S(0) = e$ (the identity of G), and $S(t)$ has a symmetric Gaussian distribution for every t.

We call $\{\varphi_{s,t}\colon 0\leqslant s\leqslant t\leqslant 1\}$ a *continuous semigroup of continuous nonnegative quadratic forms on* $\hat{G}$ if each $\varphi_{s,t}$ is a continuous nonnegative quadratic form on $\hat{G}$, $(s, t)\mapsto\varphi_{s,t}(y)$ is continuous for each $y\in\hat{G}$, and

(3) $$\varphi_{s,t}(y)+\varphi_{t,u}(y) = \varphi_{s,u}(y)$$

whenever $0\leqslant s\leqslant t\leqslant u\leqslant 1$ and $y\in\hat{G}$. ■

Note that because of (3) we can write

$$\varphi_{t,u}(y) = \varphi_u(y)-\varphi_t(y), \quad \text{where } \varphi_t(y) := \varphi_{0,t}(y).$$

Given a Wiener-type process $S:=\{S(t)\colon t\in[0,1]\}$ on G, there exists a corresponding continuous semigroup of continuous nonnegative quadratic forms such that

$$\mathrm{E}[\langle S(t)-S(s), y\rangle] = \exp[-\tfrac{1}{2}\varphi_{s,t}(y)]$$

for $0\leqslant s\leqslant t\leqslant 1$ and $y\in\hat{G}$. The converse is also true; see Bingham [2]. (In particular, there exists a nontrivial Wiener-type process on G whenever there is a nondegenerate Gaussian measure on G, or, equivalently, whenever there is a continuous nonnegative quadratic form on $\hat{G}$ which is not identically zero.)

DEFINITION 3. In the above situation the distribution of S will be called a *Wiener-type measure with continuous semigroup of continuous nonnegative quadratic forms* $\{\varphi_{s,t}\colon 0\leqslant s\leqslant t\leqslant 1\}$. ∎

Given a triangular array $\{X_{n,j}\colon j=1,2,\ldots,k_n;\ n=1,2,\ldots\}$ of G-valued random variables, define the stochastic process $S_n:=\{S_n(t)\colon 0\leqslant t\leqslant 1\}$ with sample paths in D by

$$S_n(t):=\sum_{j=1}^{[tk_n]} X_{n,j},$$

where, for each real number r, $[r]$ denotes the largest integer not exceeding r. We can now state the functional central limit theorem.

THEOREM 2. *Let $\{X_{n,j}\colon j=1,2,\ldots,k_n;\ n=1,2,\ldots\}$ be a triangular array of G-valued random variables and suppose that $\varphi_t=\varphi_{0,t}$, where $\{\varphi_{s,t}\colon 0\leqslant s\leqslant t\leqslant 1\}$ is a continuous semigroup of continuous nonnegative quadratic forms on $\hat{G}$. Suppose that the following conditions hold as $n\to\infty$ for every neighbourhood N of the identity in G and for every $y\in\hat{G}$:*

(i) $$\sum_{j=1}^{k_n} P(X_{n,j}\notin N)\to 0,$$

(ii) $$\sum_{j=1}^{k_n} |\mathrm{E}g(X_{n,j}, y)|\to 0,$$

(iii) $$\mathrm{E}\Big[\Big(\sum_{j=1}^{[tk_n]} g(X_{n,j}, y)\Big)^2\Big]\to\varphi_t(y)\quad \textit{for each } t\in[0,1].$$

Suppose also that

(iv) $$\sup_n \psi_n(j)\to 0\quad \textit{as } j\to\infty.$$

Then, as $n\to\infty$, the distribution of S_n converges weakly on D to the Wiener-type measure with continuous semigroup of continuous nonnegative quadratic forms $\{\varphi_{s,t}\colon 0\leqslant s\leqslant t\leqslant 1\}$. ∎

The proof of Theorem 2 will use the standard technique of proving the appropriate convergence of finite-dimensional distributions and establishing weak conditional compactness of the distributions of the processes $\{S_n\}$ on $\boldsymbol{D}$. The first of these steps is accomplished by Lemma 3 below. First, however, we require some preliminary results.

LEMMA 1. *Let* $\{X_{n,j}\colon j=1,2,\ldots,k_n;\ n=1,2,\ldots\}$ *be a triangular array of G-valued random variables satisfying the assumptions of Theorem 2. Define* $U_{n,j} := g(X_{n,j}, y) - \mathrm{E}g(X_{n,j}, y)$. *Then, for each* $y \in \hat{G}$,

$$\mathrm{E}\Big[\Big(\sum_{j=1}^{[tk_n]} U_{n,j}\Big)^2\Big] \to \varphi_t(y)$$

uniformly in $t \in [0, 1]$ *as* $n \to \infty$. *Consequently,*

$$\mathrm{E}\Big[\Big(\sum_{j=[sk_n]+1}^{[tk_n]} U_{n,j}\Big)^2\Big] \to \varphi_t(y) - \varphi_s(y)$$

uniformly in $0 \leqslant s \leqslant t \leqslant 1$ *as* $n \to \infty$. ∎

Proof. By Lemma 3 in Bingham [2], the convergence in assumption (3) of Theorem 2 is in fact uniform in t. Therefore

$$\begin{aligned}\Big|\mathrm{E}\Big[\Big(\sum_{j=1}^{[tk_n]} U_{n,j}\Big)^2\Big] - \varphi_t(y)\Big| &\leqslant \Big|\mathrm{E}\Big[\Big(\sum_{j=1}^{[tk_n]} g(X_{n,j}, y)\Big)^2\Big] - \varphi_t(y)\Big| + \Big(\sum_{j=1}^{[tk_n]} \mathrm{E}g(X_{n,j}, y)\Big)^2 \\ &\leqslant \Big|\mathrm{E}\Big[\Big(\sum_{j=1}^{[tk_n]} g(X_{n,j}, y)\Big)^2\Big] - \varphi_t(y)\Big| + \Big(\sum_{j=1}^{k_n} |\mathrm{E}g(X_{n,j}, y)|\Big)^2\end{aligned}$$

which tends to 0 uniformly in t as $n \to \infty$. This proves the first part of the lemma.

Let $0 \leqslant s \leqslant t \leqslant 1$. Then

$$\begin{aligned}&\mathrm{E}\Big[\Big(\sum_{j=1}^{[tk_n]} U_{n,j}\Big)^2\Big] \\ &\quad = \mathrm{E}\Big[\Big(\sum_{j=1}^{[sk_n]} U_{n,j}\Big)^2\Big] + \mathrm{E}\Big[\Big(\sum_{j=[sk_n]+1}^{[tk_n]} U_{n,j}\Big)^2\Big] + 2\mathrm{E}\Big[\Big(\sum_{j=1}^{[sk_n]} U_{n,j}\Big)\Big(\sum_{j=[sk_n]+1}^{[tk_n]} U_{n,j}\Big)\Big],\end{aligned}$$

so the second result will follow from the first if we show that

$$\mathrm{E}\Big[\Big(\sum_{j=1}^{[sk_n]} U_{n,j}\Big)\Big(\sum_{j=[sk_n]+1}^{[tk_n]} U_{n,j}\Big)\Big] \to 0$$

uniformly in s, t as $n \to \infty$.

For any positive integer r

$$\begin{aligned}(4)\quad &\mathrm{E}\Big[\Big(\sum_{j=1}^{[sk_n]} U_{n,j}\Big)\Big(\sum_{j=[sk_n]+1}^{[tk_n]} U_{n,j}\Big)\Big] \\ &\quad = \mathrm{E}\Big[\Big(\sum_{j=1}^{[sk_n]} U_{n,j}\Big)\Big(\sum_{j=[sk_n]+1}^{[sk_n]+r-1} U_{n,j}\Big)\Big] + \mathrm{E}\Big[\Big(\sum_{j=1}^{[sk_n]} U_{n,j}\Big)\Big(\sum_{j=[sk_n]+r}^{[tk_n]} U_{n,j}\Big)\Big]\end{aligned}$$

$$\leqslant \Big(\mathrm{E}\big[\big(\sum_{j=1}^{[sk_n]} U_{n,j}\big)^2\big]\,\mathrm{E}\big[\big(\sum_{j=[sk_n]+1}^{[sk_n]+r-1} U_{n,j}\big)^2\big]\Big)^{1/2}$$

$$+2\psi_n(r)^{1/2}\Big(\mathrm{E}\big[\big(\sum_{j=1}^{[sk_n]} U_{n,j}\big)^2\big]\,\mathrm{E}\big[\big(\sum_{j=[sk_n]+r}^{[tk_n]} U_{n,j}\big)^2\big]\Big)^{1/2}.$$

Here we interpret empty sums as 0 and we have used the inequality given in Lemma 1 on p. 170 of Billingsley [1].

Now

$$\mathrm{E}\big[\big(\sum_{j=[sk_n]+r}^{[tk_n]} U_{n,j}\big)^2\big] = \mathrm{E}\big[\big(\sum_{j=1}^{[tk_n]} U_{n,j}-\sum_{j=1}^{[sk_n]+r-1} U_{n,j}\big)^2\big] \tag{5}$$

$$\leqslant 2\mathrm{E}\big[\big(\sum_{j=1}^{[tk_n]} U_{n,j}\big)^2\big]+2\mathrm{E}\big[\big(\sum_{j=1}^{[sk_n]+r-1} U_{n,j}\big)^2\big].$$

Note that $[sk_n]+r-1 = [(s+(r-1)/k_n)\,k_n]$ and, by the first part of Lemma 1, it follows that the expression on the left-hand side of inequality (5) is bounded above by a finite constant that does not depend upon n, r, s, nor t. The same is therefore true of the coefficient of $\psi_n(r)^{1/2}$ in the last line of inequality (4). Consequently, because of the mixing condition (assumption (iv)), the last expression in inequality (4) can be made less than an arbitrary positive ε for all n, s, t by choosing r sufficiently large. Fix such an r.

To complete the proof it is therefore enough to show that

$$\mathrm{E}\big[\big(\sum_{j=[sk_n]+1}^{[sk_n]+r-1} U_{n,j}\big)^2\big]\to 0$$

uniformly in s as $n\to\infty$. But $|U_{n,j}|\leqslant c := 2\sup_{x\in G}|g(x, y)|$ and, for any $\varepsilon>0$,

$$\mathrm{E}\big[\big(\sum_{j=[sk_n]+1}^{[sk_n]+r-1} U_{n,j}\big)^2\big]\leqslant \varepsilon^2+(rc)^2 P\big(\big|\sum_{j=[sk_n]+1}^{[sk_n]+r-1} U_{n,j}\big|>\varepsilon\big)$$

$$\leqslant \varepsilon^2+(rc)^2\sum_{j=[sk_n]+1}^{[sk_n]+r-1} P(|U_{n,j}|>\varepsilon/r).$$

The last sum is dominated by $\sum_{j=1}^{k_n} P(|U_{n,j}|>\varepsilon/r)$, which, as can be seen from the proof of Theorem 1, goes to 0 as $n\to\infty$. Because ε was arbitrary, Lemma 1 is now proved. ∎

LEMMA 2. *Let* $\{X_{n,j}\colon j=1, 2, \ldots, k_n;\ n=1, 2, \ldots\}$ *be a triangular array of G-valued random variables satisfying the assumptions of Theorem 2. Let* $y_1, y_2\in\hat{G}$. *Then*

$$\mathrm{E}\big[\big(\sum_{j=[sk_n]+1}^{[tk_n]} g(X_{n,j}, y_1)+\sum_{j=[uk_n]+1}^{[vk_n]} g(X_{n,j}, y_2)\big)^2\big]\to\varphi_{s,t}(y_1)+\varphi_{u,v}(y_2)$$

uniformly in $0\leqslant s\leqslant t\leqslant u\leqslant v\leqslant 1$ *as* $n\to\infty$. ∎

Proof. Write

$$U_{n,j} := g(X_{n,j}, y_1) - \mathrm{E}g(X_{n,j}, y_1) \quad \text{and} \quad V_{n,j} := g(X_{n,j}, y_2) - \mathrm{E}g(X_{n,j}, y_2).$$

The difference between

$$\mathrm{E}\Big[\Big(\sum_{j=[sk_n]+1}^{[tk_n]} g(X_{n,j}, y_1) + \sum_{j=[uk_n]+1}^{[vk_n]} g(X_{n,j}, y_2)\Big)^2\Big]$$

and

$$\mathrm{E}\Big[\Big(\sum_{j=[sk_n]+1}^{[tk_n]} U_{n,j} + \sum_{j=[uk_n]+1}^{[vk_n]} V_{n,j}\Big)^2\Big]$$

is dominated by

$$\Big[\sum_{j=[sk_n]+1}^{[tk_n]} |\mathrm{E}g(X_{n,j}, y_1)| + \sum_{j=[uk_n]+1}^{[vk_n]} |\mathrm{E}g(X_{n,j}, y_2)|\Big]^2$$

which, by assumption (ii), converges to 0 uniformly in s, t, u, v as $n \to \infty$. Therefore, it is enough to show that

$$\mathrm{E}\Big[\Big(\sum_{j=[sk_n]+1}^{[tk_n]} U_{n,j} + \sum_{j=[uk_n]+1}^{[vk_n]} V_{n,j}\Big)^2\Big] \to \varphi_{s,t}(y_1) + \varphi_{u,v}(y_2)$$

uniformly as $n \to \infty$. Since

$$\mathrm{E}\Big[\Big(\sum_{j=[sk_n]+1}^{[tk_n]} U_{n,j} + \sum_{j=[uk_n]+1}^{[vk_n]} V_{n,j}\Big)^2\Big]$$

$$= \mathrm{E}\Big[\Big(\sum_{j=[sk_n]+1}^{[tk_n]} U_{n,j}\Big)^2 + \Big(\sum_{j=[uk_n]+1}^{[vk_n]} V_{n,j}\Big)^2\Big] + 2\mathrm{E}\Big[\Big(\sum_{j=[sk_n]+1}^{[tk_n]} U_{n,j}\Big)\Big(\sum_{j=[uk_n]+1}^{[vk_n]} V_{n,j}\Big)\Big],$$

it is enough, by Lemma 1, to show that

$$\mathrm{E}\Big[\Big(\sum_{j=[sk_n]+1}^{[tk_n]} U_{n,j}\Big)\Big(\sum_{j=[uk_n]+1}^{[vk_n]} V_{n,j}\Big)\Big] \to 0$$

uniformly in s, t, u, v as $n \to \infty$. But this follows by using an argument similar to the one in the last part of the proof of Lemma 1. ∎

We can now prove the required convergence of the finite-dimensional distributions. Note that, for any positive integer k, the dual group of G^k can be identified with $\hat{G}^k$.

LEMMA 3. *Let $\{X_{n,j}\colon j = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots\}$ be a triangular array of G-valued random variables that satisfies the assumptions of Theorem 2. Then, for every positive integer k and points $0 = t_0 \leqslant t_1 < t_2 < \ldots < t_k$ in $[0, 1]$, the distribution of the G^k-valued random variable $(S_n(t_1), S_n(t_2), \ldots, S_n(t_k))$ converges* *weakly to the Gaussian distribution with characteristic function*

$$(y_1, y_2, \ldots, y_k) \mapsto \exp\Big[-\frac{1}{2}\sum_{j=1}^{k} \varphi_{t_{j-1},t_j}\Big(\sum_{m=j}^{k} y_m\Big)\Big], \qquad (y_1, y_2, \ldots, y_k) \in \hat{G}^k,$$

which is the appropriate finite-dimensional distribution for the limiting Wiener--type measure in Theorem 2. ∎

Proof. Choose a positive integer k and points $t_1 < t_2 < \ldots < t_k$ in $[0, 1]$. Define a local inner product g_k on $G^k \times \hat{G}^k$ by

$$g_k(\boldsymbol{x}, \boldsymbol{y}) := \sum_{j=1}^{k} g(x_j, y_j)$$

for all $\boldsymbol{x} = (x_1, x_2, \ldots, x_k) \in G^k$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_k) \in \hat{G}^k$, where g is the local inner product already fixed on $G \times \hat{G}$. Define

$$\boldsymbol{X}_{n,j} = \begin{cases} (X_{n,j}, e, \ldots, e) & \text{for } 1 \leqslant j \leqslant [t_1 k_n], \\ (e, X_{n,j}, e, \ldots, e) & \text{for } [t_1 k_n] < j \leqslant [t_2 k_n], \\ \ldots\ldots\ldots\ldots\ldots & \\ (e, \ldots, e, X_{n,j}) & \text{for } [t_{k-1} k_n] < j \leqslant [t_k k_n], \\ (e, \ldots, e) & \text{for } [t_k k_n] < j \leqslant k_n. \end{cases}$$

We now apply Theorem 1 to the new triangular array of G^k-valued random variables $\{\boldsymbol{X}_{n,j}\colon j = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots\}$. This array satisfies assumptions (i) and (iv) of Theorem 1 for G^k-valued random variables as immediate consequences of the corresponding assumptions for the original array. Also

$$g_k(\boldsymbol{X}_{n,j}, \boldsymbol{y}) = g(X_{n,j}, y_m) \quad \text{for } [t_{m-1} k_n] < j \leqslant [t_m k_n],$$

where $t_0 = 0$, so assumption (ii) holds for $\{\boldsymbol{X}_{n,j}\}$ with g_k in place of g. It remains to prove the appropriate version of condition (iii); namely that

$$\mathrm{E}\Big[\Big(\sum_{j=1}^{k_n} g_k(\boldsymbol{X}_{n,j}, \boldsymbol{y})\Big)^2\Big] \to \psi(\boldsymbol{y}) := \varphi_{t_1}(y_1) + \varphi_{t_1,t_2}(y_2) + \ldots + \varphi_{t_{k-1},t_k}(y_k)$$

as $n \to \infty$. This follows, however, from Lemma 2 (extended from two to k summands) and the fact that

$$\sum_{j=1}^{k_n} g_k(\boldsymbol{X}_{n,j}, \boldsymbol{y}) = \sum_{j=1}^{[t_1 k_n]} g(X_{n,j}, y_1) + \sum_{j=[t_1 k_n]+1}^{[t_2 k_n]} g(X_{n,j}, y_2) + \ldots$$
$$\ldots + \sum_{j=[t_{k-1} k_n]+1}^{[t_k k_n]} g(X_{n,j}, y_k).$$

From Theorem 1 we can now deduce that the distribution of

$$\sum_{j=1}^{k_n} X_{n,j} = \big(S_n(t_1),\, S_n(t_2)-S_n(t_1),\, \ldots,\, S_n(t_k)-S_n(t_{k-1})\big)$$

converges weakly on G^k to the symmetric Gaussian distribution with continuous nonnegative quadratic form ψ.

By straightforward algebraic manipulation of the characteristic functions, this is seen to be equivalent to the conclusion of Lemma 3. ∎

The following lemma will be used to establish the weak conditional compactness of the distributions of the processes $\{S_n\}$ on $\boldsymbol{D}$.

LEMMA 4. *Let the assumptions and notation be as in Theorem 2 and let μ_n denote the distribution of the process S_n on $\boldsymbol{D}$. For each positive integer m and $l = 0, 1, 2, \ldots, m$ let $\beta_l := [lk_n/m]$ and $\alpha_l = \beta_{l-1}+1$. Suppose that the following condition holds, where $U_{n,j} = g(X_{n,j}, y) - \mathrm{E}g(X_{n,j}, y)$:*

For every $y \in \hat{G}$, $\varepsilon > 0$ and $\eta > 0$ there exist integers m_0 and n_0 such that

$$P\Big(\max_{1\leqslant l\leqslant m}\ \max_{\alpha_l\leqslant k\leqslant\beta_l} \Big|\sum_{j=\alpha_l}^{k} U_{n,j}\Big| \geqslant \varepsilon\Big) \leqslant \eta \tag{6}$$

whenever $m \geqslant m_0$ and $n \geqslant n_0$.

Then $\{\mu_n\}$ is weakly conditionally compact and if μ is the weak limit of any subsequence of $\{\mu_n\}$, then $\mu(\boldsymbol{C}) = 1$, where $\boldsymbol{C}$ is the subspace of $\boldsymbol{D}$ consisting of all continuous G-valued functions on $[0, 1]$. ∎

Proof. The condition (6) in Lemma 4 is essentially the same as (6.13) in Bingham [2] and in the same way, using assumption (ii) of Theorem 2 in place of the approximate martingale condition, we can show that it implies the second condition of Proposition 3 in Bingham [2]. That Proposition therefore implies the present result, because by Lemma 3 we know that for each $t \in [0, 1]$ the distribution of $S_n(t)$ converges weakly. ∎

Proof of Theorem 2. In view of what has already been shown, the proof of Theorem 2 will be complete if it can be verified that the condition (6) given in Lemma 4 holds.

Fix $y \in \hat{G}$ and let $\varepsilon > 0$. For $\alpha_l \leqslant k \leqslant \beta_l$ define

$$T_k := \sum_{j=\alpha_l}^{k} U_{n,j}, \qquad A_k := [|T_j| < 3\varepsilon \text{ for every } \alpha_l \leqslant j < k,\ |T_k| \geqslant 3\varepsilon],$$

where, as before,

$$U_{n,j} := g(X_{n,j}, y) - \mathrm{E}g(X_{n,j}, y).$$

Then, for any positive integer r,

$$P\big(\max_{\alpha_l\leqslant k\leqslant\beta_l} |T_k| \geqslant 3\varepsilon\big) \leqslant P(|T_{\beta_l}| \geqslant \varepsilon) + \sum_{k=\alpha_l}^{\beta_l-1} P\big(A_k \cap [|T_{\beta_l} - T_k| \geqslant 2\varepsilon]\big) \tag{7}$$

$$\leqslant P(|T_{\beta_l}| \geqslant \varepsilon) + \sum_{k=\alpha_l}^{\beta_l - r - 1} P(|T_{k+r} - T_k| \geqslant \varepsilon)$$

$$+ \sum_{k=\alpha_l}^{\beta_l - r - 1} P(A_k \cap [|T_{\beta_l} - T_{k+r}| \geqslant \varepsilon]) + \sum_{k=\beta_l - r}^{\beta_l - 1} P(A_k \cap [|T_{\beta_l} - T_k| \geqslant \varepsilon])$$

$$\leqslant P(|T_{\beta_l}| \geqslant \varepsilon) + \sum_{k=\alpha_l}^{\beta_l - r - 1} P(|T_{k+r} - T_k| \geqslant \varepsilon)$$

$$+ \sum_{k=\alpha_l}^{\beta_l - r - 1} P(A_k)\big(P(|T_{\beta_l} - T_{k+r}| \geqslant \varepsilon) + \psi_n(r)\big) + \sum_{k=\beta_l - r}^{\beta_l - 1} P(A_k \cap [|T_{\beta_l} - T_k| \geqslant \varepsilon])$$

$$\leqslant P(|T_{\beta_l}| \geqslant \varepsilon) + \sum_{k=\alpha_l}^{\beta_l - r - 1} P(|T_{k+r} - T_k| \geqslant \varepsilon)$$

$$+ \max_{\alpha_l \leqslant k \leqslant \beta_l - r - 1} P(|T_{\beta_l} - T_{k+r}| \geqslant \varepsilon) + \psi_n(r) + r \max_{\beta_l - r \leqslant k \leqslant \beta_l - 1} P(|T_{\beta_l} - T_k| \geqslant \varepsilon).$$

Given $\delta > 0$ choose r large enough to make $\psi_n(r) < \delta$ for every n. Then for this fixed r we prove that, as $n \to \infty$,

$$(8) \qquad r \max_{\beta_l - r \leqslant k \leqslant \beta_l - 1} P(|T_{\beta_l} - T_k| \geqslant \varepsilon) \to 0.$$

For each positive integer n and $0 \leqslant s \leqslant t \leqslant 1$ define

$$F_n(s, t) := \mathrm{E}\Big[\Big(\sum_{j=[sk_n]+1}^{[tk_n]} U_{n,j}\Big)^2\Big].$$

Since

$$(9) \qquad \max_{\beta_l - r \leqslant k \leqslant \beta_l - 1} P(|T_{\beta_l} - T_k| \geqslant \varepsilon) \leqslant \frac{1}{\varepsilon^2} \max_{\beta_l - r \leqslant k \leqslant \beta_l - 1} \mathrm{E}[(T_{\beta_l} - T_k)^2],$$

it is enough to show that the right-hand side of inequality (9) goes to 0 as $n \to \infty$. But, if the right-hand side of inequality (9) did not go to 0, there would be an $\eta > 0$ and a sequence of points $\{s_n\}$ such that

$$l/m - r/k_n \leqslant s_n \leqslant l/m \quad \text{and} \quad F(s_n, l/m) > \eta \quad \text{for all } n.$$

But then $s_n \to l/m$; so, by Lemma 1,

$$F_n(s_n, l/m) \to \varphi_{l/m, l/m}(y) = 0,$$

which yields a contradiction. Therefore (8) must hold.

Next, as $U_{n,j} = g(X_{n,j}, y) - \mathrm{E}g(X_{n,j}, y)$, if we choose a neighbourhood N of e in G such that $|g(x, y)| < \varepsilon/(2r)$ whenever $x \in N$, then assumption (ii) of Theorem 2 implies that, if $X_{n,j} \in N$ and n is sufficiently large, then $|U_{n,j}| < \varepsilon/r$. Therefore $|U_{n,j}| \geqslant \varepsilon/r$ implies that $X_{n,j} \notin N$ when n is sufficiently large. Hence, for large n,

$$\sum_{k=\alpha_l}^{\beta_l-r-1} P(|T_{k+r}-T_k| \geqslant \varepsilon) = \sum_{k=\alpha_l}^{\beta_l-r-1} P\Big(\Big|\sum_{j=k+1}^{k+r} U_{n,j}\Big| \geqslant \varepsilon\Big)$$
$$\leqslant \sum_{k=\alpha_l}^{\beta_l-r-1} \sum_{j=k+1}^{k+r} P(|U_{n,j}| \geqslant \varepsilon/r)$$
$$\leqslant r \sum_{j=\alpha_l}^{\beta_l} P(|U_{n,j}| \geqslant \varepsilon/r) \leqslant r \sum_{j=\alpha_l}^{\beta_l} P(X_{n,j} \notin N).$$

Assumption (i) now gives

$$\limsup_{n\to\infty} \sum_{k=\alpha_l}^{\beta_l-r-1} P(|T_{k+r}-T_k| \geqslant \varepsilon) = 0. \tag{10}$$

Since δ was arbitrary, (7), (8) and (10) imply that

$$\limsup_{n\to\infty} P\Big(\max_{\alpha_l\leqslant k\leqslant\beta_l} \Big|\sum_{j=\alpha_l}^{k} U_{n,j}\Big| \geqslant 3\varepsilon\Big) \leqslant 2\limsup_{n\to\infty} \max_{\alpha_l\leqslant k\leqslant\beta_l} P\Big(\Big|\sum_{j=k}^{\beta_l} U_{n,j}\Big| \geqslant \varepsilon\Big). \tag{11}$$

Let A be a constant such that $\varepsilon < A < \infty$. Then

$$\max_{\alpha_l\leqslant k\leqslant\beta_l} P\Big(\Big|\sum_{j=k}^{\beta_l} U_{n,j}\Big| \geqslant \varepsilon\Big) \tag{12}$$
$$\leqslant \max_{\alpha_l\leqslant k\leqslant\beta_l} P\Big(\Big|\sum_{j=k}^{\beta_l} U_{n,j}\Big| \geqslant A\Big) + \max_{\alpha_l\leqslant k\leqslant\beta_l} P\Big(A > \Big|\sum_{j=k}^{\beta_l} U_{n,j}\Big| \geqslant \varepsilon\Big)$$
$$\leqslant \frac{1}{A^2} \max_{\alpha_l\leqslant k\leqslant\beta_l} \mathrm{E}\Big[\Big(\sum_{j=k}^{\beta_l} U_{n,j}\Big)^2\Big] + \frac{1}{\varepsilon^4} \max_{\alpha_l\leqslant k\leqslant\beta_l} \mathrm{E}\Big[\Big(\sum_{j=k}^{\beta_l} U_{n,j}\Big)^4 1\Big(\Big|\sum_{j=k}^{\beta_l} U_{n,j}\Big| < A\Big)\Big].$$

Let s_n be such that

$$(l-1)/m \leqslant s_n \leqslant l/m \quad \text{and} \quad \max_{\alpha_l\leqslant k\leqslant\beta_l} \mathrm{E}\Big[\Big(\sum_{j=k}^{\beta_l} U_{n,j}\Big)^2\Big] = F_n(s_n, l/m).$$

Then

$$F_n(s_n, l/m) \leqslant |F_n(s_n, l/m) - (\varphi_{l/m}(y) - \varphi_{s_n}(y))| + \varphi_{l/m}(y) - \varphi_{s_n}(y)$$
$$\leqslant |F_n(s_n, l/m) - (\varphi_{l/m}(y) - \varphi_{s_n}(y))| + \varphi_{l/m}(y) - \varphi_{(l-1)/m}(y) \to \varphi_{l/m}(y) - \varphi_{(l-1)/m}(y)$$

as $n \to \infty$ by the uniform convergence of F_n, which was proved in Lemma 1. Thus

$$\limsup_{n\to\infty} \sum_{l=1}^{m} \frac{1}{A^2} \max_{\alpha_l\leqslant k\leqslant\beta_l} \mathrm{E}\Big[\Big(\sum_{j=k}^{\beta_l} U_{n,j}\Big)^2\Big] \leqslant \frac{\varphi_1(y)}{A^2} \tag{13}$$

which can be made as small as we like by choosing A sufficiently large.

Write $\Delta_l\varphi := \varphi_{l/m}(y) - \varphi_{(l-1)/m}(y)$. We now prove that

$$\limsup_{n\to\infty} \max_{\alpha_l \leqslant k \leqslant \beta_l} \mathrm{E}\big[\big(\sum_{j=k}^{\beta_l} U_{n,j}\big)^4 1\big(\big|\sum_{j=k}^{\beta_l} U_{n,j}\big| < A\big)\big] \leqslant 3(\Delta_l\varphi)^2. \tag{14}$$

Suppose that inequality (14) is false. Then there exists $\delta > 0$ such that the maximum on the left is at least $3(\Delta_l\varphi)^2 + \delta$ for infinitely many n. For each n let $t_n \in [(l-1)/m, l/m]$ be such that the maximum on the left-hand side is attained at $k = t_n k_n$. We shall derive a contradiction. In order to do so, we can assume without loss of generality that $t_n \to t \in [(l-1)/m, l/m]$ as $n \to \infty$.

Define

$$X'_{n,j} := \begin{cases} X_{n,j} & \text{for } t_n k_n \leqslant j \leqslant \beta_l, \\ e & \text{otherwise}, \end{cases}$$

$$U'_{n,j} := g(X'_{n,j}, y) - \mathrm{E}g(X'_{n,j}, y).$$

We claim that, as $n \to \infty$,

$$\sum_{j=\alpha_l}^{\beta_l} U'_{n,j} \text{ converges weakly in distribution to } \mathcal{N}(0, \varphi_{t,l/m}(y)), \tag{15}$$

the normal distribution with mean 0 and variance $\varphi_{t,l/m}(y)$. To prove this we can apply Theorem 1 to the triangular array $\{U'_{n,j}\colon j = \alpha_l, \ldots, \beta_l; n = 1, 2, \ldots\}$ of real-valued random variables, taking G as the real line R. Assumption (i) holds for $\{U'_{n,j}\}$ as a consequence of assumptions (i) and (ii) for $\{X'_{n,j}\}$.

Note that

$$|U'_{n,j}| \leqslant 2 \sup_{x\in G} |g(x, y)| =: c < \infty.$$

Take a local inner product g_R on $R \times \hat{R} \cong R^2$ such that $g_R(\xi_1, \xi_2) = \xi_1 \xi_2$ for $|\xi_1| \leqslant c$ and all $\xi_2 \in R$. For any $\xi \in R$ we have

$$\mathrm{E}[g_R(U'_{n,j}, \xi)] = \xi \mathrm{E}[U'_{n,j}] = 0,$$

so assumption (ii) of Theorem 1 holds, and

$$\mathrm{E}\big[\big(\sum_{j=\alpha_l}^{\beta_l} g_R(U'_{n,j}, \xi)\big)^2\big] = \xi^2 \mathrm{E}\big[\big(\sum_{j=\alpha_l}^{\beta_l} U'_{n,j}\big)^2\big] \to \xi^2 \varphi_{t,l/m}(y)$$

as $n \to \infty$, so assumption (iii) holds with the quadratic form $\xi \mapsto \xi^2 \varphi_{t,l/m}(y)$. The mixing condition for the $U'_{n,j}$'s is implied by the corresponding condition for the $X_{n,j}$'s. Therefore the claim (15) is valid.

Consequently,

$$\limsup_{n\to\infty} \max_{\alpha_l \leqslant k \leqslant \beta_l} \mathrm{E}\big[\big(\sum_{j=k}^{\beta_l} U_{n,j}\big)^4 1\big(\big|\sum_{j=k}^{\beta_l} U_{n,j}\big| < A\big)\big]$$
$$= \limsup_{n\to\infty} \mathrm{E}\big[\big(\sum_{j=\alpha_l}^{\beta_l} U'_{n,j}\big)^4 1\big(\big|\sum_{j=\alpha_l}^{\beta_l} U'_{n,j}\big| < A\big)\big] \leqslant 3\big(\varphi_{t,l/m}(y)\big)^2 \leqslant 3(\Delta_l \varphi)^2.$$

But this contradicts the choice of the sequence $\{t_n\}$, so inequality (14) is proved.
Using (14), we have

$$\limsup_{n\to\infty} \frac{1}{\varepsilon^4} \sum_{l=1}^{m} \max_{\alpha_l \leqslant k \leqslant \beta_l} \mathrm{E}\big[\big(\sum_{j=k}^{\beta_l} U_{n,j}\big)^4 1\big(\big|\sum_{j=k}^{\beta_l} U_{n,j}\big| < A\big)\big] \tag{16}$$
$$\leqslant \frac{3}{\varepsilon^4} \sum_{l=1}^{m} (\Delta_l \varphi)^2 \leqslant \frac{3}{\varepsilon^4} \big(\max_{1 \leqslant l \leqslant m} \Delta_l \varphi\big) \sum_{l=1}^{m} \Delta_l \varphi \leqslant \frac{3}{\varepsilon^4} \big(\max_{1 \leqslant l \leqslant m} \Delta_l \varphi\big) \varphi_1(y).$$

By the (uniform) continuity of $t \mapsto \varphi_t(y)$, the last expression in inequality (16) goes to 0 as $m \to \infty$. Combining this with (11), (12) and (13) we see that the premises of Lemma 4 are satisfied, so Theorem 2 is proved. ∎

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