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DISTRIBUTION PROCESSES OF THE FRACTIONAL ARMA TYPE, MIXING PROPERTIES

BY

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Abstract. In this paper we firstly study f, the inverse Laplace transform of $F(s) = \prod_{k=1}^{K} (s-a_k)^{d_k}$. The distribution f is then used to define a family of linear distribution processes. This family generalizes the so-called fractional ARMA processes $X_t = \int_{-\infty}^{t} f(t-s)\,dW_s$ which were introduced by Viano et al. [14] in the case of square integrable f. Using previous results [1] we describe the regularity properties of the distribution process associated with F. Finally, we give a definition of the mixing coefficients suitable for distribution processes, and we obtain conditions on the parameters with F needed for fractional ARMA distribution processes to be mixing.

1. INTRODUCTION

The concept of stochastic distribution processes was introduced by Itô [9] and by Gelfand and Vilenkin [6], and these processes have been widely studied since this time. See for instance Fernique [5]. In [6] Gelfand and Vilenkin point out the interest of stochastic distribution processes in domain such as physics where distributions are successfully used as an alternative to ordinary functions. This idea is developed in Meidan [11] where the connection between ordinary and distribution processes is investigated.

In this paper we present a parametric family of such generalized processes. This family is a particular case of the more general so-called linear distribution processes introduced in [1], and is directly derived from the family of fractional continuous time ARMA ordinary processes recently studied by Viano et al. [14].

Recall that till now long memory second order continuous processes are mostly of the form

$$X_t = \int_{-\infty}^{t} f(t-s)\,dW_s, \tag{1}$$

where W is the standard Brownian motion and f is such that the behaviour of the spectral density of X is like $|\lambda-\lambda_0|^{-2d}$ for some λ_0. This is the case for increments of the fractional Brownian motion (Mandelbrot and Van Ness [10]) and for the fractional ARMA processes introduced and studied by Viano et al. [14]. In the later case the filter f is chosen amongst the inverse Laplace transforms of the parametrized family of F having the form

$$F(s) = \prod_{k=1}^{K} (s-a_k)^{d_k}. \tag{2}$$

The investigation carried out in [14] remains within the framework of second order ordinary stationary processes, and hence supposes that F is the Laplace transform of a square-integrable function on $\boldsymbol{R}^+$. This imposes then certain conditions, in particular, on the sum of the exponents d_k.

The aim of this paper is to investigate what happens when this assumption is dropped. We show that in this case the process can be defined as a distribution process.

In Section 2 we study the distribution f, defined as the Laplace transform of a function F with properties of holomorphy and polynomial growth at infinity. Transfer functions of the fractional ARMA type (2) satisfy these assumptions.

In Section 3 we study the regularity of the paths and the covariance of fractional ARMA distribution processes.

Finally, we propose a definition of mixing for distribution processes that generalizes the usual definition for ordinary processes. We then show that if the transfer function F is holomorphic in a domain which strictly contains the right-half plane and F is bounded above and below by a polynomial at infinity, then the mixing coefficient sequence of the associated distribution process is geometrically decaying. When f has a singularity on the imaginary axis, we show that there is no mixing property. These results are compared to those of Ibragimov and Rozanov [8], Rozanov [12] and Viano et al. [14] for ordinary time continuous processes.

The family of fractional distribution ARMA processes presents all the advantages of a parametrized family: it can be used in fields as diverse as finance, rugosity of surfaces and phenomena connected to viscoelasticity. As a family of distribution processes it is a good tool to describe diffusion phenomena across complex surfaces.

2. DESCRIPTION OF THE FILTER

We deal with linear distribution processes on $\boldsymbol{R}$ with the distribution filter f defined by

$$\forall \varphi \in \mathscr{C}_0^\infty(\boldsymbol{R}),\ \langle X, \varphi\rangle = \int \check{f} * \varphi(s)\, dW_s. \tag{3}$$

These processes have been widely studied in [1] and, in particular, we know that the expression (3) is well defined if and only if f belongs to the Sobolev space $H^{-\infty}(\boldsymbol{R}) = \bigcup_{s\in\boldsymbol{R}} H^s(\boldsymbol{R})$, where $H^s(\boldsymbol{R})$, $s\in\boldsymbol{R}$, is defined by

$$H^s(\boldsymbol{R}) = \{f\in\mathscr{S}'(\boldsymbol{R});\ \langle\xi\rangle^s\hat{f}(\xi)\in L^2(\boldsymbol{R}^n)\},$$

where $\langle\xi\rangle^s = (1+|\xi|)^{s/2}$ and $\mathscr{S}'(\boldsymbol{R})$ is the space of tempered distributions.

In this paper we focus on the case where f is the Laplace transform of a transfer function F. We will make the following hypotheses on F:

(H1) F is holomorphic in the domain

$$\mathscr{D} = \boldsymbol{C}\backslash\{z;\ \mathrm{Re}(z) \leqslant a \text{ and } |\mathrm{Im}(z)| \leqslant K|\mathrm{Re}(z)|\}.$$

(H2) $\exists N > 0,\ \exists C,\ |F(z)| \leqslant C\langle|z|\rangle^N$ in $\mathscr{D}$.

The Laplace transform is defined by

$$\forall t > 0,\ \mathscr{L}(f)(s) = F(s) = \langle f(t), e^{-st}\rangle \quad \text{for } s\in\boldsymbol{C},\ \mathrm{Re}(s) > a.$$

When F satisfies the hypotheses (H1) and (H2), the inverse Laplace transform f, written $f = \mathscr{L}^{-1}(F)$ in $\mathscr{D}'(\boldsymbol{R}_+)$, is well defined (see [13]). We will pay a particular attention to the case where

$$F(s) = \prod_{k=1}^{K} (s-a_k)^{d_k} \quad \text{with } a_k\in\boldsymbol{C},\ d_k\in\boldsymbol{C},\ k\in\boldsymbol{N}.$$

Distribution processes obtained in this way for admissible f generalize the fractional ARMA processes as introduced by Viano et al. [14].

We firstly describe the behaviour of the distribution f. The distribution process (3) being well defined only if $f\in H^{-\infty}(\boldsymbol{R})$, we study the conditions for F under which this property is attained.

Proposition 1. *If F satisfies the assumptions* (H1) *and* (H2), *then $f\in H^{-\infty}(\boldsymbol{R})$ if $a < 0$ or $a = 0$ and $F(i\xi)\in L^2_{\mathrm{loc}}(\boldsymbol{R})$. In this case f belongs to the Besov $B_{2,\infty}^{-N-1/2}(\boldsymbol{R})$.*

Proof. If $F(i\xi)$ is locally integrable, $\hat{f}(\xi) = F(i\xi)$ and there exists $q\in\boldsymbol{Z}$ such that $\langle\xi\rangle^q F(i\xi)\in L^2(\boldsymbol{R})$. Hence $f\in H^q(\boldsymbol{R})$ and $f\in H^{-\infty}(\boldsymbol{R})$, and if $a < 0$, then $F(i\xi)$ is locally integrable.

We denote by $B(0, r)$ the ball with radius r in $\boldsymbol{R}^n$. Let $\psi\in\mathscr{C}_0^\infty(B(0, 2))$, $0 \leqslant \psi \leqslant 1$ and $\psi \equiv 1$ on $B(0, 1)$. Then we have in $\mathscr{C}^\infty(\boldsymbol{R}^n)$

$$\lim_{q\to+\infty} \psi(2^{-q}\xi) = 1.$$

We write $\chi(\xi) = \psi(2^{-1}\xi)-\psi(\xi)$, $\chi_k(\xi) = \chi(2^{-k}\xi)$ to give

$$\lim_{q\to+\infty} \psi(2^{-q}\xi) = \psi(\xi)+ \sum_{k=0}^{\infty} \chi_k(\xi). \tag{4}$$

For $k = -1, 0, \dots$ we define the operator Δ_k by: for $u \in \mathscr{S}'(\mathbf{R})$,

$$\mathscr{F}(\Delta_k u)(\xi) = \chi_k(\xi)\hat{u}(\xi),\ k \geqslant 0, \qquad \mathscr{F}(\Delta_{-1}u)(\xi) = \psi(\xi)\hat{u}(\xi).$$

We write $\hat{\phi}(\xi) = \chi(\xi)$ and, for $k \geqslant 0$, $\phi_k(t) = 2^k\phi(2^k t)$. We then have $\hat{\phi}_k(\xi) = \chi_k(\xi)$ and $\Delta_k u = \phi_k * u$ for $k \geqslant 0$, $\Delta_{-1}u = \mathscr{F}^{-1}(\psi) * u$.

The Besov space $B^s_{p,q}(\mathbf{R})$, $s \in \mathbf{R}$, $1 \leqslant p \leqslant +\infty$, $1 \leqslant q \leqslant +\infty$, is defined by

$$B^s_{p,q}(\mathbf{R}) = \{u \in \mathscr{S}'(\mathbf{R});\ \exists (c_k) \in l^q,\ \forall k \geqslant -1,\ \|\Delta_k u\|_{L^p(\mathbf{R})} \leqslant c_k 2^{-ks}\}.$$

If F satisfies the assumptions (H1) and (H2), then we have

$$\|\Delta_k f\|_{L^2(\mathbf{R}^n)} = \|\mathscr{F}(\Delta_k f)\|_{L^2(\mathbf{R}^n)} = \|\chi_k(\xi) F(i\xi)\|_{L^2(\mathbf{R}^n)}.$$

Now

$$|\chi_k(\xi) F(i\xi)| \leqslant C|\xi|^N \mathbf{1}_{[2^k, 2^{k+2}]} \leqslant C 2^{kN} \mathbf{1}_{[2^k, 2^{k+2}]}$$

and

$$\|\Delta_k f\|_{L^2(\mathbf{R}^n)} \leqslant K 2^{-k(-N-1/2)} \quad \text{for all } k \geqslant -1,$$

and therefore $f \in B^{-N-1/2}_{2,\infty}(\mathbf{R})$.

We now study the conditions for F under which f is analytic.

PROPOSITION 2. *If F satisfies the assumptions* (H1) *and* (H2) *and if $a < 0$, then f is an analytic function for $t > 0$. More precisely, there exists $K' > 1$ such that f can be extended holomorphically to* $\{|\mathrm{Im}(t)| \leqslant (1/K')\,\mathrm{Re}(t)\}$.

Proof. Let $\delta > 0$; then for $\varphi \in \mathscr{C}^\infty_0([\delta, +\infty[)$ we have

$$\langle f(t), \varphi\rangle = \langle F(i\xi), \hat{\varphi}(\xi)\rangle = \lim_{\varepsilon \to 0} \langle \exp[-\varepsilon\xi^2] F(i\xi), \hat{\varphi}(-\xi)\rangle$$

$$= \lim_{\varepsilon \to 0} \Big\langle \int_{\mathbf{R}} \exp[i\xi t]\exp[-\varepsilon\xi^2] F(i\xi)\, d\xi,\ \varphi(t)\Big\rangle.$$

For $t > \delta$, let

$$f_\varepsilon(t) = \int \exp[i\xi t]\exp[-\varepsilon\xi^2] F(i\xi)\, d\xi = \frac{1}{i}\int_\Gamma \exp[zt]\exp[\varepsilon z^2] F(z)\, dz,$$

where $\Gamma = i\mathbf{R}$. Let $a' > a$ and $K' > \sup(K, 1)$. We write $\Gamma' = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ with

$$\Gamma_1 = \{z = \xi + i\zeta;\ \xi < a',\ \zeta = -K'\xi\},$$

$$\Gamma_2 = \{z = \xi + i\zeta;\ \xi = a',\ \zeta \in [K'a',\ -K'a']\},$$

$$\Gamma_3 = \{z = \xi + i\zeta;\ \xi < a',\ \zeta = K'\xi\}.$$

The function $\exp[zt]\exp[\varepsilon z^2] F(z)$ is holomorphic in $\mathscr{D}$ and

$$\lim_{\beta \to +\infty} \left| \frac{1}{i} \int_{-\beta}^{0} \exp[(\xi + iK\beta)t]\exp[\varepsilon(\xi + iK\beta)^2] F(\xi + iK\beta)\, d\xi \right|$$

$$\leqslant \lim_{\beta \to +\infty} \int_{-\beta}^{0} \exp[\xi t]\exp[\varepsilon(\xi^2 - (K\beta)^2)]\, |F(\xi + iK\beta)|\, d\xi = 0.$$

Hence we have

$$f_\varepsilon(t) = \frac{1}{i} \int\limits_{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3} \exp[zt] \exp[\varepsilon z^2] F(z)\, dz$$

$$= \int\limits_{\xi < a'} \exp[(\xi - iK'\xi)t] \exp[\varepsilon(\xi - iK'\xi)^2] F(\xi - iK'\xi) \frac{1 - iK'}{i} d\xi$$

$$+ \int\limits_{K'a'}^{-K'a'} \exp[(a' + i\xi)t] \exp[\varepsilon(a' + i\xi)^2] F(a + i\xi)\, d\xi$$

$$+ \int\limits_{\xi < a'} \exp[(\xi + iK'\xi)t] \exp[\varepsilon(\xi + iK'\xi)^2] F(\xi + iK'\xi) \frac{1 + iK'}{i} d\xi.$$

In each of the three integrals we can let ε tend to 0. Hence for $t > 0$ we have

$$f(t) = \int\limits_{\xi < a'} \exp[(\varepsilon - iK'\xi)t] F(\xi - iK'\xi) \frac{1 - iK'}{i} d\xi$$

$$+ \int\limits_{-K'a'}^{K'a'} \exp[(a' + i\xi)t] F(a + i\xi)\, d\xi$$

$$+ \int\limits_{\xi < a'} \exp[(\xi + iK'\xi)t] F(\xi + iK'\xi) \frac{1 + iK'}{i} d\xi.$$

The first integral can be extended to $t \in \boldsymbol{C}$ if, for all ξ, $\xi < a'$ and $\mathrm{Re}(\xi t - iK'\xi t) < 0$, i.e. if

$$\mathrm{Im}(t) > -(1/K')\,\mathrm{Re}(t).$$

The second integral is an entire function of t. The third integral can be extended to $t \in \boldsymbol{C}$ if, for all ξ, $\xi < a'$ and $\mathrm{Re}(\xi t - iK'\xi t) < 0$, i.e. if

$$\mathrm{Im}(t) < (1/K')\,\mathrm{Re}(t).$$

Hence the proposition is proved.

We now study the case where

$$F(s) = \prod_{k=1}^{K} (s - a_k)^{d_k} \quad \text{with } a_k \in \boldsymbol{C},\ d_k \in \boldsymbol{C},\ k \in \boldsymbol{N}.$$

The singular points of F are the points a_k such that $d_k \notin \boldsymbol{N}$. We use the following notation:

E* denotes the set of indices of the singular points of F;

E** denotes the set of indices of the singular points that have the largest real part;

$a = \sup_{k \in E^*} \{\mathrm{Re}(a_k)\}$;

$d = \inf_{k \in E^{**}} \{\mathrm{Re}(d_k)\}$;

$D = \sum_{k=1}^{K} d_k$; for the sake of simplicity we assume that D is real.

Then F is well defined for $\mathrm{Re}(s) > a$ and satisfies the assumptions (H1) and (H2).

Let δ be the Dirac mass in $t=0$ and $\delta^{(j)}$ its j-th derivative in the distribution sense. Later on, $\mathrm{vp}(t^\lambda)$ will denote the principal value of t^λ, i.e.

$$\mathrm{vp}(t^\lambda) = t^\lambda \quad \text{if } \mathrm{Re}(\lambda) > -1,$$

$$\mathrm{vp}(t^\lambda) = \frac{(-1)^n}{\prod_{j=1}^n (-\lambda-j)} \partial^{(n)}(t^{\lambda+n}) \quad \text{if } \lambda \notin \boldsymbol{Z},\ -\mathrm{Re}(\lambda)+n > -1,\ n \in \boldsymbol{N}.$$

PROPOSITION 3. *Let* $F(s) = \prod_{k=1}^K (s-a_k)^{d_k}$, $D = \sum_{k=1}^K d_k$, *and* f *be the inverse Laplace transform of* F. *Then*

$$f(t) = \delta^{(D)}(t) + \sum_{j=1}^{D} \gamma_j \delta^{(D-j)}(t) + \sum_{j=D+1}^{\infty} \gamma_j \frac{t^{-(D-j+1)}}{\Gamma(j-D)} \quad \textit{for } D \in \boldsymbol{N},$$

$$f(t) = \frac{1}{\Gamma(-D)} \mathrm{vp}(t^{-(D+1)}) + \sum_{j=1}^{\infty} \gamma_j \frac{\mathrm{vp}(t^{-(D-j+1)})}{\Gamma(-D+j)} \quad \textit{for } D \in \boldsymbol{R}\backslash\boldsymbol{N}.$$

Proof. We have

$$\mathscr{L}^{-1}(s^\lambda) = \begin{cases} \delta^{(\lambda)} & \text{if } \lambda \in \boldsymbol{N}, \\ \dfrac{1}{\Gamma(-\lambda)} \mathrm{vp}(t^{-\lambda-1}) & \text{if } \lambda \in \boldsymbol{R}\backslash\boldsymbol{N}. \end{cases}$$

We can write

$$F(s) = s^D \prod_{k=1}^{K} \left(1 - \frac{a_k}{s}\right)^{d_k} = s^D \left(1 + \sum_{j=1}^{\infty} \frac{\gamma_j}{s^j}\right),$$

and the series converges in the domain $\{|s| > \max_{k\in E^*} |a_k|\}$. Let $J = [D]$ if $D \geqslant 0$ and $J = 0$ otherwise. Then

$$F(s) = s^D + \sum_{j=1}^{\infty} \frac{\gamma_j}{s^{j-D}} = s^D + \sum_{j=1}^{J} \frac{\gamma_j}{s^{j-D}} + \sum_{j=J+1}^{\infty} \frac{\gamma_j}{s^{j-D}}.$$

The series $\sum_{j=J+1}^{\infty} (\gamma_j/s^{j-D})$ converges absolutely for $|s| > r_0$ such that

$$\{\mathrm{Re}(s) > r_0\} \subset \{|s| > \max_{k\in E^*} |a_k|\}.$$

Hence (see [2])

$$\mathscr{L}^{-1}\left(\sum_{j=J+1}^{\infty} \frac{\gamma_j}{s^{j-D}}\right) = \sum_{j=J+1}^{\infty} \gamma_j \frac{t^{j-D-1}}{\Gamma(j-D)} = t^{-D-1} \sum_{j=J+1}^{\infty} \gamma_j \frac{t^j}{\Gamma(j-D)}$$

and we obtain the assertion of the proposition.

We now study the behaviour of $f(t)$ at infinity.

PROPOSITION 4. *For $N > 0$, there exist complex numbers $c_j^{(k)}$ and a constant C such that for $t > 1$*

$$f(t) = \sum_{k\in E^{**}} \exp[a_k t] \sum_{j=0}^{J_k} \frac{c_j^{(k)}}{\Gamma(-d_k-j)} t^{-(d_k+j+1)} + R(t) \qquad \text{with } |R(t)| \leqslant \frac{C}{t^{N+1}},$$

where $J_k = [N - \mathrm{Re}(d_k)] - 1$. In particular, $f(t) \sim A e^{at} t^{-(d+1)}$ in the neighbourhood of infinity.

Remark. This proposition shows that in order that $f \in L^2([1, +\infty[)$ the following conditions are to be satisfied: $a < 0$ or ($a = 0$ and $d > -\frac{1}{2}$).

The proof of Proposition 4 is very tedious and is presented in Appendix A. The following corollary is derived directly from Proposition 1.

COROLLARY 5. *$f \in H^{-\infty}(\boldsymbol{R})$ if and only if $a < 0$ or if ($a = 0$ and $d > -\frac{1}{2}$).*

Proof. If $a > 0$ or if ($a = 0$ and $d \leqslant -\frac{1}{2}$), then according to Proposition 4, f is not locally integrable, and hence does not belong to $H^{-\infty}(\boldsymbol{R})$.

PROPOSITION 6. *If $F(s) = \prod_{k=1}^{K} (s-a_k)^{d_k}$, then $f = \mathscr{L}^{-1}(F)$ satisfies a K-th order differential equation whose coefficients are affine with respect to t.*

Proof. If $F(s) = \prod_{k=1}^{K} (s-a_k)^{d_k}$, then $F'(s)/F(s) = B(s)/C(s)$, where $B(s)$ and $C(s)$ are polynomials of s:

$$C(s) = \prod_{k=1}^{K} (s-a_k), \qquad B(s) = \sum_{k=1}^{K} d_k \prod_{j\neq k} (s-a_j) = Ds^{K-1} + \sum_{j\leqslant K-2} b_j s^j.$$

Hence we have

$$\mathscr{L}^{-1}(C(s)F'(s)) = \mathscr{L}^{-1}(B(s)F(s)) \qquad \text{and} \qquad C(\partial_t)(-tf) = B(\partial_t)f(t).$$

Now

$$C(\partial_t)(tf) = t\partial_t^K f + K\partial_t^{K-1} f - t\Big(\sum_{k=1}^{K} a_k\Big)\partial_t^{K-1} f + \sum_{j\leqslant K-2} \alpha_j(t)\,\partial_t^j f$$

and

$$B(\partial_t)f = D\partial_t^{K-1} f + \sum_{j\leqslant K-2} \beta_j(t)\,\partial_t^j f.$$

Therefore

$$t\partial_t f + \Big(-t\Big(\sum_{k=1}^{K} a_k\Big) + (K+D)\Big)\partial_t^{K-1} f + \sum_{j\leqslant K-2} \gamma_j(t)\,\partial_t^j f = 0,$$

where the coefficients $\gamma_j(t)$ are affine with respect to t, proving the proposition.

3. FRACTIONAL DISTRIBUTION PROCESSES

DEFINITION 1. Let F be a function on C that satisfies the assumptions (H1) and (H2) such that the inverse Laplace transform f of F belongs to $H^{-\infty}(\boldsymbol{R})$. The distribution process with transfer function F is the process with filter f defined by

$$\forall \varphi \in \mathscr{C}_0^\infty(\boldsymbol{R}^n), \ \langle X, \varphi \rangle = \int \check{f} * \varphi(s)\, dW_s.$$

If F is of the form $F(s) = \prod_{k=1}^{K} (s-a_k)^{d_k}$, where $a_k \in \boldsymbol{C}$, $d_k \in \boldsymbol{C}$, $k \in \boldsymbol{N}$, then the process is called a *fractional distribution process.*

3.1. Regularity and covariance. The regularity of the distribution process X can be obtained directly from the parameters of the transfer function F.

PROPOSITION 7. *Let X be a fractional distribution process with transfer function*

$$F(s) = \prod_{k=1}^{K} (s-a_k)^{d_k}, \qquad D = \sum_{k=1}^{K} d_k.$$

Then X belongs to $C^{-D-1/2}(\boldsymbol{R}, L^2(\Omega))$.

Proof. This result follows immediately from Proposition 1 and the results in [1] (i.e. the distribution process X with filter f belongs to $C^s(\boldsymbol{R}^n, L^2(\Omega))$ if and only if $f \in B^s_{2,\infty}(\boldsymbol{R}^n)$. In particular we obtain the following results of [14]:

If $-3/2 < D < -1/2$, then X has Hölderian paths.

If $D < -3/2$, then X has continuous and differentiable paths.

THEOREM 8. *The covariance of the fractional distribution process with transfer function $F(s) = \prod_{k=1}^{K} (s-a_k)^{d_k}$, $D = \sum_{k=1}^{K} d_k$, has the following form:*

If $2D$ is not an integer, then

$$\sigma = -\frac{\Gamma(2D+1)}{\pi}\sin(D\pi)\,|t|^{-(2D+1)}$$

$$+i \sum_{\substack{j\,odd\\ j\geq 1}} c_j \frac{\Gamma(2D-j+1)}{\pi}\cos\frac{(2D-j)\pi}{2}\left(t_+^{-(2D+1-j)} - t_-^{-(2D+1-j)}\right)$$

$$-\sum_{\substack{j\,even\\ j\geq 2}} c_j \frac{\Gamma(2D-j+1)}{\pi}\sin\frac{(2D-j)\pi}{2}\,|t|^{-(2D+1-j)} + h(t).$$

If $2D$ is an even positive integer, then

$$\sigma = (-1)^D \delta^{(2D)} + (-1)^D \sum_{1\leq j\leq 2D} c_j i^j \delta^{(2D-j)}$$

$$+(-1)^D \sum_{\substack{j\,odd\\ j\geq 2D+1}} c_j \frac{i^j}{2(-2D+j-1)!}\left(t_+^{-(2D+1-j)} - t_-^{-(2D+1-j)}\right)$$

$$+(-1)^D \sum_{\substack{j\,even\\ j>2D+1}} c_j \frac{i^j}{2(-2D+j-1)!}|t|^{-(2D+1-j)}+h(t).$$

If 2*D is an odd positive integer, then*

$$\sigma = (i)^{2D+1}\frac{(2D)!}{\pi}\mathrm{vp}(t^{-(2D+1)})$$

$$+(-1)^D \sum_{1\leqslant j\leqslant 2D} c_j \frac{i^{-j+1}}{\pi}(2D-j)!\,\mathrm{vp}(t^{-(2D+1-j)})$$

$$+(-1)^D \sum_{j\geqslant 2D+1} c_j i^{-j+1}\frac{-\log|t|}{\pi(-2D+j-1)!}t^{-(2D+1-j)}+h(t).$$

If 2*D is an even negative integer, then*

$$\sigma = (-1)^D \sum_{j\,odd} c_j \frac{i^j}{2(-2D+j-1)!}(t_+^{-(2D+1-j)}-t_-^{-(2D+1-j)})$$

$$+(-1)^D \sum_{j\,even} c_j \frac{i^j}{2(-2D+j-1)!}|t|^{-(2D+1-j)}+h(t).$$

If 2*D is an odd negative integer, then*

$$\sigma = (-1)^D \sum_{j\geqslant 0} c_j i^{j-1}\frac{\log|t|}{\pi(-2D+j-1)!}t^{-(2D+1-j)}+h(t),$$

where $h(t)$ *is an analytic function and the coefficients* c_j *are given by the development of* $|F(i\lambda)|^2$ *at infinity.*

The proof of the theorem is given in Appendix B because the calculation of the covariance is very long.

Remark. When f is real and if $D < -\frac{1}{2}$, then we obtain the result given in [14], i.e.

$$\sigma = \sigma_0 - \frac{\Gamma(2D+1)}{\Gamma(D)}|t|^{-(2D+1)}+t^2\varepsilon(t) \quad \text{if } -\tfrac{3}{2}<D<-\tfrac{1}{2},$$

$$\sigma = \sigma_0+\sigma_2 t+t^2\varepsilon(t) \quad \text{if } D<-\tfrac{3}{2},$$

$$\sigma \sim \frac{1}{2\pi}t^2\log|t| \quad \text{if } D=-\tfrac{3}{2}.$$

The behaviour of the covariance in the neighbourhood of infinity is determined by the parameter a. Viano et al. [14] showed that if $a<0$, then

$$\sigma(t) = o(e^{(a+\varepsilon)t}) \quad \text{with } \varepsilon>0 \text{ and } a+\varepsilon<0,$$

and if $a=0$, then

$$\sigma(t) \sim \sum_{j\in E^{**}} \lambda_j e^{a_j t} t^{-2d_j-1}.$$

This proves in particular that σ is not summable (hence X is with long range) if and only if $a=0$ and $d>-\frac{1}{2}$.

3.2. Mixing properties. In order to define mixing coefficients and properties for distribution processes that extend the usual definitions for temporal processes [4] we will replace the remoteness in time by remoteness on the support of the test functions. Let X be a stationary distribution process, and let

$\mathscr{H}_{-\infty}^{0}$ and $\mathscr{H}_{T}^{+\infty}$ be the vectorial subspaces spanned respectively by

$$\{X(\varphi);\ \varphi\in\mathscr{C}_0^\infty(]-\infty,0])\} \quad \text{and} \quad \{X(\psi);\ \psi\in\mathscr{C}_0^\infty([T,+\infty[)\};$$

$\mathscr{U}_{-\infty}^{0}$ and $\mathscr{U}_{T}^{+\infty}$ be the algebras respectively spanned by the two previous families of variables;

$\mathscr{M}_{-\infty}^{0}$ and $\mathscr{M}_{T}^{+\infty}$ be the spaces of $L^2(\Omega)$ variables, respectively measurable for the two previous algebras.

Definition 2. The *linear mixing, ϱ-mixing* and *strong mixing coefficients* for the distribution process X are defined by

$$r_T=\sup\{|\mathrm{corr}(Y,Z)|;\ Y\in\mathscr{H}_{-\infty}^{0},\ Z\in\mathscr{H}_{T}^{+\infty}\},$$

$$\varrho_T=\sup\{|\mathrm{corr}(Y,Z)|;\ Y\in\mathscr{M}_{-\infty}^{0};\ Z\in\mathscr{M}_{T}^{+\infty}\},$$

$$\alpha_T=\sup\{|P(A\cap B)-P(A)P(B)|;\ A\in\mathscr{U}_{-\infty}^{0},\ B\in\mathscr{U}_{T}^{+\infty}\},$$

$$\phi_T=\sup\{|P(B|A)-P(B)|;\ P(A)\neq 0,\ A\in\mathscr{U}_{-\infty}^{0},\ B\in\mathscr{U}_{T}^{+\infty}\}.$$

Remark. When X is a temporal process, these coefficients coincide with the usual linear mixing, ϱ-mixing, strong mixing and ϕ-mixing coefficients. Other mixing coefficients can be defined in the same way.

These processes can be considered as a family of random variables indexed by the set $\mathscr{C}_0^\infty(\boldsymbol{R})$ of test functions. It then follows from [12] that when the process is Gaussian, the links between the several conditions for mixing are the same as for temporal processes:

– Coefficients r_T and ϱ_T are identical.

– The estimation $\alpha_T\leqslant\varrho_T\leqslant 2\pi\alpha_T$ holds, and it follows that for Gaussian distribution processes X, linear mixing, ϱ-mixing and α-mixing are equivalent.

– Likewise we can prove directly, using the same argument as in the temporal case [4], that ϕ-mixing is equivalent to m-dependence, i.e.

$$\lim_{T\to+\infty}\phi_T=0$$

if and only if there exists $m>0$ such that $\mathscr{U}_{-\infty}^{0}$ and $\mathscr{U}_{m}^{+\infty}$ are independent.

Assume now that X has a spectral density of the form $|F(i\lambda)|^2$, where F satisfies the assumptions (H1) and (H2). Our aim is to give a sufficient condition for the process X to be ϱ-mixing and a sufficient condition for X not to be ϱ-mixing. A necessary and sufficient condition for ARMA distribution processes to be mixing then holds.

THEOREM 9. *If the spectral density of the distribution process X is $|F(i\lambda)|^2$, and if F satisfies the assumptions* (H1) *and* (H2), *if $a<0$ and if moreover there exist C, A such that, for $|z|>A$, $C|z|^N \leqslant |F(z)|$, then the ϱ-mixing coefficient of the distribution process X tends to* 0 *when T tends to infinity. In this case we have*

$$\varrho_T = O(e^{bT}) \quad \textit{for all } b\in]a, 0[.$$

Proof. Assume first that $F(z)$ does not vanish on the imaginary axis. Let $H(z) = F(z)\bar{F}(-\bar{z})$. Then $H(z)$ is holomorphic in the set

$$\{z;\ a<\operatorname{Re}(z)<-a\} \cup \{|\operatorname{Im}(z)| > K|\operatorname{Re}(z)|\}$$

and $|F(i\xi)|^2 = H(i\xi)$. Assume that $\varphi\in\mathscr{C}_0^\infty(]-\infty, 0])$, $\psi\in\mathscr{C}_0^\infty([T, +\infty[)$, and $\chi\in\mathscr{C}^\infty$ are such that $\operatorname{Supp}(\chi)\subset]-\infty, -1+\delta[$, $0<\delta<1$, with $\chi\equiv 1$ on $]-\infty, -1[$. Using the Fubini result we have

$$\begin{aligned}
\operatorname{cov}(X(\varphi), X(\psi)) &= \int \hat{\varphi}(\xi)\overline{\hat{\psi}}(\xi)|F(i\xi)|^2\,d\xi = \int \hat{\varphi}(\xi)\overline{\hat{\psi}}(\xi) H(i\xi)\,d\xi \\
&= \lim_{\varepsilon\to 0}\int \hat{\varphi}(\xi)\overline{\hat{\psi}}(\xi)\exp[-\varepsilon\xi^2] H(i\xi)\,d\xi \\
&= \lim_{\varepsilon\to 0}\int\Big(\int \exp[-it\xi]\,\varphi(t)\,dt \int \exp[is\xi]\,\overline{\psi}(s)\,ds\Big)\exp[-\varepsilon\xi^2] H(i\xi)\,d\xi \\
&= \lim_{\varepsilon\to 0}\iint \varphi(t)\overline{\psi}(s)\chi\Big(\frac{t-s}{T}\Big)\int \exp[-i(t-s)\xi]\exp[-\varepsilon\xi^2] H(i\xi)\,d\xi\,ds\,dt \\
&= \lim_{\varepsilon\to 0}\iint \varphi(t)\overline{\psi}(s) K_\varepsilon^T(s, t)\,dt\,ds
\end{aligned}$$

with

$$\begin{aligned}
K_\varepsilon^T(s, t) &= \int_{\boldsymbol{R}} \chi\Big(\frac{t-s}{T}\Big)\exp[-i(t-s)\xi]\exp[-\varepsilon\xi^2] H(i\xi)\,d\xi \\
&= \int_{\Gamma} \chi\Big(\frac{t-s}{T}\Big)\exp[-(t-s)z]\exp[\varepsilon z^2] H(z)\frac{dz}{i},
\end{aligned}$$

where $\Gamma = i\boldsymbol{R}$. We can change the contour:

$$K_\varepsilon^T(s, t) = \int_{\Gamma'} \chi\Big(\frac{t-s}{T}\Big)\exp[-(t-s)z]\exp[\varepsilon z^2] H(z)\frac{dz}{i},$$

where $\Gamma' = \Gamma_1\cup\Gamma_2\cup\Gamma_3$ with

$$\begin{aligned}
\Gamma_1 &= \{z=\lambda+i\zeta;\ \lambda<b,\ \zeta=-K'\lambda\}, \\
\Gamma_2 &= \{z=\lambda+i\zeta;\ \lambda=b,\ \zeta\in[K'b, -K'b]\}, \\
\Gamma_3 &= \{z=\lambda+i\zeta;\ \lambda<b,\ \zeta=K'\lambda\},
\end{aligned}$$

where $b > a$, $K' > K$ and $K' > 1$. We let ε tend to 0 in each of the integrals and

$$\lim_{\varepsilon\to 0} K_\varepsilon^T(s,t) = K_0^T(s,t) = \int_{\Gamma'} \chi\left(\frac{t-s}{T}\right) e^{-(t-s)z} H(z)\frac{dz}{i}.$$

It then follows that

$$\operatorname{cov}\big(X(\varphi), X(\psi)\big) = \iint \varphi(t)\overline{\psi}(s) K_0^T(s,t)\,dt\,ds.$$

LEMMA 10. *K_0^T is a continuous operator from $H^m(\mathbf{R})$ to $H^{-m}(\mathbf{R})$ for all $m \in \mathbf{R}$. Moreover*

$$\|K_0^T\|_{H^m(\mathbf{R})\to H^{-m}(\mathbf{R})} \leqslant Ce^{b(1-\delta)T}.$$

Assume the lemma has been proved. Since $F(i\xi)$ does not vanish for $\xi \in \mathbf{R}$, there exist constants $A, B > 0$ such that

$$A\langle\xi\rangle^{2N} \leqslant |F(i\xi)|^2 \leqslant B\langle\xi\rangle^{2N}.$$

On the other hand,

$$\|X(\varphi)\|^2_{L^2(\Omega)} = \|\check{f} * \varphi\|^2_{L^2(\mathbf{R})} = \|F(i\xi)\hat{\varphi}(\xi)\|^2_{L^2(\mathbf{R})}.$$

Hence

$$A|\langle\xi\rangle^N \hat{\varphi}(\xi)|^2 \leqslant |F(i\xi)\hat{\varphi}(\xi)|^2 \leqslant B|\langle\xi\rangle^N \hat{\varphi}(\xi)|^2$$

and

$$A\|\varphi\|_{H^N(\mathbf{R})} \leqslant \|X(\varphi)\|_{L^2(\Omega)} \leqslant B\|\varphi\|_{H^N(\mathbf{R})}.$$

From the lemma we have

$$\begin{aligned}
\big|\operatorname{cov}\big(X(\varphi), X(\psi)\big)\big| &= |\langle K_0^T\varphi, \psi\rangle| \leqslant \|K_0^T\varphi\|_{H^{-N}(\mathbf{R})}\|\psi\|_{H^N(\mathbf{R})} \\
&\leqslant \|K_0^T\|\,\|\varphi\|_{H^N(\mathbf{R})}\|\psi\|_{H^N(\mathbf{R})} \\
&\leqslant Ce^{b(1-\delta)T}\|X(\varphi)\|_{L^2(\Omega)}\|X(\psi)\|_{L^2(\Omega)}.
\end{aligned}$$

Therefore

$$\operatorname{corr}\big(X(\varphi), X(\psi)\big) = \frac{\operatorname{cov}\big(X(\varphi), X(\psi)\big)}{\|X(\varphi)\|_{L^2(\Omega)}\|X(\psi)\|_{L^2(\Omega)}} \leqslant Ce^{b(1-\delta)T},$$

and from Definition 2 we obtain

(5) $$\begin{aligned}\varrho_T = r_T &= \sup\{\operatorname{corr}\big(X(\varphi), X(\psi)\big);\ \varphi \in \mathscr{C}_0^\infty(]-\infty, 0]),\ \psi \in \mathscr{C}_0^\infty(]T, +\infty])\} \\ &= O(e^{b(1-\delta)T}).\end{aligned}$$

P r o o f o f L e m m a 10. Assume we have proved that, for all $\alpha, \beta \in \mathbf{N}$, $\partial_t^\alpha \partial_s^\beta K_0(s,t)$ operates continuously from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$ with

$$\|\partial_t^\alpha \partial_s^\beta K_0(s,t)\|_{L^2(\mathbf{R})\to L^2(\mathbf{R})} \leqslant Ce^{b(1-\delta)T}.$$

Let $m \in \mathbf{R}$ for all u in $H^m(\mathbf{R})$. There exists $M \in \mathbf{N}$ such that $u \in H^{-2M}(\mathbf{R})$. Thus we have $u_M = (1-\Delta)^{-M} u \in L^2(\mathbf{R})$, where Δ denotes the Laplacian operator, and

$$\|u\|_{H^{-2M}(\mathbf{R})} = \|(1-\Delta)^{-M} u\|_{L^2(\mathbf{R})} \leqslant C\,\|u\|_{H^m(\mathbf{R})}.$$

Let $v \in H^m(\mathbf{R})$; then

$$|\langle K_0^T u, v\rangle| = \langle (1-\Delta_s)^M K_0^T (1-\Delta_t)^M u_M, v_M\rangle$$
$$= \langle \sum_{\alpha,\beta} C_{\alpha,\beta}\, \partial_t^\alpha \partial_s^\beta K_0 u_M, v_M\rangle$$

and we have

$$|\langle K_0^T u, v\rangle| \leqslant C e^{b(1-\delta)T} \|u\|_{H^{-2M}(\mathbf{R})} \|v\|_{H^{-2M}(\mathbf{R})} \leqslant C e^{b(1-\delta)T} \|u\|_{H^m(\mathbf{R})} \|v\|_{H^m(\mathbf{R})},$$

which proves the lemma.

We now show that, for all $\alpha, \beta \in N$, $\partial_t^\alpha \partial_s^\beta K_0(s, t)$ operates continuously from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$ with $\|\partial_t^\alpha \partial_s^\beta K_0(s, t)\|_{L^2(\mathbf{R})\to L^2(\mathbf{R})} \leqslant C e^{b(1-\delta)T}$. We have

$$\partial_t^\alpha \partial_s^\beta K_0(s, t) = \int_{\Gamma'} \sum_{l,j} C_{l,j} \frac{1}{T^{l+j}} (\partial^{l+j}\chi)\left(\frac{t-s}{T}\right) z^{\alpha+\beta-l-j} e^{-(t-s)z} H(z) \frac{dz}{i}$$
$$= \sum_k C_k \tilde{K}_k^p(s, t),$$

where

$$\tilde{K}_k^p(s, t) = \int_{\Gamma_p} \frac{1}{T^k} \chi_k\left(\frac{t-s}{T}\right) z^{\alpha+\beta-k} e^{-(t-s)z} H(z) \frac{dz}{i} \quad \text{with } \chi_k = \partial^k \chi.$$

For $z \in \Gamma_1 \cup \Gamma_3$ there exist C and $N > 0$ such that

$$|H(z)| \leqslant C\,|z|^N$$

and

$$\int_{\mathbf{R}} |\tilde{K}_k^1(s, t)|\,dt \leqslant C \int_{-\infty}^{b} \int_{t-s<T(\delta-1)} \frac{1}{T^k} \left|\chi_k\left(\frac{t-s}{T}\right)\right| |z|^{\alpha+\beta-k+N} e^{(s-t)z}\,dt dz$$
$$\leqslant C \int_{-\infty}^{b} \frac{1}{T^k} |z|^{\alpha+\beta-k+N} \int_{u<-T(1-\delta)} e^{-uz}\,du dz$$
$$\leqslant \frac{C}{T^k} \int_{-\infty}^{b} |z|^{\alpha+\beta-k+N-1} e^{T(1-\delta)z}\,dz$$
$$\leqslant \frac{C}{T^k} \int_{-\infty}^{b} |z|^\gamma e^{(T(1-\delta)+\varepsilon-\varepsilon)z}\,dz \leqslant \frac{C}{T^k} e^{bT(1-\delta)-\varepsilon b}.$$

In the same way we have

$$\int_{\mathbf{R}} |\tilde{K}_k^3(s, t)|\,dt \leqslant \frac{C}{T^k} e^{T(1-\delta)b}.$$

On the other hand,

$$\int_{\boldsymbol{R}} |\tilde{K}_k^2(s,t)|\,dt = C \int_{K'b}^{-K'b} \int_{t-s<-T(1-\delta)} \left| \frac{1}{T^k} \chi_k\left(\frac{t-s}{T}\right)(b+iz)^{\alpha+\beta-k} e^{-(t-s)(b+iz)} \right| dtdz$$

$$\leqslant C \int_{K'b}^{-K'b} \int_{t-s<-T(1-\delta)} \frac{1}{T^k} e^{-(t-s)b}\,dtdz \leqslant C \int_{K'b}^{-K'b} \frac{1}{T^k} \int_{u<-T(1-\delta)} e^{-ub}\,dudz$$

$$\leqslant \frac{C}{bT^k} \int_{Kb}^{-Kb} e^{T(1-\delta)b}\,dz \leqslant \frac{C}{T^k} e^{T(1-\delta)b}.$$

Hence we have

$$\sup_s \int |\partial_t^\alpha \partial_s^\beta K_0(s,t)|\,dt \leqslant \frac{C}{T^k} e^{T(1-\delta)b}$$

and it follows that $\|\partial_t^\alpha \partial_s^\beta K_0(s,t)\|_{L^2(\boldsymbol{R})\to L^2(\boldsymbol{R})} \leqslant Ce^{b(1-\delta)T}$.

Assume now that $F(i\xi)$ vanishes on the imaginary axis. From the assumptions concerning F, $F(i\xi)$ admits a finite number of zeros and we can set

$$|F(i\xi)|^2 = |P(\xi)|^2\,|G(\xi)|^2,$$

where P is a polynomial and $G(\xi) \neq 0$ for all $\xi \in \boldsymbol{R}$. We have

$$\int \hat{\varphi}(\xi)\,\overline{\hat{\psi}}(\xi)\,|F(i\xi)|^2\,d\xi = \int \mathscr{F}(P(D)\varphi)(\xi)\,\mathscr{F}(P(D)\psi)(\xi)\,|G(\xi)|^2\,d\xi$$

with

$$\operatorname{Supp} P(D)\varphi \subset]-\infty, 0] \quad \text{and} \quad \operatorname{Supp} P(D)\psi \subset [T, +\infty[.$$

$|G(\xi)|^2$ does not vanish and there exist $\tilde{N}$, A, and B such that

$$A\langle\xi\rangle^{\tilde{N}} \leqslant |G(\xi)|^2 \leqslant B\langle\xi\rangle^{\tilde{N}}.$$

We come back to the case where $|F(i\xi)|^2$ does not vanish by replacing $|F(i\xi)|^2$ with $|G(\xi)|^2$, φ with $P(D)\varphi$, and ψ with $P(D)\psi$. It then follows that

$$\operatorname{cov}(X(\varphi), X(\psi)) = \int \mathscr{F}(P(D)\varphi(\xi))\,\mathscr{F}(P(D)\psi(\xi))\,|G(\xi)|^2\,d\xi$$

$$\leqslant \frac{C}{T^k} e^{T(1-\delta)b} \left\|\mathscr{F}(P(D)\varphi)\,G\right\|_{L^2(\boldsymbol{R})} \left\|\mathscr{F}(P(D)\psi)\,G(\xi)\right\|_{L^2(\boldsymbol{R})}$$

$$\leqslant \frac{C}{T^k} e^{T(1-\delta)b} \|P\hat{\varphi}G\|_{L^2(\boldsymbol{R})} \|P\hat{\psi}G\|_{L^2(\boldsymbol{R})}$$

$$\leqslant \frac{C}{T^k} e^{T(1-\delta)b} \|X(\varphi)\|_{L^2(\Omega)} \|X(\psi)\|_{L^2(\Omega)}$$

and in this case we still have $r_T = O(e^{b(1-\delta)T})$.

We now show that if $F(i\xi)$ has singularities on the real axis, then the process X is not ϱ-mixing.

THEOREM 11. *If the spectral density of the distribution process X is $|F(i\lambda)|^2$ and if F can be written as $F(z) = (z-i\alpha)^d G(z)$ with $d \in C \backslash N$, $\alpha \in R$, and G is continuous in the neighbourhood of $i\alpha$ with $G(i\alpha) \neq 0$, then the distribution process X is not ϱ-mixing.*

Proof. We can set $\alpha = 0$. Thus we have

$$\forall \xi \in R,\ |F(i\xi)|^2 = |(i\xi)^d|^2 \tilde{G}(\xi) = C(\xi)\tilde{G}(\xi)$$

and

$$C(\xi) = |(i\xi)^d|^2 = \exp[2d_1 \log|\xi| - d_2 \pi \operatorname{sign}(\xi)] = \begin{cases} C_1 |\xi|^{2d_1} & \text{if } \xi > 0, \\ C_2 |\xi|^{2d_1} & \text{if } \xi < 0, \end{cases}$$

where $d = d_1 + id_2$.

Assume that φ and $\psi \in \mathscr{C}_0^\infty(R)$ are such that $\operatorname{Supp}\varphi \subset]-\infty, 0]$, $\operatorname{Supp}\psi \subset [\delta, +\infty[$, $\delta > 0$, and $\varphi_T(t) = \varphi(t/T)$, $\psi_T(t) = \psi(\delta t T^{-1})$. We have

$$\operatorname{Supp}\varphi_T \subset]-\infty, 0] \quad \text{and} \quad \operatorname{Supp}\psi_T \subset [T, +\infty[.$$

Assume that X is ϱ-mixing; then we have $\lim_{T\to\infty} r_T = 0$. Now

$$\left|\int \hat{\varphi}_T(\xi)\overline{\hat{\psi}}_T(\xi)|F(i\xi)|^2 d\xi\right| \leqslant r_T \left[\int |\hat{\varphi}(\xi)|^2 |F(i\xi)|^2 d\xi\right]^{1/2} \left[\int |\hat{\psi}(\xi)|^2 |F(i\xi)|^2 d\xi\right]^{1/2}.$$

Moreover,

$$\hat{\varphi}_T(\xi) = \int e^{-it\xi}\varphi(t/T)\,dt = \int e^{-iTs\xi}\varphi(s)\,T ds = T\hat{\varphi}(T\xi)$$

and

$$\begin{aligned} \int |\hat{\varphi}_T(\xi)|^2 |F(i\xi)|^2 d\xi &= \int T^2 |\hat{\varphi}(T\xi)|^2 |F(i\xi)|^2 d\xi \\ &= \int T^2 |\hat{\varphi}(\xi)|^2 C(\xi/T)\tilde{G}(\xi/T)\,d\xi \\ &= \int T^{1-2d_1} |\hat{\varphi}(\xi)|^2 C(\xi)\tilde{G}(\xi/T)\,d\xi \\ &= T^{1-2d_1}\left[\int |\hat{\varphi}(\xi)|^2 C(\xi)\tilde{G}(0)\,d\xi + \varepsilon(T)\right] \\ &= T^{1-2d_1}(C_\varphi + \varepsilon(T)), \end{aligned}$$

where $\lim_{T\to 0}\varepsilon(T) = 0$ and $C_\varphi = \int |\hat{\varphi}(\xi)|^2 C(\xi)\tilde{G}(0)\,d\xi$.

In the same way we obtain

$$\int |\hat{\psi}_T(\xi)|^2 |F(i\xi)|^2 d\xi = T^{1-2d_1}(C_\psi + \varepsilon(T))$$

with $\lim_{T\to 0}\varepsilon(T) = 0$ and $C_\psi = \int |\hat{\psi}(\xi/\delta)|^2 C(\xi)\tilde{G}(0)\,d\xi$, and

$$r_T \left[\int |\hat{\varphi}_T(\xi)|^2 |F(i\xi)|^2 d\xi\right]^{1/2} \left[\int |\hat{\psi}_T(\xi)|^2 |F(i\xi)|^2 d\xi\right]^{1/2} = T^{1-2d_1}\left(\sqrt{C_\varphi C_\psi} + \varepsilon(T)\right)$$

while

$$\begin{aligned} \int \hat{\varphi}_T(\xi)\overline{\hat{\psi}}_T(\xi)|F(i\xi)|^2 d\xi &= T^{1-2d_1}\left\{\int \hat{\varphi}(\xi)\overline{\hat{\psi}}(\xi)C(\xi)\tilde{G}(0)\,d\xi + \varepsilon T)\right\} \\ &= T^{1-2d_1}(C_{\varphi,\psi} + \varepsilon(T)) \end{aligned}$$

with $\lim_{T\to 0}\varepsilon(T)=0$ and $C_{\varphi,\psi}=\int\hat\varphi(\xi)\hat\psi(\xi/\delta)C(\xi)\tilde G(0)\,d\xi$. Hence we have

$$T^{1-2d_1}(C_{\varphi,\psi}+\varepsilon(T))\leqslant r_T\,T^{1-2d_1}(\sqrt{C_\varphi C_\psi}+\varepsilon(T)),$$

and since $\lim_{T\to\infty} r_T=0$, we obtain

$$\lim_{T\to+\infty} C_{\varphi,\psi}+\varepsilon(T)\leqslant \lim_{T\to+\infty} r_T(\sqrt{C_\varphi C_\psi}+\varepsilon(T))=0.$$

Consequently,

$$C_{\varphi,\psi}=0 \quad\text{and}\quad \int\hat\varphi(\xi)\overline{\hat\psi}(\xi/\delta)C(\xi)\tilde G(0)\,d\xi=0,$$

and thus

$$\int\hat\varphi(\xi)\overline{\hat\psi}(\xi/\delta)C(\xi)\,d\xi=0. \tag{6}$$

In order to prove that there is no mixing, it suffices to show that if φ and ψ, with compact support as above, are such that (6) holds, then $C(\xi)$ is necessarily a polynomial. We now construct a family of φ and ψ such that (6) is satisfied.

Let $\chi\in\mathscr{C}_0^\infty([-1,1])$ be such that $\int\chi(t)\,dt=1$ and

$$\varphi(t)=\chi[n(t+x/2)] \quad\text{with } x>0,$$
$$\psi(t)=\chi[n(t-y/2)] \quad\text{with } y>2\delta.$$

For n large enough, $\operatorname{Supp}\varphi\subset\,]-\infty,0]$ and $\operatorname{Supp}\psi\subset[\delta,+\infty[$. We have

$$\hat\varphi(\xi)=\int e^{-it\xi}\chi\left(n\left(t+\frac{x}{2}\right)\right)dt=\int\exp\left[-i\left(\frac{s}{n}-\frac{x}{2}\right)\xi\right]\chi(s)\,ds$$
$$=\frac{\exp[i2^{-1}x\xi]}{n}\int\exp\left[-i\frac{s}{n}\xi\right]\chi(s)\,ds=\frac{\exp[i2^{-1}x\xi]}{n}\hat\chi\left(\frac{\xi}{n}\right)$$

and

$$\hat\psi\left(\frac{\xi}{\delta}\right)=\frac{\exp[-iy(1-\delta)\xi]}{n}\hat\chi\left(\frac{\xi}{n\delta}\right),$$

and therefore

$$\int\hat\varphi(\xi)\overline{\hat\psi}\left(\frac{\xi}{\delta}\right)C(\xi)\,d\xi=\int\frac{\exp[i(x/2+y(1-\delta))\xi]}{n}\hat\chi\left(\frac{\xi}{n}\right)\overline{\hat\chi\left(\frac{\xi}{n\delta}\right)}C(\xi)\,d\xi.$$

If we assume that (6) holds, then

$$\int\frac{\exp[i(x/2+y(1-\delta))\xi]}{n}\hat\chi\left(\frac{\xi}{n}\right)\overline{\hat\chi\left(\frac{\xi}{n\delta}\right)}C(\xi)\,d\xi=0.$$

Let $W=\mathscr{S}'(\boldsymbol{R})$ be such that $\hat W(\xi)=C(\xi)$ and put $u=x/2+y/2$. Then $u>\delta$ and

$$\int e^{iu\xi}\hat\chi(\xi/n)\overline{\hat\chi(\xi/(n\delta))}\,\hat W(\xi)\,d\xi=0.$$

Let $\phi \in \mathscr{C}_0^\infty([\delta, +\infty[)$. We then have

$$\int \phi(u) \int e^{iu\xi} \hat{\chi}(\xi/n) \overline{\hat{\chi}(\xi/(n\delta))} \hat{W}(\xi) \, d\xi du = 0.$$

Thus

$$\int (\mathscr{F}^{-1}\phi)(\xi) \hat{\chi}(\xi/n) \overline{\hat{\chi}(\xi/(n\delta))} \hat{W}(\xi) \, d\xi = 0$$

and

$$\lim_{n\to+\infty} \int (\mathscr{F}^{-1}\phi)(\xi) \hat{\chi}(\xi/n) \overline{\hat{\chi}(\xi/(n\delta))} \hat{W}(\xi) \, d\xi = \int (\mathscr{F}^{-1}\phi)(\xi) \hat{W}(\xi) \, d\xi = 0.$$

Consequently, for all $\phi \in \mathscr{C}_0^\infty([\delta, +\infty[)$

$$\langle \hat{W}, \mathscr{F}^{-1}\phi \rangle = \langle W, \phi \rangle = \langle \check{W}, \check{\phi} \rangle = \overline{\langle W, \overline{\check{\phi}} \rangle} = 0$$

and it follows that, for all $\delta > 0$, $\operatorname{Supp} W \subset [-\delta, \delta]$. Hence $\operatorname{Supp} W = \{0\}$, where W is necessarily of the form

$$W = \sum_{\alpha \in N} C_\alpha \delta_0^{(\alpha)}$$

and C is expressed as

$$C(\xi) = \hat{W}(\xi) = \sum_{\alpha \in N} \mathscr{F}(C_\alpha \delta_0^{(\alpha)})(\xi) = \sum_{\alpha \in N} C_\alpha \xi^\alpha.$$

Therefore $C(\xi)$ is a polynomial and it follows that $C_2 = C_1$, $d_1 \in N$, $d_2 = 0$. Thus the proposition is proved.

Remark. Ibragimov and Rozanov [8], Rozanov [12], Hayashi [7] and Dominguez [3] gave conditions on the spectral density for temporal processes to be mixing. Theorem 9 gives a better mixing rate than that of Lemma 10.6 of [12] but for more restrictive assumptions. Theorem 11 is a consequence of Corollary 2 (chap. IV, § 3) of [8] for continuous time processes. Hayashi determined the necessary and sufficient conditions for the spectral density for the mixing coefficient to tend to 0, and Dominguez gave the necessary and sufficient conditions for the mixing coefficient to tend to 0 with a determinate rate. However, these conditions, which require the spectral density to belong to some functional spaces, are hardly verifiable in our case.

Theorems 9 and 11 lead easily to the following corollary for fractional distribution processes, that generalizes the result of [14].

COROLLARY 12. *The fractional distribution process with transfer function F is ϱ-mixing if and only if $a < 0$. In this case we have*

$$\varrho_T = O(e^{bT}) \quad \textit{for all } b \in]a, 0[.$$

In the case where $D < -\frac{1}{2}$ this corollary allows us to complete Proposition 6 of [14] by the convergence rate of ϱ-mixing when a is negative.

APPENDIX A: PROOF OF PROPOSITION 4

Assume that $a = 0$ and that there are two singularities $(a_k)_{k=1,2}$ such that $\mathrm{Re}(a_k) = 0$ for $k = 1, 2$.

Let $N > 0$. In a neighbourhood of a_k we can write

$$F(s) = (s-a_k)^{d_k} \sum_{j=0}^{+\infty} c_j^{(k)} (s-a_k)^j$$
$$= (s-a_k)^{d_k} \sum_{j=0}^{J_k} c_j^{(k)} (s-a_k)^j + (s-a_k)^{\tilde{N}_k} \sum_{j=0}^{+\infty} \tilde{c}_j^{(k)} (s-a_k)^j$$

with $J_k = [N - \mathrm{Re}(d_k)] - 1$ and $\tilde{N}_k = d_k + J_k + 1$, and hence $0 < N < \mathrm{Re}(\tilde{N}_k) \leqslant N+1$. Thus we have

$$F(s) = (s-a_k)^{d_k} \sum_{j=0}^{J_k} c_j^{(k)} (s-a_k)^j + (s-a_k)^{\tilde{N}_k} \tilde{F}_k(s)$$

with $\tilde{F}_k(s)$ holomorphic for s in the neighbourhood of a_k.

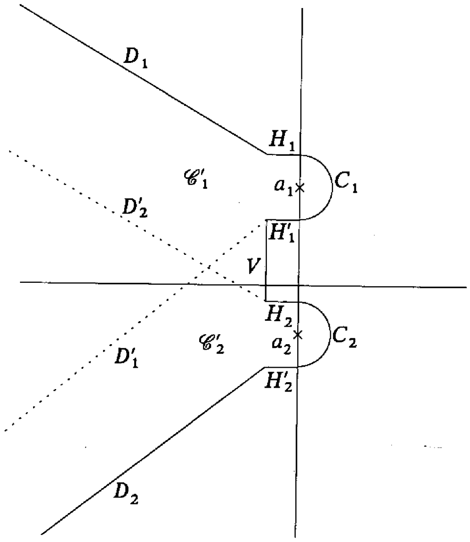

Fig. 1

Let $x > 0$. The inverse Laplace transform f of F is defined by

$$f(t) = \int_{x-i\infty}^{x+i\infty} e^{st} F(s)\, ds.$$

We integrate on the new contour $\mathscr{C}$ (see Fig. 1) such that all singularities of F are located on the left-hand side of $\mathscr{C}$, and we obtain

$$f(t) = \int_{\mathscr{C}} e^{st} F(s)\,ds = \int_{D_1} e^{st} F(s)\,ds + \int_{D_2} e^{st} F(s)\,ds + \int_{V} e^{st}(s)\,ds$$
$$+ \sum_{k=1}^{2} \Big(\int_{H_k \cup H'_k \cup C_k} e^{st}(s-a_k)^{d_k} \sum_{j=0}^{J_k} c_j^{(k)} (s-a_k)^j ds$$
$$+ \int_{H_k \cup H'_k \cup C_k} e^{st}(s-a_k)^{\tilde{N}_k} \tilde{F}_k(s)\,ds\Big).$$

This can be written in the form

$$f(t) = \sum_{k=1}^{2} \int_{\mathscr{C}'_k} e^{st}(s-a_k)^{d_k} \sum_{j=0}^{J_k} c_j^{(k)} (s-a_k)^j\,ds + \int_{D_1 \cup D_2 \cup V} e^{st} F(s)\,ds$$
$$+ \sum_{k=1}^{2} \Big(\int_{H_k \cup H'_k} e^{st}(s-a_k)^{\tilde{N}_k} \tilde{F}_k(s)\,ds + \int_{C_k} e^{st}(s-a_k)^{\tilde{N}_k} \tilde{F}_k(s)\,ds$$
$$- \int_{D_k \cup D'_k} e^{st}(s-a_k)^{d_k} \sum_{j=0}^{J_k} c_j^{(k)} (s-a_k)^j\,ds\Big).$$

We have

$$\int_{\mathscr{C}'_k} e^{st}(s-a_k)^{d_k} \sum_{j=0}^{J_k} c_j^{(k)} (s-a_k)^j\,ds = \exp[a_k t] \sum_{j=0}^{J_k} \frac{c_j^{(k)}}{\Gamma(-d_k-j)} t^{-(d_k+j+1)}.$$

We now estimate the remaining terms.

- Term $\int_{D_k \cup V} e^{st} F(s)\,ds$.

For $s = x+iy \in D_k$ we have

$$y = ax+b, \qquad x \in]-\infty, \delta],$$

$$\int_{D_k} e^{st} F(s)\,ds = \pm \int_{-\delta}^{-\infty} \exp[xt + it(ax+b)]\, F\big(x+i(ax+b)\big)(1+ia)\,dx.$$

For $s \in D_k$ there exists $L > 0$ such that

$$|F(s)| \leqslant C(1+|s|)^L$$

and

$$\Big|\int_{D_k} e^{st} F(s)\,ds\Big| \leqslant C \int_{\delta}^{+\infty} e^{-xt}(1+s)^L\,dx \leqslant C \exp\Big[-\frac{\delta}{2}t\Big] \int_{\delta/2}^{+\infty} e^{-xt}(1+x)^L\,dx$$
$$\leqslant C \exp\Big[-\frac{\delta}{2}t\Big] \quad \text{for } t > 1.$$

In the same way we obtain

$$\int_{V} e^{st} F(s)\,ds = \int_{\alpha_1}^{\alpha_2} \exp[-\delta t + ity]\, F(\delta+iy)\, i\,dy \quad \text{and} \quad \Big|\int_{V} e^{st} F(s)\,ds\Big| \leqslant Ce^{-\delta t}.$$

• Terms $\int_{D_k} e^{st}(s-a_k)^{d_k} \sum_{j=0}^{J_k} c_j^{(k)}(s-a_k)^j\,ds$.
For $s \in D_k$ we have

$$\left|(s-a_k)^{d_k} \sum_{j=0}^{J_k} c_j^{(k)}(s-a_k)^j\,ds\right| \leqslant C(1+|s|)^L.$$

Hence, as previously seen,

$$\left|\int_{D_k} e^{st}(s-a_k)^{d_k} \sum_{j=0}^{J} c_j^{(k)}(s-a_k)^j\,ds\right| \leqslant C\exp\left[-\frac{\delta}{2}t\right].$$

• Terms $\int_{H_k} e^{st}(s-a_k)^{\tilde{N}_k} \tilde{F}_k(s)\,ds$.
For $s = x+iy \in H_k$ we have

$$y = a_k/i \pm 1/t, \qquad x \in]-\delta, 0],$$

$$\int_{H_k} e^{st}(s-a_k)^{\tilde{N}_k} \tilde{F}_k(s)\,ds = \pm \int_0^{-\delta} e^{xt} e^{iyt} (x \pm i/t)^{\tilde{N}_k} \tilde{F}_k(x+iy)\,dx$$

and

$$\left|\int_{H_k} e^{st}(s-a_k)^{\tilde{N}_k} \tilde{F}_k(s)\,ds\right| \leqslant C \int_0^{\delta} e^{-xt} \left|x \pm \frac{i}{t}\right|^N dx$$

$$\leqslant \frac{C}{t^{N+1}} \int_0^{\delta t} e^{-u} |u \pm i|^N\,du \leqslant \frac{C}{t^{N+1}}.$$

• Terms $\int_{C_k} e^{st}(s-a_k)^{\tilde{N}_k} \tilde{F}_k(s)\,ds$.
For $s = a_k + re^{i\theta} \in C_k$ we have

$$r = 1/t, \qquad \theta \in]-\pi/2, \pi/2[,$$

$$\int_{C_k} e^{st}(s-a_k)^{\tilde{N}_k} \tilde{F}_k(s)\,ds = \int_{-\pi/2}^{\pi/2} \exp[t(a_k + re^{i\theta})]\, r^{\tilde{N}_k} \exp[i\theta \tilde{N}_k]\, \tilde{F}_k(s)\, ire^{i\theta}\,d\theta$$

and

$$\left|\int_{C_k} e^{st}(s-a_k)^{\tilde{N}_k} \tilde{F}_k(s)\,ds\right| \leqslant C \int_{-\pi/2}^{\pi/2} r^{\mathrm{Re}(\tilde{N}_k)+1}\,d\theta \leqslant \frac{C}{t^{N+1}}.$$

Finally,

$$f(t) = \sum_{k=1}^{2} \exp[a_k t] \sum_{j=0}^{J_k} \frac{c_j^{(k)}}{\Gamma(-d_k-j)} t^{-(d_k+j+1)} + R(t) \qquad \text{with } |R(t)| < \frac{C}{t^{N+1}}.$$

We generalize easily this result to the case where $a \neq 0$ and to case where there are more than two singularities a_k such that $\mathrm{Re}(a_k) = a$. Thus the proposition is proved.

APPENDIX B: PROOF OF THEOREM 8

Let $\lambda_0 > \sup_{k\in E^*}\{|a_k|\}$. For $\lambda > \lambda_0$

$$g(\lambda) = |F(i\lambda)|^2 = \Big|\prod_{k=1}^{K}(i\lambda - a_k)^{d_k}\Big|^2 = |\lambda|^{2D}\big(1+\sum_{j=1}^{\infty} c_j\lambda^{-j}\big).$$

Let χ be even and such that

$$\chi(\lambda) = \begin{cases} 1 & \text{for } |\lambda| > 2\lambda_0, \\ 0 & \text{for } |\lambda| < \frac{3}{2}\lambda_0, \end{cases}$$

$$\sigma = \mathscr{F}^{-1}(g) = \mathscr{F}^{-1}\big((1-\chi)g\big) + \mathscr{F}^{-1}(\chi g)$$

$$= \mathscr{F}^{-1}\big((1-\chi)g\big) + \mathscr{F}^{-1}\big(|\lambda|^{2D}\big(1+\sum_{j=1}^{\infty} c_j\lambda^{-j}\big)\chi\big).$$

$\mathscr{F}^{-1}\big((1-\chi)g\big)$ is analytic since $\big((1-\chi)g\big)$ has a compact support. Hence

$$\sigma = \mathscr{F}^{-1}\big(|\lambda|^{2D}\big(1+\sum_{j=1}^{\infty} c_j\lambda^{-j}\big) + |\lambda|^{2D}\big(1+\sum_{j=1}^{\infty} c_j\lambda^{-j}\big)(\chi-1)\big) + h(t)$$

with $h(t)$ analytic. $\mathscr{F}^{-1}\big(|\lambda|^{2D}\big(1+\sum_{j=1}^{\infty} c_j\lambda^{-j}\big)(\chi-1)\big)$ is analytic, and thus

$$\sigma = \mathscr{F}^{-1}\big(|\lambda|^{2D} + \sum_{j\ \mathrm{odd}}^{\infty} c_j|\lambda|^{2D}\lambda^{-j} + \sum_{j\ \mathrm{even}}^{\infty} c_j|\lambda|^{2D-j}\big) + h(t). \tag{7}$$

Let λ_+^s and λ_-^s be distributions defined by

$$\langle \lambda_+^s, \varphi\rangle = \int_0^{+\infty} \lambda^s\varphi(\lambda)\,d\lambda \quad \text{if } s > -1,$$

$$\langle \lambda_+^s, \varphi\rangle = \int_0^1 \lambda^s\Big(\varphi(\lambda) - \sum_{k=0}^{[-s]-1}\varphi^{(k)}(0)\frac{\lambda^k}{k!}\Big)d\lambda + \int_1^{+\infty}\lambda^s\varphi(\lambda)\,d\lambda \tag{8}$$

$$+\sum_{k=0}^{[-s]-1}\frac{\varphi^{(k)}(0)}{k!(k+1+s)} \quad \text{if } s\in \boldsymbol{R}^-\backslash \boldsymbol{Z}^-,$$

$$\langle \lambda_-^s, \varphi\rangle = \langle \lambda_+^s, \check{\varphi}\rangle.$$

We have $|\lambda|^s = \lambda_+^s + \lambda_-^s$, $|\lambda|^{2D}\lambda^{-j} = \lambda_+^{2D-j} - \lambda_-^{2D-j}$ if j is odd. We compute now $\mathscr{F}^{-1}(|\lambda|^s)$ and $\mathscr{F}^{-1}(\lambda_+^s - \lambda_-^s)$.

- Case $s \in \boldsymbol{R}\backslash\boldsymbol{Z}$.

From the equalities

$$\mathscr{F}(t_+^s) = \Gamma(s+1)\Big(\exp\Big[-i(s+1)\frac{\pi}{2}\Big]\lambda_+^{-(s+1)} + \exp\Big[i(s+1)\frac{\pi}{2}\Big]\lambda_-^{-(s+1)}\Big),$$

$$\mathscr{F}(t_-^s) = \Gamma(s+1)\Big(\exp\Big[-i(s+1)\frac{\pi}{2}\Big]\lambda_-^{-(s+1)} + \exp\Big[i(s+1)\frac{\pi}{2}\Big]\lambda_+^{-(s+1)}\Big)$$

it follows that

$$\lambda_+^{-(s+1)} = i\frac{\Gamma(-s)}{2\pi}\mathscr{F}\left(\exp\left[-i(s+1)\frac{\pi}{2}\right]t_+^s - \exp\left[i(s+1)\frac{\pi}{2}\right]t_-^s\right),$$

$$\lambda_-^{-(s+1)} = -i\frac{\Gamma(-s)}{2\pi}\mathscr{F}\left(\exp\left[i(s+1)\frac{\pi}{2}\right]t_+^s - \exp\left[-i(s+1)\frac{\pi}{2}\right]t_-^s\right).$$

Therefore

$$\lambda_+^s = i\frac{\Gamma(s+1)}{2\pi}\mathscr{F}\left(\exp\left[is\frac{\pi}{2}\right]t_+^{-(s+1)} - \exp\left[-is\frac{\pi}{2}\right]t_-^{-(s+1)}\right),$$

$$\lambda_-^s = -i\frac{\Gamma(s+1)}{2\pi}\mathscr{F}\left(\exp\left[-is\frac{\pi}{2}\right]t_+^{-(s+1)} - \exp\left[is\frac{\pi}{2}\right]t_-^{-(s+1)}\right)$$

and

$$\mathscr{F}^{-1}(|\lambda|^s) = -\frac{\Gamma(s+1)}{\pi}\sin\left(\frac{s\pi}{2}\right)|t|^{-(s+1)},$$

$$\mathscr{F}^{-1}(\lambda_+^s - \lambda_-^s) = i\frac{\Gamma(s+1)}{\pi}\cos\left(\frac{s\pi}{2}\right)(t_+^{-(s+1)} - t_-^{-(s+1)}).$$

- Case $s \in \boldsymbol{Z}^{-*}$.

If $s \in -l$, $l \in \boldsymbol{N}$, then the term $k = l-1$ is not defined in the sum of the expression (8). Now

$$\frac{\varphi^{(l-1)}(0)}{(l-1)!\,(l+s)} = (-1)^{l-1}\left\langle\frac{\delta^{(l-1)}}{(l-1)!\,(l+s)};\,\varphi\right\rangle,$$

and hence the distribution

$$T_+^s = \lambda_+^s - (-1)^{l-1}\frac{\delta^{(l-1)}}{(l-1)!\,(l+s)}$$

is holomorphic with respect to s in the neighbourhood of $-l$, and

$$\mathscr{F}^{-1}(T_+^s) = i\frac{\Gamma(s+1)}{2\pi}\left(\exp\left[is\frac{\pi}{2}\right]t_+^{-(s+1)} - \exp\left[-is\frac{\pi}{2}\right]t_-^{-(s+1)}\right)$$

$$-(-1)^{l-1}\frac{(-it)^{l-1}}{2\pi(l-1)!\,(l+s)}.$$

We now develop this expression in the neighbourhood of $-l$. We set $z = s+l$, and we have

$$\Gamma(z-l+1) = \frac{(-1)^{l-1}}{(l-1)!\,z} + \gamma_l(z)$$

with $\gamma_l(z)$ holomorphic in the neighbourhood of 0.

For $t>0$

$$i\frac{\Gamma(z-l+1)}{2\pi}\exp\left[i(z-l)\frac{\pi}{2}\right]t_+^{-(z-l+1)}-(-1)^{l-1}\frac{(-it)^{l-1}}{2\pi(l-1)!\,z}$$
$$=\frac{(it)^{l-1}}{2\pi}\left(\left(i\frac{\pi}{2}-\log t\right)\frac{1}{(l-1)!}+\varepsilon(z)-(-1)^l\gamma_l(z)\big(1+\varepsilon(z)\big)\right)$$

and for $z=0$ we have

$$\mathscr{F}^{-1}(T_+^{-l})=\frac{(it)^{l-1}}{2\pi}\left(\left(i\frac{\pi}{2}-\log t\right)\frac{1}{(l-1)!}+(-1)^{l-1}\gamma_l(0)\right).$$

In the same way for $t<0$ we have

$$\mathscr{F}^{-1}(T_+^{-l})=\frac{(it)^{l-1}}{2\pi}\left(-\frac{1}{(l-1)!}\left(i\frac{\pi}{2}+\log|t|\right)+(-1)^{l-1}\gamma_l(0)\right).$$

A similar calculation leads to

$$\mathscr{F}^{-1}(T_-^{-l})=-i\frac{\Gamma(s+1)}{2\pi}\left(\exp\left[-is\frac{\pi}{2}\right]t_+^{-(s+1)}-\exp\left[is\frac{\pi}{2}\right]t_-^{-(s+1)}\right)$$
$$-\frac{(-it)^{l-1}}{2\pi(l-1)!\,(l+s)}\quad\text{for } t>0,$$

$$\mathscr{F}^{-1}(T_-^{-l})=\frac{(it)^{l-1}}{2\pi}\left(\frac{(-1)^l}{(l-1)!}\left(i\frac{\pi}{2}+\log t\right)+\gamma_l(0)\right)\quad\text{for } t<0,$$

$$\mathscr{F}^{-1}(T_-^{-l})=\frac{(it)^{l-1}}{2\pi}\left(\frac{(-1)^{l-1}}{(l-1)!}\left(i\frac{\pi}{2}-\log|t|\right)+\gamma_l(0)\right).$$

$\mathscr{F}^{-1}(\delta^{(j)})=(2\pi)^{-1}(-it)^j$ is analytic with respect to t. On the other hand, λ_+^{-l} and λ_-^{-l} are square-integrable at infinity. Hence $\mathscr{F}^{-1}(|\lambda|^{-l})$ and $\mathscr{F}^{-1}(\lambda_+^{-l}-\lambda_-^{-l})$ are without Dirac mass at the origin, and by noting that for l odd $|t|^{l-1}$ is analytic, we write

$$\mathscr{F}^{-1}(|\lambda|^{-l})=\begin{cases}\dfrac{i^l|t|^{l-1}}{2(l-1)!}+h(t) & \text{if } l \text{ is even},\\[2ex] i^{l-1}|t|^{l-1}\dfrac{-\log|t|}{\pi(l-1)!}+h(t) & \text{if } l \text{ is odd},\end{cases}$$

$$\mathscr{F}^{-1}(\lambda_+^{-l}-\lambda_-^{-l})=\begin{cases}\dfrac{i^l}{2(l-1)!}+(t_+^{l-1}-t_-^{l-1})+h(t) & \text{if } l \text{ is odd},\\[2ex] (it)^{l-1}\dfrac{-\log|t|}{\pi(l-1)!}+h(t) & \text{if } l \text{ is even},\end{cases}$$

where $h(t)$ is an analytic function.

• Case $s \in N$.

For $l \in N$ the distributions $t_+^{-(l+1)}$ and $t_-^{-(l+1)}$ are not defined. Let s be in the neighbourhood of $l \in N$. Then

$$\langle t_+^{-(s+1)}, \varphi \rangle = \int_0^1 t^{-(s+1)} \left(\varphi(t) - \sum_{k=0}^{l} \varphi^{(k)}(0) \frac{t^k}{k!} \right) dt + \int_1^{+\infty} t^{-(s+1)} \varphi(t)\, dt$$

$$+ \sum_{k=0}^{l} \frac{\varphi^{(k)}(0)}{k!\,(k-s)},$$

$$\langle t_-^{-(s+1)}, \varphi \rangle = \int_0^1 t^{-(s+1)} \left(\check{\varphi}(t) - \sum_{k=0}^{l} \check{\varphi}^{(k)}(0) \frac{t^k}{k!} \right) dt + \int_1^{+\infty} t^{-(s+1)} \check{\varphi}(t)\, dt$$

$$+ \sum_{k=0}^{l} \frac{\check{\varphi}^{(k)}(0)}{k!\,(k-s)}$$

and

$$\frac{2\pi}{i\Gamma(s+1)} \langle \mathscr{F}^{-1}(\lambda_+^s), \varphi \rangle = \left\langle \exp\left[is\frac{\pi}{2} \right] t_+^{-(s+1)} - \exp\left[-is\frac{\pi}{2} \right] t_-^{-(s+1)}; \varphi \right\rangle$$

$$= \int_0^1 t^{-(s+1)} \exp\left[is\frac{\pi}{2} \right] \left(\varphi(t) - \sum_{k=0}^{l} \varphi^{(k)}(0) \frac{t^k}{k!} \right) dt$$

$$- \int_0^1 t^{-(s+1)} \exp\left[-is\frac{\pi}{2} \right] \left(\check{\varphi}(t) - \sum_{k=0}^{l} \check{\varphi}^{(k)}(0) \frac{t^k}{k!} \right) dt$$

$$+ \int_1^{+\infty} t^{-(s+1)} \left(\exp\left[is\frac{\pi}{2} \right] \varphi(t)\, dt - \exp\left[-is\frac{\pi}{2} \right] \check{\varphi}(t) \right) dt$$

$$+ \sum_{k=0}^{l} \frac{\exp[is\,\pi/2] \varphi^{(k)}(0) - \exp[-is\,\pi/2]\, \check{\varphi}^{(k)}(0)}{k!\,(k-s)}.$$

For $s = l + z$ the term $k = l$ in the sum can be written as

$$\left(\exp\left[is\frac{\pi}{2} \right] - (-1)^l \exp\left[-is\frac{\pi}{2} \right] \right) \frac{\varphi^{(l)}(0)}{l!\,(l-s)} = -(i)^{l+1} \pi \frac{\varphi^{(l)}(0)}{l!} + \varepsilon(z).$$

In the same way we obtain

$$\frac{2\pi}{i\Gamma(s+1)} \langle \mathscr{F}^{-1}(\lambda_-^s), \varphi \rangle = \left\langle \exp\left[is\frac{\pi}{2} \right] t_-^{-(s+1)} - \exp\left[-is\frac{\pi}{2} \right] t_+^{-(s+1)}; \varphi \right\rangle$$

$$= \int_0^1 t^{-(s+1)} \exp\left[is\frac{\pi}{2} \right] \left(\check{\varphi}(t) - \sum_{k=0}^{l} \check{\varphi}^{(k)}(0) \frac{t^k}{k!} \right) dt$$

$$- \int_0^1 t^{-(s+1)} \exp\left[-is\frac{\pi}{2} \right] \left(\varphi(t) - \sum_{k=0}^{l} \varphi^{(k)}(0) \frac{t^k}{k!} \right) dt$$

$$+\int_1^{+\infty} t^{-(s+1)}\left(\exp\left[is\frac{\pi}{2}\right]\check{\varphi}(t)\,dt-\exp\left[-is\frac{\pi}{2}\right]\varphi(t)\right)dt$$

$$+\sum_{k=0}^{l}\frac{\exp[is\pi/2]\,\check{\varphi}^{(k)}(0)-\exp[-is\pi/2]\,\varphi^{(k)}(0)}{k!\,(k-s)}.$$

For $s=l+z$ the term $k=l$ in the sum can be written as

$$\left((-1)^l\exp\left[is\frac{\pi}{2}\right]-\exp\left[-is\frac{\pi}{2}\right]\right)\frac{\varphi^{(l)}(0)}{l!\,(l-s)}=(-i)^{l+1}\pi\frac{\varphi^{(l)}(0)}{l!}+\varepsilon(z).$$

Therefore, by noting $\mu(s)=\exp[is\pi/2]-\exp[-is\pi/2]$, we have

$$\frac{2\pi}{i\Gamma(s+1)}\langle\mathscr{F}^{-1}(\lambda_+^s+\lambda_-^s);\,\varphi\rangle=\int_0^1 t^{-(s+1)}\mu(s)\left(\varphi(t)-\sum_{k=0}^{l}\varphi^{(k)}(0)\frac{t^k}{k!}\right)dt$$

$$-\int_0^1 t^{-(s+1)}\mu(s)\left(\check{\varphi}(t)-\sum_{k=0}^{l}\check{\varphi}^{(k)}(0)\frac{t^k}{k!}\right)dt+\int_1^{+\infty} t^{-(s+1)}\mu(s)\left(\varphi(t)+\check{\varphi}(t)\right)dt$$

$$+\sum_{k=0}^{l-1}\mu(s)\frac{\varphi^{(k)}(0)+\check{\varphi}^{(k)}(0)}{k!\,(k-s)}+\mu(l)\,\pi\frac{\varphi^{(l)}(0)}{l!}+\varepsilon(z).$$

Since

$$\mu(l)=\begin{cases}0 & \text{if } l \text{ is even},\\ 2(i)^l & \text{if } l \text{ is odd},\end{cases}$$

letting s tend to l, if l is even, we have

$$\langle\mathscr{F}^{-1}(|\lambda|^l),\,\varphi\rangle=(i)^l\,\varphi^{(l)}(0),$$

and if l is odd, setting $\psi=(\varphi+\check{\varphi})/2$, we obtain

$$\frac{2\pi}{(i)^{l+1}l!}\langle\mathscr{F}^{-1}(|\lambda|^l);\,\varphi\rangle=2\int_{-1}^{1} t^{-(l+1)}\left(\psi(t)-\sum_{k=0}^{l}\psi^{(k)}(0)\frac{t^k}{k!}\right)dt$$

$$+2\int_1^{+\infty} t^{-(l+1)}\psi(t)\,dt+2\int_{-\infty}^{-1} t^{-(l+1)}\psi(t)\,dt+4\sum_{k=0}^{l-1}\frac{\psi^{(k)}(0)}{k!\,(k-s)}$$

$$=2\langle\mathrm{vp}(t^{-(l+1)});\,\psi\rangle=2\langle\mathrm{vp}(t^{-(l+1)});\,\varphi\rangle.$$

It follows that

$$\mathscr{F}^{-1}(|\lambda|^l)=\begin{cases}(i)^l\,\delta^{(l)} & \text{if } l \text{ is even},\\[2ex] \dfrac{i^{l+1}\,l!}{\pi}\mathrm{vp}(t^{-(l+1)}) & \text{if } l \text{ is odd}.\end{cases}$$

In the same way we get

$$\mathscr{F}^{-1}(\lambda_+^l - \lambda_-^l) = \begin{cases} \dfrac{i^{l+1}\, l!}{\pi} \mathrm{vp}\,(t^{-(l+1)}) & \text{if } l \text{ is even,} \\ (-i)^l\, \delta^{(l)} & \text{if } l \text{ is odd.} \end{cases}$$

We obtain the desired assertion by putting this result in (7)

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