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OPERATOR VALUED STOCHASTIC CONTROL IN FOCK SPACE WITH APPLICATIONS TO ORBIT TRACKING AND NOISE FILTERING

BY

ANDREAS BOUKAS (ATHENS)

Abstract. The control process that minimizes the quadratic performance functional associated with a quantum system whose evolution is described by a Hudson–Parthasarathy type stochastic differential equation in Fock space is explicitly computed. A "noisy" infinite-dimensional Riccati equation appears for the first time and it is shown to have a unique solution. The solution to the control problem is used to derive the Fock space analogue of the Bucy–Kalman filter. The solution to an associated optimal trajectory problem is also obtained.

1. Introduction to stochastic calculus in Fock space. Using the ideas of the fundamental paper of Accardi et al. [1], Hudson and Parthasarathy constructed in [11] a simple noncommutative stochastic calculus in boson Fock space that was shown to include the classical Brownian motion and Poisson processes. Their theory has been particularly useful in the study of phenomena related to quantum optics and the study of the dynamics of quantum particles in the presence of noise (see [4] and [5]). We present here a brief summary of the basic ideas and results of the Hudson–Parthasarathy calculus that we are going to use, as found in [12].

Suppose that H_0 is a complex separable *initial* Hilbert space describing *events* (projection operators) and *observables* (self-adjoint operators) concerning a system. For $H = L^2((-\infty, \infty), \boldsymbol{C})$ let

$$\Gamma = \Gamma(H) = \boldsymbol{C} \oplus H \oplus H \otimes H \oplus \ldots \oplus H \otimes \ldots \otimes H \oplus \ldots$$

be the *symmetric* (or *boson*) Fock space describing events and observables concerning a *noise process* (or *heat bath*). We use $H_0 \otimes \Gamma$ to describe events and observables concerning system plus noise.

The span of the *exponential vectors*

$$\psi(f) = 1 \oplus f \oplus \frac{f \otimes f}{\sqrt{2!}} \oplus \ldots \oplus \frac{f \otimes \ldots \otimes f}{\sqrt{n!}} \oplus \ldots,$$

where $f \in H$, is dense in Γ with respect to the inner product $\langle \psi(f), \psi(g) \rangle_\Gamma = \exp\langle f, g \rangle_H$. We assume that inner products are linear in the second and antilinear in the first component. The set

$$\{v \otimes \psi(f) \mid v \in H_0, f \in H\} \subset H_0 \otimes \Gamma$$

is linearly independent and its span is dense in $H_0 \otimes \Gamma$.

Let β denote the σ-algebra of Borel measurable subsets of $[0, \infty]$. The notions of *time* and *adaptedness* are introduced through the concept of a $[0, \infty)$-*valued observable* which is defined as a spectral measure ξ on $([0, \infty), \beta)$ whose values are projection operators on H such that $\xi([0, \infty)) = I$, the identity operator on H. We assume that *time* is a $[0, \infty)$-valued observable ξ with no jump points, i.e. $\xi(\{t\}) = 0$ for every $t \geqslant 0$. To define the notion of *adaptedness*, let $D_0 \subset H_0$ and $\Lambda \subset H$ be linear manifolds such that $\xi([s, t]) f \in \Lambda$ whenever $f \in M$ for all $0 \leqslant s < t < \infty$. Let also, for each $t \geqslant 0$, $H_{t]}$ denote the range of $\xi([0, t])$ and let $f_{t]}$ and $f_{[t}$ stand for $\xi([0, t]) f$ and $\xi([t, +\infty)) f$, respectively. A family $X = \{X(t) \mid t \geqslant 0\}$ of operators from $H_0 \otimes \Gamma$ to $H_0 \otimes \Gamma$ is called an *adapted process* with respect to the triple (ξ, D_0, Λ) if for all $t \geqslant 0$, $v \in D_0$, and $f \in \Lambda$:

(i) $\operatorname{dom}(X(t)) \supset \operatorname{span}\{v \otimes \psi(f) \mid v \in D_0, f \in \Lambda\}$;

(ii) $X(t) v \otimes \psi(f_{t]}) \in H_0 \otimes \Gamma(H_{t]})$;

(iii) $X(t) v \otimes \psi(f) = (X(t) v \otimes \psi(f_{t]})) \otimes \psi(f_{[t})$.

In addition, X is said to be *regular* if for every $v \in D_0$ and $f \in \Lambda$ the map $t \in [0, \infty) \to X(t) v \otimes \psi(f)$ is continuous.

For a $[0, \infty)$-valued observable ξ we define a ξ-*martingale* to be a map m: $t \in [0, \infty) \to m_t \in H$ such that, for every t, $m_t \in H_{t]}$ and $\xi([0, s]) m_t = m_s$ for $s < t$. This notion will play a key role in the solution of the Fock space filtering problem through a suitable formulation of the concept of *orthogonal Brownian motions*. If m, m' are ξ-martingales, then there exists a complex valued measure $\langle\langle m, m' \rangle\rangle$ in $[0, \infty)$ which has finite variation in every bounded interval and satisfies

$$\langle\langle m, m' \rangle\rangle([0, t]) = \langle m_t, m'_t \rangle_H \quad \text{for all } t \geqslant 0.$$

If $f, g \in H$ and m is a ξ-martingale, we denote the ξ-martingales $t \to f_{t]}$ and $t \to g_{t]}$ also by f and g, respectively, and we denote the corresponding measures by $\langle\langle f, g \rangle\rangle$, $\langle\langle f, m \rangle\rangle$ and $\langle\langle m, g \rangle\rangle$.

For a ξ-martingale m we define, for each $t \geqslant 0$, linear operators $a^+(m_t)$, $a(m_t)$: $\Gamma \to \Gamma$ on $\operatorname{span}\{\psi(f) \mid f \in H\}$ by

$$a^+(m_t) \psi(f) = \left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \psi(f_{t]} + \varepsilon m_t) \right) \otimes \psi(f_{[t}),$$

$$a(m_t) \psi(f) = (\langle\langle m, f \rangle\rangle([0, t]) \psi(f_{t]})) \otimes \psi(f_{[t}).$$

The *basic regular adapted noise processes* in $H_0 \otimes \Gamma$ with respect to which stochastic integration will be considered are the following:

$$A_m^+ = \{A_m^+(t) = I \otimes a^+(m_t) \mid t \geq 0\} \quad (\textit{creation process}),$$

$$A_m = \{A_m(t) = I \otimes a(m_t) \mid t \geq 0\} \quad (\textit{annihilation process}),$$

where I denotes the identity operator on H_0.

For each $t \geq 0$, $A_m^+(t)$ and $A_m(t)$ are dual operators. The process

$$B_m = \{B_m(t) = A_m(t) + A_m^+(t) \mid t \geq 0\}$$

is a standard Brownian motion formulated in the language of Fock space theory (see [11] and [12]).

The *stochastic integral* from 0 to t of a regular adapted process X with respect to the noise process $N \in \{A_m^+, A_m\}$ is the linear operator $\int_0^t X(s)\,dN(s)\colon H_0 \otimes \Gamma \to H_0 \otimes \Gamma$ defined on span$\{v \otimes \psi(f) \mid v \in D_0, f \in \Lambda\}$ with *matrix elements*

$$(1.1) \quad \langle v \otimes \psi(f), \int_0^t X(s)\,dN(s)\,\omega \otimes \psi(g)\rangle = \int_0^t \langle v \otimes \psi(f), X(s)\,\omega \otimes \psi(g)\rangle\,d\mu(s),$$

where $\mu = \langle\langle f, m\rangle\rangle$ or $\langle\langle m, g\rangle\rangle$ according to whether $N = A_m^+$ or A_m.

If Y is another regular adapted process, then the joint matrix elements of $\int_0^t X(s)\,dN_1(s)$ and $\int_0^t Y(s)\,dN_2(s)$, where $N_i \in \{A_{m_i}^+, A_{m_i}\}$, $i = 1, 2$, are given by

$$\begin{aligned}(1.2) \quad & \langle \int_0^t X(s)\,dN_1(s)\,v \otimes \psi(f), \int_0^t Y(s)\,dN_2(s)\,\omega \otimes \psi(g)\rangle \\ &= \int_0^t \langle X(s)\,v \otimes \psi(f), \int_0^s Y(z)\,dN_2(z)\,\omega \otimes \psi(g)\rangle\,d\mu_1(s) \\ &\quad + \int_0^t \langle X(z)\,dN_1(z)\,v \otimes \psi(f), Y(s)\,\omega \otimes \psi(g)\rangle\,d\mu_2(s) \\ &\quad + \int_0^t \langle X(s)\,v \otimes \psi(f), Y(s)\,\omega \otimes \psi(g)\rangle\,d\mu_{12}(s),\end{aligned}$$

where μ_1, μ_2 and μ_{12} are defined by the following tables:

(1.3)

N_1	$A_{m_1}^+$	A_{m_1}
μ_1	$\langle\langle m_1, g\rangle\rangle$	$\langle\langle f, m_1\rangle\rangle$

N_2	$A_{m_2}^+$	A_{m_2}
μ_2	$\langle\langle f, m_2\rangle\rangle$	$\langle\langle m_2, g\rangle\rangle$

$N_1 \diagdown N_2$	$A_{m_2}^+$	A_{m_2}
$A_{m_1}^+$	$\langle\langle m_1, m_2\rangle\rangle$	0
A_{m_1}	0	0

If X is a regular adapted process and τ is a complex-valued measure of the form $\langle\langle m, m'\rangle\rangle$, where m, m' are ξ-martingales, then

$$(1.4) \quad \langle v \otimes \psi(f), \int_0^t X(s)\,d\tau(s)\,\omega \otimes \psi(g)\rangle = \int_0^t \langle v \otimes \psi(f), X(s)\,\omega \otimes \psi(g)\rangle\,d\tau(s),$$

while for two such measures τ_1, τ_2 and a regular adapted process Y

$$(1.5)\quad \langle \int_0^t X(s)\, d\tau_1(s)\, v\otimes\psi(f), \int_0^t Y(s)\, d\tau_2(s)\, \omega\otimes\psi(g)\rangle$$

$$= \int_0^t \langle X(s)\, v\otimes\psi(f), \int_0^s Y(z)\, d\tau_2(z)\, \omega\otimes\psi(g)\rangle\, d\tau_1(s)$$

$$+ \int_0^t \langle \int_0^s X(z)\, d\tau_1(z)\, v\otimes\psi(f), Y(s)\, \omega\otimes\psi(g)\rangle\, d\tau_2(s).$$

Moreover, for $N \in \{A_m^+, A_m\}$ and $\tau = \langle\langle m', m''\rangle\rangle$

$$(1.6)\quad \langle \int_0^t X(s)\, dN(s)\, v\otimes\psi(f), \int_0^t Y(s)\, d\tau(s)\, \omega\otimes\psi(g)\rangle$$

$$= \int_0^t \langle X(s)\, v\otimes\psi(f), \int_0^s Y(z)\, d\tau(z)\, \omega\otimes\psi(g)\rangle\, d\mu(s)$$

$$+ \int_0^t \langle \int_0^s X(z)\, dN(z)\, v\otimes\psi(f), Y(s)\, \omega\otimes\psi(g)\rangle\, d\tau(s),$$

where $\mu = \langle\langle m, g\rangle\rangle$ or $\langle\langle f, m\rangle\rangle$ according to whether $N = A_m^+$ or A_m.

An adapted process $X = \{X(t) \mid t \geqslant 0\}$ defined as a stochastic integral by

$$X(t) = \int_0^t L_1(s)\, d\tau(s) + L_2(s)\, dA_m(s) + L_3(s)\, dA_m^+(s)$$

can be written in differential form as

$$dX = \{dX(t) \mid t \geqslant 0\},$$

where $dX(t) = L_1(t)\, d\tau(t) + L_2(t)\, dA_m(t) + L_3(t)\, dA_m^+(t)$.

As a consequence of (1.2), (1.5), and (1.6) the products of the stochastic differentials of the basic noise processes are given by the following *quantum Itô's table*:

(1.7)

	$dA_{m_2}^+$	dA_{m_2}	$d\tau_{m_2}$
$dA_{m_1}^+$	0	0	0
dA_{m_1}	$d\langle\langle m_1, m_2\rangle\rangle$	0	0
$d\tau_{m_1}$	0	0	0

where $\tau_{m_i} = \langle\langle m_i, m_i\rangle\rangle$, $i = 1, 2$.

We remark also that if X and Y are adapted processes, then

$$(1.8)\qquad d(XY) = X\cdot dY + dX\cdot Y + dX\cdot dY,$$

and that adapted processes commute with the stochastic differentials of the basic noise processes A_m and A_m^+.

Moreover, if τ_m is a positive measure on $[0, \infty]$ absolutely continuous with respect to Lebesgue measure dt with Radon–Nikodym derivative $d\tau_m/dt$, and if $\{\lambda(t, s)/t$ (respectively, $s) \geqslant 0\}$ is for each s (respectively, t) a regular adapted process, then in the sense of equality of matrix elements we have

$$d\int_0^t \lambda(t, s)\, d\tau_m(s) = d\int_0^t \lambda(t, s)\frac{d\tau_m(s)}{ds}\, ds$$

$$= \lambda(t, t)\frac{d\tau_m(t)}{dt}\, dt + \int_0^t \frac{d}{dt}\lambda(t, s)\frac{d\tau_m(s)}{ds}\, ds\cdot dt,$$

which implies that

$$d\int_0^t \lambda(t, s)\, d\tau_m(s) = \lambda(t, t)\, d\tau_m(t) + \int_0^t d\lambda(t, s)\, d\tau_m(s). \tag{1.9}$$

Finally, we denote the dual of an operator K by K^* while the unique positive square root of a positive operator K is denoted by $K^{1/2}$. If K is positive and invertible, then $K^{-1/2}$ denotes the square root of K^{-1}. The real part ReK of K is defined by $\langle \mathrm{Re}Kx, y\rangle = \mathrm{Re}\langle Kx, y\rangle$.

2. The optimal control problem. We suppose that the evolution of the state of a physical system (for example, the position of a quantum particle) is described by a Hudson–Parthasarathy stochastic differential equation of the form

$$\begin{gathered} dX(t) = -[(FX+Gu+L)(t)\, d\tau_m(t) + (\Phi X+\sigma)(t)\, dA_m^+(t) + (\Psi X+\varrho)(t)\, dA_m(t)], \\ X(T) = C, \quad 0 \leqslant t \leqslant T < \infty, \end{gathered} \tag{2.1}$$

where $\tau_m = \langle\langle m, m\rangle\rangle$ is a positive measure on $[0, \infty)$ absolutely continuous with respect to Lebesgue measure dt. Here and in what follows we use the notation and terminology established in Section 1. Equation (2.1) can equivalently be written in the integral form

$$\begin{aligned} X(t) = C + \int_t^T & (FX+Gu+L)(s)\, d\tau_m(s) \\ & + (\Phi X+\sigma)(s)\, dA_m^+(s) + (\Psi X+\varrho)(s)\, dA_m(s). \end{aligned} \tag{2.2}$$

We assume that the coefficients of (2.2) satisfy the following conditions:

(a) ADAPTEDNESS. F, G, L, Φ, σ, Ψ, ϱ, u are stochastic processes adapted with respect to the triple (ξ, D_0, Λ), where the time observable ξ is as in Section 1.

(b) BOUNDEDNESS. For every $t \in [0, T]$, $F(t)$, $G(t)$, $L(t)$, $\Phi(t)$, $\sigma(t)$, $\Psi(t)$, $\varrho(t)$, $u(t)$, $C \in B(H_0 \otimes \Gamma)$, the space of bounded linear operators from $H_0 \otimes \Gamma$ to itself, and

$$\sup_{0\leqslant t\leqslant T} \|F(t)\| < \infty, \ldots, \sup_{0\leqslant t\leqslant T} \|u(t)\| < \infty.$$

(c) Strong continuity. For every $h \in H_0 \otimes \Gamma$, the maps $t \to F(t)h$, $t \to G(t)h, \ldots, t \to u(t)h$ are continuous in $[0, T]$.

(d) Tameness condition. D_0, Λ, Φ, Ψ are such that

$$\operatorname{span}\big(D_0 \otimes \{y(f) \mid f \in \Lambda\}\big)$$

is dense in $H_0 \otimes \Gamma$ and for every $s \geqslant 0$ and adapted process

$$W = \{W(t) \in B(H_0 \otimes \Gamma) \mid t \geqslant s\} \quad \text{with} \sup_{s \leqslant t \leqslant T} \|W(t)\|_{B(H_0 \otimes \Gamma)} < \infty,$$

the solution $K = \{K(t) \mid s \leqslant t \leqslant T\}$ of the equation

$$dK(t) = [W(t)\, d\tau_m(t) + \Phi^*(t)\, dA_m(t) + \Psi^*(t)\, dA_m^+(t)]\, K(t), \; K(s) = I, \; s \leqslant t \leqslant T,$$

satisfies

$$\sup_{s \leqslant t \leqslant T} \|K(t)\|_{B(H_0 \otimes \Gamma)} < \infty.$$

The tameness condition will play a role in the proof of Lemma 2.1.

By modifying the proof of Proposition 7.1 of [11], or of Proposition 26.1 of [12], we can show that under conditions (a), (b), (c), (d) above, equation (2.2) has a unique regular (ξ, D_0, Λ)-adapted solution $X = \{X(t) \mid 0 \leqslant t \leqslant T\}$. This is accomplished using a Picard iterations type argument by showing that the sequence $\{X_n\}_{n=1}^{\infty}$ of regular (ξ, D_0, Λ)-adapted processes defined recursively by

$$\lambda(t) = \int_t^T (Gu + L)(s)\, d\tau_m(s) + \sigma(s)\, dA_m^+(s) + \varrho(s)\, dA_m(s), \qquad X_0(t) = C + \lambda(t),$$

and for $n = 1, 2, 3, \ldots$

$$X_n(t) = X_0(t) + \int_t^T (FX_{n-1})(s)\, d\tau_m(s) + (\Phi X_{n-1})(s)\, dA_m^+(s) + (\Psi X_{n-1})(s)\, dA_m(s)$$

has the property that for each $h \in \operatorname{span}\{v \otimes y(f) \mid v \in D_0, f \in \Lambda\}$ the sequence $\{X_n(t)h\}_{n=1}^{\infty}$ converges uniformly in $[0, T]$. The limiting process X can be shown to be the unique solution of (2.2). We notice that this solution depends on the *control process* u.

By generalizing the ideas of classical stochastic control theory we associate with (2.1) and (2.2) the *quadratic performance criterion*

$$(2.3) \qquad J(u) = \int_0^T [\langle X(t)h, Q(t)X(t)h\rangle + \langle u(t)h, R(t)u(t)h\rangle]\, d\tau_m(t) + \langle X(0)h, MX(0)h\rangle,$$

where $h \in \operatorname{span}\{v \otimes y(f) \mid v \in D_0, f \in \Lambda\}$ is arbitrary and the processes Q, R, M satisfy the following conditions:

(a) $M = \tilde{M} \otimes I \in B(H_0 \otimes \Gamma)$, $\tilde{M} \geqslant 0$.

(b) Q and R are (ξ, D_0, Λ)-adapted processes with $Q(t)$, $R(t) \in B(H_0 \otimes \Gamma)$, $Q(t)$, $R(t) \geqslant 0$ for each $t \in [0, T]$, and

$$\sup_{0 \leqslant t \leqslant T} \|Q(t)\| < \infty, \qquad \sup_{0 \leqslant t \leqslant T} \|R(t)\| < \infty.$$

(c) R has an inverse $R^{-1} = \{R^{-1}(t) \mid 0 \leqslant t \leqslant T\}$ which is also (ξ, D_0, Λ)-adapted and such that, for every $t \in [0, T]$, $R^{-1}(t) \in B(H_0 \otimes \Gamma)$, $R^{-1}(t) > 0$ and

$$\sup_{0 \leqslant t \leqslant T} \|R^{-1}(t)\|_{B(H_0 \otimes \Gamma)} < \infty.$$

(d) For each $h \in H_0 \otimes \Gamma$ the maps $t \to Q(t)h$, $t \to R(t)h$, and $t \to R^{-1}(t)h$ are continuous in $[0, T]$.

Our goal is to find the control process $u = \{u(t) \mid 0 \leqslant t \leqslant T\}$ that minimizes (2.3). In the case when $Q = I$, the identity process, the problem is equivalent to that of finding the control process u that will minimize the $L^2_{\tau_m}$-norm of $X(\cdot)h$ and the energy inner product $\langle u(t)h, R(t)u(t)h\rangle$. This viewpoint is particularly appropriate when $X = \{X(t) \mid 0 \leqslant t \leqslant T\}$ represents the deviation from a desired trajectory, and the overall objective is to keep the system as much as possible "on target."

In general, the sizes of Q and R represent the cost of having large values of X (deviations) and u (control force or energy) on the average over the time interval $[0, T]$, while M represents the cost of having a large initial deviation.

We notice that (2.1) is solved backwards in time. This is particularly suitable when one tries to solve the Fock space analogue of the Bucy–Kalman filtering problem by reducing it to a control problem as demonstrated in Section 4.

For early work on the control and filtering problems associated with quantum dynamical systems we refer to the papers of Belavkin [6]–[8].

As in the classical stochastic control, the solution of the control problem (2.1), (2.3) utilizes an appropriate infinite-dimensional Riccati equation. In the classical theory the Riccati equation is deterministic (see [3] and [9]). However, in the Fock space control problem considered here, there is a need for a "noisy" Riccati equation, and to the author's best knowledge this is the first time that such a need has arised.

LEMMA 2.1 (Riccati). *There exists a unique regular positive (ξ, D_0, Λ)-adapted process $P = \{P(t) \mid 0 \leqslant t \leqslant T\}$ defined on all of $H_0 \otimes \Gamma$ such that for every $h_1, h_2 \in H_0 \otimes \Gamma$*

$$\begin{aligned}(2.4)\quad \langle P(t)h_1, h_2\rangle = \langle Mh_1, h_2\rangle + \Big\langle \Big[\int_0^t \big((F+\Psi\Phi)^* P + P(F+\Psi\Phi) + \Phi^* P\Phi \\ + Q - PGR^{-1}G^* P\big)(s)\,d\tau_m(s) + (\Phi^* P + P\Psi)(s)\,dA_m(s) \\ + (\Psi^* P + P\Phi)(s)\,dA_m^+(s)\Big] h_1, h_2\Big\rangle, \quad t \in [0, T],\ P(0) = M.\end{aligned}$$

Proof. Equation (2.4) can be written as

$$(2.5)\quad dP(t) = [(F+\Psi\Phi - GR^{-1}G^*P)(t)\,d\tau_m(t) + \Phi(t)\,dA_m^+(t) + \Psi(t)\,dA_m(t)]^*P(t) + P(t)[(F+\Psi\Phi - GR^{-1}G^*P)(t)\,d\tau_m(t) + \Phi(t)\,dA_m^+(t) + \Psi(t)\,dA_m(t)] + \Phi^*P\Phi(t)\,d\tau_m(t) + (Q+PGR^{-1}G^*P)(t)\,d\tau_m(t)$$

or, equivalently, as

$$(2.6)\quad P(t) = K(t,0)MK(t,0)^* + \int_0^t K(t,s)[Q+PGR^{-1}G^*P](s)K(t,s)^*\,d\tau_m(s),$$

where $K(t,s)$ is the unique solution of the Hudson–Parthasarathy equation

$$(2.7)\quad dK(t,s) = [(F+\Psi\Phi - GR^{-1}G^*P)(t)\,d\tau_m(t) + \Phi(t)\,dA_m^+(t) + \Psi(t)\,dA_m(t)]^*K(t,s),$$
$$K(s,s) = I, \qquad s \leqslant t \leqslant T.$$

Indeed, by (1.8) and (1.9), equation (2.6) implies

(2.8)

$$dP(t) = dK(t,0)\cdot M\cdot K(t,0)^* + K(t,0)\cdot M\cdot dK(t,0)^* + dK(t,0)\cdot M\cdot dK(t,0)^* + K(t,t)(Q+PGR^{-1}G^*P)(t)K(t,t)^*\,d\tau_m(t) + \int_0^t dK(t,s)\cdot(Q+PGR^{-1}G^*P)(s)\cdot K(t,s)^*\,d\tau_m(s) + \int_0^t K(t,s)\cdot(Q+PGR^{-1}G^*P)(s)\cdot dK(t,s)^*\,d\tau_m(s) + \int_0^t dK(t,s)\cdot(Q+PGR^{-1}G^*P)(s)\cdot dK(t,s)^*\,d\tau_m(s).$$

Replacing $dK(t,0)$, $dK(t,0)^*$, $dK(t,s)$, and $dK(t,s)^*$ by (2.7) and using quantum Itô's table (1.7) to compute the products of differentials we see that the right-hand side of (2.8) is the same as that of (2.5).

We interpret (2.6) through its matrix elements form

$$\langle h_1, P(t)h_2\rangle = \langle K(t,0)^*h_1, MK(t,0)^*h_2\rangle + \int_0^t \langle K(t,s)^*h_1, (Q+PGR^{-1}G^*P)(s)K(t,s)^*h_2\rangle\,d\tau_m(s),$$

where $h_1, h_2 \in \operatorname{span}\{v\otimes y(f) \mid v\in D_0, f\in \Lambda\}$ and the existence of $K(t,s)^*$ follows from the fact that $K(t,s)$ is defined on $\operatorname{span}\{v\otimes y(f) \mid v\in D_0, f\in\Lambda\}$ which is dense in $H_0\otimes\Gamma$.

Based on (2.6) we define the iteration scheme

$$(2.9)\quad P_1(t) = M,$$

$$P_{n+1}(t) = K_n(t, 0)MK_n(t, 0)^* + \int_0^t K_n(t, s)(Q+P_n GR^{-1}G^*P_n)(s)K_n(t, s)^* d\tau_m(s) \quad \text{for } n \geqslant 1,$$

where $K_n(t, s)$ is the solution of the equation

$$(2.10)\quad dK_n(t, s) = [(F+\Psi\Phi - GR^{-1}G^*P_n)(t)d\tau_m(t) + \Phi(t)dA_m^+(t) + \Psi(t)dA_m(t)]^* K_n(t, s),$$
$$K_n(s, s) = I, \quad s \leqslant t \leqslant T,$$

where, by the tameness assumption, $K_n(t, s) \in B(H_0 \otimes \Gamma)$ for every t, s.

Since M, $Q(t) \geqslant 0$ and $R^{-1}(t) > 0$, (2.9) defines in each step a positive operator $P_n(t)$ on all of $H_0 \otimes \Gamma$.

We will now show that, for all $t \in [0, T]$ and $n = 1, 2, \ldots$, we have $0 \leqslant P_{n+1}(t) \leqslant P_n(t)$. To that end we notice that (2.9) can be written, as in the proof of the equivalence of (2.5) and (2.6), in the form

$$(2.11)\quad dP_{n+1}(t) = [(F+\Psi\Phi - GR^{-1}G^*P_n)(t)d\tau_m(t) + \Phi(t)dA_m^+(t) + \Psi(t)dA_m(t)]^* P_{n+1}(t)$$
$$+ P_{n+1}(t)[(F+\Psi\Phi - GR^{-1}G^*P_n)(t)d\tau_m(t) + \Phi(t)dA_m^+(t) + \Psi(t)dA_m(t)]$$
$$+ \Phi^*(t)P_{n+1}(t)\Phi(t)d\tau_m(t) + (Q+P_n GR^{-1}G^*P_n)(t)d\tau_m(t).$$

Letting $\Lambda_n(t) = P_{n+1}(t) - P_n(t)$ and using (2.11) we obtain

$$(2.12)\quad d\Lambda_n(t) = [(F+\Psi\Phi - GR^{-1}G^*P_n)(t)d\tau_m(t) + \Phi(t)dA_m^+(t) + \Psi(t)dA_m(t)]^* \Lambda_n(t)$$
$$+ \Lambda_n(t)[(F+\Psi\Phi - GR^{-1}G^*P_n)(t)d\tau_m(t) + \Phi(t)dA_m^+(t) + \Psi(t)dA_m(t)]$$
$$+ (\Phi^*\Lambda_n\Phi)(t)d\tau_m(t) - (\Lambda_{n-1}GR^{-1}G^*\Lambda_{n-1})(t)d\tau_m(t), \quad \Lambda_n(0) = 0.$$

Thus

$$(2.13)\quad \Lambda_n(t) = -\int_0^t K_n(t, s)(\Lambda_{n-1}GR^{-1}G^*\Lambda_{n-1})(s)K_n(t, s)^* d\tau_m(s),$$

which, in view of the positivity of $R^{-1}(s)$, implies that $\Lambda_n(t) \leqslant 0$, thus proving $\{P_n(t)\}_{n=1}^{\infty}$ to be a positive decreasing sequence, which therefore converges strongly to a nonnegative operator $P(t)$ on $H_0 \otimes \Gamma$. As a strong limit of adapted processes the process $P = \{P(t) \mid 0 \leqslant t \leqslant T\}$ is itself adapted.

Moreover, letting $n \to \infty$ in (2.10) and (2.9) we obtain (2.6). Thus P solves (2.4).

Now, if P_1 and P_2 are two such solutions, then letting $\Lambda(t) = P_1(t) - P_2(t)$ we obtain

$$\begin{aligned} d\Lambda(t) = \Lambda(t)\,&[(F + \Psi\Phi - GR^{-1}G^* P_2)(t)\,d\tau_m(t) + \Phi(t)\,dA_m^+(t) + \Psi(t)\,dA_m(t)] \\ &+ [(F + \Psi\Phi - GR^{-1}G^* P_2)(t)\,d\tau_m(t) + \Phi(t)\,dA_m^+(t) + \Psi(t)\,dA_m(t)]^* \Lambda(t) \\ &+ (\Phi^* \Lambda\Phi)(t)\,d\tau_m(t) - (\Lambda GR^{-1}G^* \Lambda)(t)\,d\tau_m(t), \qquad \Lambda(0) = 0. \end{aligned}$$

Thus, as in (2.12) and (2.13), $\Lambda(t) \leqslant 0$, i.e. $P_1(t) \leqslant P_2(t)$. Interchanging P_1 with P_2 we obtain the same result. Thus $P_2(t) \leqslant P_1(t)$, and so $P_1(t) = P_2(t)$ for every $t \in [0, T]$, which proves the uniqueness. ∎

THEOREM 2.1. *The performance criterion* (2.3) *associated with the Hudson –Parthasarathy stochastic differential equation* (2.1) *is minimized by the control process* $u = R^{-1}G^*(g - PX)$, *where* P *is the solution of* (2.4) *and* g *is the solution of the equation*

$$\begin{aligned} (2.14) \quad dg(t) = \big((-PGR^{-1}G^* + \Phi^*\Psi^* + F^*)\,g - \Phi^* P\sigma + P\Psi\sigma - PL\big)(t)\,d\tau_m(t) \\ + (\Psi^* g - P\sigma)(t)\,dA_m^+(t) + (\Phi^* g - P\varrho)(t)\,dA_m(t), \qquad g(0) = 0. \end{aligned}$$

Moreover, the minimum value is

$$(2.15) \qquad \min_u J(u) = \langle Ch, P(T)Ch\rangle - 2\mathrm{Re}\,\langle Ch, g(T)h\rangle + \langle b(T)h, h\rangle,$$

where

$$\begin{aligned} (2.16) \quad b(t) = \int_0^t \big[\big(\sigma^* P\sigma - g^* GR^{-1}G^* g - 2\mathrm{Re}(g^*\Psi\sigma) - 2\mathrm{Re}(g^* L)\big)(t)\,d\tau_m(t) \\ - 2\mathrm{Re}(g^*\sigma)(t)\,dA_m^+(t) - 2\mathrm{Re}(g^*\varrho)(t)\,dA_m(t)\big]. \end{aligned}$$

Proof. Let $h \in \mathrm{span}\{v \otimes y(f) \mid v \in D_0, f \in \Lambda\}$ and for $t \in [0, T]$ define

$$(2.17) \quad H(t) = \langle X(t)h, P(t)X(t)h\rangle - 2\mathrm{Re}\,\langle X(t)h, g(t)h\rangle + \langle b(t)h, h\rangle.$$

By (1.8) we have

$$\begin{aligned} dH(t) = \langle dX(t)h,\, &P(t)X(t)h\rangle \\ &+ \langle X(t)h, [dP(t)\cdot X(t) + P(t)\cdot dX(t) + dP(t)\cdot dX(t)]\,h\rangle \\ &+ \langle dX(t)h, [dP(t)\cdot X(t) + P(t)\cdot dX(t) + dP(t)\cdot dX(t)]\,h\rangle \\ &- 2\mathrm{Re}\,\langle dX(t)h, g(t)h\rangle - 2\mathrm{Re}\,\langle X(t)h, dg(t)h\rangle \\ &- 2\mathrm{Re}\,\langle dX(t)h, dg(t)h\rangle + \langle db(t)h, h\rangle. \end{aligned}$$

Replacing $dX(t)$, $dP(t)$, $dg(t)$, $db(t)$ by (2.1), (2.4), (2.14), (2.16), respectively, and using (1.7) to multiply the stochastic differentials dA_m, dA_m^+, $d\tau_m$, after several cancellations we obtain

$$(2.18)\quad \begin{aligned} dH(t) = [&\langle X(t)h, Q(t)X(t)h\rangle - \langle X(t)h, (PGR^{-1}G^*P)(t)X(t)h\rangle \\ &- \langle (Gu)(t)h, P(t)X(t)h\rangle - \langle P(t)X(t)h, (Gu)(t)h\rangle \\ &+ \langle (Gu)(t)h, g(t)h\rangle + \langle g(t)h, (Gu)(t)h\rangle \\ &+ \langle X(t)h, (PGR^{-1}G^*g)(t)h\rangle + \langle (PGR^{-1}G^*g)(t)h, X(t)h\rangle \\ &- \langle (g^*GR^{-1}G^*g)(t)h, h\rangle]\, d\tau_m(t). \end{aligned}$$

By (2.18), using $P(0) = M$, $g(0) = 0$ and $b(0) = 0$, we obtain

$$(2.19)\quad \begin{aligned} H(T) - H(0) = &\langle Ch, P(T)Ch\rangle - 2\mathrm{Re}\langle Ch, g(T)h\rangle + \langle b(T)h, h\rangle \\ &- \langle X(0)h, MX(0)h\rangle. \end{aligned}$$

Moreover, by (2.18),

(2.20)

$$\begin{aligned} H(T) - H(0) = \int_0^T [&\langle X(t)h, Q(t)X(t)h\rangle - \langle X(t)h, (PGR^{-1}G^*P)(t)X(t)h\rangle \\ &- \langle (Gu)(t)h, P(t)X(t)h\rangle - \langle P(t)X(t)h, (Gu)(t)h\rangle \\ &+ \langle (Gu)(t)h, g(t)h\rangle + \langle g(t)h, (Gu)(t)h\rangle + \langle X(t)h, (PGR^{-1}G^*g)(t)h\rangle \\ &+ \langle (PGR^{-1}G^*g)(t)h, X(t)h\rangle - \langle (g^*GR^{-1}G^*g)(t)h, h\rangle]\, d\tau_m(t). \end{aligned}$$

Now,

(2.21)

$$\begin{aligned} J(u) = [&\langle Ch, P(T)Ch\rangle - 2\mathrm{Re}\langle Ch, g(T)h\rangle + \langle b(T)h, h\rangle \\ &- \langle X(0)h, MX(0)h\rangle] + J(u) - [\langle Ch, P(T)Ch\rangle - 2\mathrm{Re}\langle Ch, g(T)h\rangle \\ &+ \langle b(T)h, h\rangle - \langle X(0)h, MX(0)h\rangle]. \end{aligned}$$

On the right-hand side of (2.21) we replace $J(u)$ by (2.3), and in view of (2.19) we replace the second brackets by (2.20) to obtain after cancellations

$$\begin{aligned} J(u) = &\langle Ch, P(T)Ch\rangle - 2\mathrm{Re}\langle Ch, g(T)h\rangle + \langle b(T)h, h\rangle \\ &+ \int_0^T [\langle u(t)h, (Ru)(t)h\rangle + \langle X(t)h, (PGR^{-1}G^*P)(t)X(t)h\rangle \\ &+ \langle (Gu)(t)h, P(t)X(t)h\rangle + \langle P(t)X(t)h, (Gu)(t)h\rangle \\ &- \langle (Gu)(t)h, g(t)h\rangle - \langle g(t)h, (Gu)(t)h\rangle - \langle X(t)h, (PGR^{-1}G^*g)(t)h\rangle \\ &- \langle (PGR^{-1}G^*g)(t)h, X(t)h\rangle + \langle (g^*GR^{-1}G^*g)(t)h, h\rangle]\, d\tau_m(t) \end{aligned}$$

$$= \langle Ch, P(T)Ch\rangle - 2\mathrm{Re}\langle Ch, g(T)h\rangle + \langle b(T)h, h\rangle$$

$$+\int_0^T \|(R^{1/2}u - R^{-1/2}G^* g + R^{-1/2}G^* PX)(t)h\|^2 \, d\tau_m(t)$$

which is minimum when the integrand is zero for all h and t, i.e. when

$$u = R^{-1}G^*(g-PX). \blacksquare$$

A separate place in classical stochastic control literature is reserved for the *linear regulator* problem in which the noise coefficients in (2.1) are independent of X. The Fock space analogue of that problem is described in the following:

COROLLARY 2.1 (*linear regulator*). *The performance criterion* (2.3) *associated with the Hudson–Parthasarathy stochastic differential equation*

$$(2.22)\qquad \begin{aligned} dX(t) &= -[(FX+Gu)(t)\,d\tau_m(t) + \sigma(t)\,dA_m^+(t) + \varrho(t)\,dA_m(t)],\\ X(T) &= C, \quad 0 \leqslant t \leqslant T < \infty, \end{aligned}$$

is minimized by the control process $u = R^{-1}G^*(g-PX)$, *where* P *is the solution of the deterministic Riccati equation*

$$(2.23)\qquad dP(t) = (F^*P + PF + Q - PGR^{-1}G^*P)(t)\,d\tau_m(t), \qquad P(0) = M,$$

and g *is the solution of*

$$(2.24)\qquad \begin{aligned} dg(t) &= (-PGR^{-1}G^* + F^*)(t)\,g(t)\,d\tau_m(t)\\ &\quad -(P\sigma)(t)\,dA_m^+(t) - (P\varrho)(t)\,dA_m(t),\\ g(0) &= 0, \end{aligned}$$

and the minimum value is

$$\min_u J(u) = \langle Ch, P(T)Ch\rangle - 2\mathrm{Re}\langle Ch, g(T)h\rangle + \langle b(T)h, h\rangle,$$

where

$$b(t) = \int_0^t [(\sigma^* P\sigma - g^* GR^{-1}G^* g)(t)\,d\tau_m(t) - 2\mathrm{Re}(g^*\sigma)(t)\,dA_m^+(t) - 2\mathrm{Re}(g^*\varrho)(t)\,dA_m(t)].$$

Proof. The corollary follows from Theorem 2.1 by taking in (2.1) the equalities $L = \Phi = \Psi = 0$. $\blacksquare$

The control process u obtained in Theorem 2.1 is of *feedback* or *closed-loop* type since it utilizes the solution X to stabilize the system. In the linear regulator case we can show that there exists an *open-loop* control process u (i.e. one that does not use X) that is as good as the feedback control process u of Corollary 2.1. In fact, we have the following

THEOREM 2.2. *Let* $K = \{K(t) = -(R^{-1} G^* P)(t) \mid 0 \leqslant t \leqslant T\}$, *where* P *is the solution of* (2.23) *and let* $\Omega = \{\Omega(t) \mid 0 \leqslant t \leqslant T\}$ *be the solution of the equation*

$$d\Omega(t) = -(F+GK)(t)\,\Omega(t)\,d\tau_m(t), \qquad \Omega(T) = I, \tag{2.25}$$

where I *is the identity operator on* $H_0 \otimes \Gamma$. *The open-loop control process*

$$\begin{aligned} u_0 = \{u_0(t) = K(t)\,\Omega(t)\big[C + \int_t^T \Omega^{-1}(s)\big((GR^{-1}G^* g)(s)\,d\tau_m(s) + \sigma(s)\,dA_m^+(s) \\ + \varrho(s)\,dA_m(s)\big)\big] \mid 0 \leqslant t \leqslant T\}, \end{aligned} \tag{2.26}$$

where g *is the solution of* (2.24), *is an optimal control process for the linear regulator problem of* (2.22) *and* (2.3).

Proof. Let $Y = \{Y(t) \mid 0 \leqslant t \leqslant T\}$ be the solution of the equation

$$\begin{aligned} dY(t) &= -[(FY+Gu_0)(t)\,d\tau_m(t) \\ &\qquad + (GR^{-1}G^* g)(t)\,d\tau_m(t) + \sigma(t)\,dA_m^+(t) + \varrho(t)\,dA_m(t)], \\ Y(T) &= C, \quad 0 \leqslant t \leqslant T, \end{aligned} \tag{2.27}$$

or, equivalently, of the equation

$$\begin{aligned} dY(t) &= -\big[\big(F(t)\,Y(t) + (GK\Omega)(t)\,Z(t)\big)\,d\tau_m(t) \\ &\qquad + (GR^{-1}G^* g)(t)\,d\tau_m(t) + \sigma(t)\,dA_m^+(t) + \varrho(t)\,dA_m(t)\big], \\ &\qquad Y(T) = C, \quad 0 \leqslant t \leqslant T, \end{aligned} \tag{2.28}$$

where

$$Z(t) = C + \int_t^T \Omega^{-1}(s)\big((GR^{-1}G^* g)(s)\,d\tau_m(s) + \sigma(s)\,dA_m^+(s) + \varrho(s)\,dA_m(s)\big).$$

We notice that $\Omega(t)Z(t)$ is also a solution of (2.28) since, by (1.8),

$$\begin{aligned} d\big(\Omega(t)Z(t)\big) &= d\Omega(t)\cdot Z(t) + \Omega(t)\cdot dZ(t) + d\Omega(t)\cdot dZ(t) \\ &= -(F+GK)(t)\,\Omega(t)\,d\tau_m(t)\,Z(t) \\ &\quad - \Omega(t)\,\Omega^{-1}(t)\big((GR^{-1}G^* g)(t)\,d\tau_m(t) + \sigma(t)\,dA_m^+(t) + \varrho(t)\,dA_m(t)\big) \\ &\quad + (F+GK)(t)\,\Omega(t)\,d\tau_m(t)\cdot\Omega^{-1}(t)\big((GR^{-1}G^* g)(t)\,d\tau_m(t) \\ &\quad + \sigma(t)\,dA_m^+(t) + \varrho(t)\,dA_m(t)\big) \\ &= -(F+GK)(t)\,\Omega(t)\,Z(t)\,d\tau_m(s) \\ &\quad - (GR^{-1}G^* g)(t)\,d\tau_m(t) - \sigma(t)\,dA_m^+(t) - \varrho(t)\,dA_m(t) + 0 \quad \text{(by (1.7))} \\ &= -\big(F(t)\,\Omega(t)\,Z(t)\,d\tau_m(t) + (GK)(t)\,\Omega(t)\,Z(t)\,d\tau_m(t) \\ &\quad + (GR^{-1}G^* g)(t)\,d\tau_m(t) + \sigma(t)\,dA_m^+(t) + \varrho(t)\,dA_m(t)\big) \end{aligned}$$

and

$$\Omega(T)\cdot C = I\cdot C = C.$$

Thus $Y(t) = \Omega(t)Z(t)$ and (2.28) becomes

$$dY(t) = -[(F+GK)(t)\,Y(t)\,d\tau_m(t)+(GR^{-1}G^*g)(t)\,d\tau_m(t)$$
$$+\sigma(t)\,dA_m^+(t)+\varrho(t)\,dA_m(t)],$$
$$Y(T)=C, \quad 0\leqslant t\leqslant T,$$

which is the same as (2.22) with u replaced by $R^{-1}G^*(g-PX)$.

Thus $Y = X$, where X is the solution of (2.22) corresponding to the optimal feedback control process u found in Corollary 2.1, and so the value of the performance functional (2.3) for the open-loop control process u_0 of (2.26) is the same as that for the closed-loop optimal control process of Corollary 2.1. ∎

3. The orbit tracking problem. Theorem 2.1 can be used to solve the problem of finding the control process that will drive the output of a physical system close to a desired trajectory in an optimal fashion.

Specifically, we assume that the evolution $X = \{X(t) \mid 0\leqslant t\leqslant T\}$ of a physical system is described by the Hudson–Parthasarathy stochastic differential equation (2.1), and that the desired trajectory is described by an adapted process

$$(3.1)\quad U = \{U(t) = \delta+\int_t^T \alpha(s)\,d\tau_m(s)+\beta(s)\,dA_m^+(s)+\gamma(s)\,dA_m(s) \mid 0\leqslant t\leqslant T\},$$

where α, β, γ are adapted processes and $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta\in B(H_0\otimes\Gamma)$ for each $t\in[0, T]$. We consider the problem of finding the control process u that minimizes the *orbit tracking functional*

$$(3.2)\quad J(u) = \int_0^T [\langle (X(t)-U(t))h,\, Q(t)(X(t)-U(t))h\rangle$$
$$+\langle u(t)h,\, R(t)u(t)h\rangle]\,d\tau_m(t)+\langle (X(0)-U(0))h,\, M(X(0)-U(0))h\rangle$$

for every $h\in\operatorname{span}\{v\otimes y(f) \mid v\in D_0, f\in A\}$, where Q, R, M, τ_m are as in Section 2. The solution is provided by the following

PROPOSITION 3.1. *The orbit tracking functional J of* (3.2) *is minimized by the control process*

$$u = \{u(t) = R^{-1}G^*(g-P(X-U))(t) \mid 0\leqslant t\leqslant T\},$$

where $P = \{P(t) \mid 0\leqslant t\leqslant T\}$ is the solution of (2.4), *$g = \{g(t) \mid 0\leqslant t\leqslant T\}$ is* *the solution of the equation*

(3.3)
$$dg(t) = \big((-PGR^{-1}G^* + \Phi^*\Psi^* + F^*)g - \Phi^* P(\Phi U + \sigma + \beta) + P\Psi(\Phi U + \sigma + \beta) - P(FU + L + \alpha)\big)(t)\,d\tau_m(t) + \big(\Psi^* g - P(\Phi U + \sigma + \beta)\big)(t)\,dA_m^+(t) + \big(\Phi^* g - P(\Psi U + \varrho + \gamma)\big)(t)\,dA_m(t), \quad g(0) = 0,$$

and the minimum is

$$(3.4) \quad \min_u J(u) = \langle (C-\delta)h, P(T)(C-\delta)h\rangle - 2\mathrm{Re}\,\langle (C-\delta)h, g(T)h\rangle + \langle b(T)h, h\rangle,$$

where

(3.5)
$$b(t) = \int_0^t \big((\Phi U + \sigma + \beta)^* P(\Phi U + \sigma + \beta) - g^* GR^{-1}G^* g - 2\mathrm{Re}\big(g^*\Psi(\Phi U + \sigma + \beta)\big) - 2\mathrm{Re}\big(g^*(FU + L + \alpha)\big)\big)(t)\,d\tau_m(t) - 2\mathrm{Re}\big(g^*(\Phi U + \sigma + \beta)\big)(t)\,dA_m^+(t) - 2\mathrm{Re}\big(g^*(\Psi U + \varrho + \gamma)\big)(t)\,dA_m(t).$$

Proof. Let $Y = X - U$. Then (2.1), (3.1), and $dY = dX - dU$ imply

$$\begin{aligned} dY(t) &= -[(FX + Gu + L + \alpha)(t)\,d\tau_m(t) + (\Phi X + \sigma + \beta)(t)\,dA_m^+(t) + (\Psi X + \varrho + \gamma)(t)\,dA_m(t)] \\ &= -[(FY + Gu + FU + L + \alpha)(t)\,d\tau_m(t) + (\Phi Y + \Phi U + \sigma + \beta)(t)\,dA_m^+(t) \\ &\quad + (\Psi Y + \Psi U + \varrho + \gamma)(t)\,dA_m(t)], \qquad Y(T) = C - \delta, \end{aligned}$$

which is of the same form as (2.1). The proof is completed by a direct appeal to Theorem 2.1. ∎

Of particular importance is the minimization of the orbit tracking functional (3.2) in the case when $U = \{U(t) \mid 0 \leqslant t \leqslant T\}$ is the unique unitary solution of the Hudson–Parthasarathy equation

$$(3.6) \quad \begin{aligned} &dU(t) = (iK - \tfrac{1}{2}NN^*)\cdot U(t)\,d\tau_m(t) - WN^*\cdot U(t)\,dA_m^+(t) + N\cdot U(t)\,dA_m(t), \\ &U(T) = I, \end{aligned}$$

where I is the identity operator on $H_0 \otimes \Gamma$, $K = \tilde{K} \otimes \mathrm{Id}$, $N = \tilde{N} \otimes \mathrm{Id}$, $W = \tilde{W} \otimes \mathrm{Id} \in B(H_0 \otimes \Gamma)$, with Id denoting the identity operator on Γ, $\tilde{W}$ is unitary, and $\tilde{K}$ is self-adjoint (see Section 7 in [11] and Section 26 in [12]). In that case we can prove the following

PROPOSITION 3.2. *With X and U as in* (2.1) *and* (3.6), *respectively, the orbit tracking functional* (3.2) *is minimized by the control process*

$$u = \{u(t) = R^{-1}G^*(g - P(X - U))(t) \mid 0 \leqslant t \leqslant T\},$$

where $P = \{P(t) \mid 0 \leqslant t \leqslant T\}$ *is the solution of* (2.4), $g = \{g(t) \mid 0 \leqslant t \leqslant T\}$ *is the solution of the equation*

$$(3.7)\quad \begin{aligned} dg(t) = \big((&-PGR^{-1}G^* + \Phi^*\Psi^* + F^*)g - \Phi^* P(\Phi U + \sigma - WN^* U) \\ &+ P\Psi(\Phi U + \sigma - WN^* U) - P\big(FU + L + (iK - \tfrac{1}{2}NN^*)U\big)\big)(t)\, d\tau_m(t) \\ &+ \big(\Psi^* g - P(\Phi U + \sigma - WN^* U)\big)(t)\, dA_m^+(t) \\ &+ \big(\Phi^* g - P(\Psi U + \varrho + NU)\big)(t)\, dA_m(t), \qquad g(0) = 0, \end{aligned}$$

and the minimum is

$$(3.8)\quad \min_u J(u) = \langle (C-I)h,\, P(T)(C-I)h\rangle - 2\mathrm{Re}\langle (C-I)h,\, g(T)h\rangle + \langle b(T)h,\, h\rangle,$$

where

$$\begin{aligned} b(t) = \int_0^t \big((&\Phi U + \sigma - WN^* U)^* P(\Phi U + \sigma - WN^* U) - g^* GR^{-1} G^* g \\ &- 2\mathrm{Re}\big(g^* \Psi(\Phi U + \sigma - WN^* U)\big) - 2\mathrm{Re}\big(g^*\big(FU + L + (iK - \tfrac{1}{2}NN^*)U\big)\big)\big)(t)\, d\tau_m(t) \\ &- 2\mathrm{Re}\big(g^*(\Phi U + \sigma - WN^* U)\big)(t)\, dA_m^+(t) - 2\mathrm{Re}\big(g^*(\Psi U + \varrho + NU)\big)(t)\, dA_m(t). \end{aligned}$$

Proof. Let $Y = X - U$. Then by (2.1) and (3.6) we have

$$\begin{aligned} dY(t) &= -\big[\big((FX + Gu + L)(t) + (iK - \tfrac{1}{2}NN^*)\cdot U(t)\, d\tau_m(t)\big) \\ &\quad + \big((\Phi X + \sigma)(t) - WN^* \cdot U(t)\big)\, dA_m^+(t) + \big((\Psi X + \varrho)(t) + N\cdot U(t)\big)\, dA_m(t)\big] \\ &= -\big[\big((FY + Gu)(t) + (FU + L)(t) + (iK - \tfrac{1}{2}NN^*)\big)\cdot U(t)\, d\tau_m(t) \\ &\quad + \big((\Phi Y + \Phi U + \sigma)(t) - WN^*\big)\cdot U(t)\, dA_m^+(t) \\ &\quad + \big((\Psi Y + \Psi U + \varrho)(t) + N\cdot U(t)\big)\, dA_m(t)\big], \\ Y(Y) &= C - I, \end{aligned}$$

which is of the same form as (2.1) and the result follows directly from Theorem 2.1. ∎

4. The noise filtering problem. The filtering problem is that of the optimal estimation of the state of a system based on noisy observations of it. A well-known method of solution of the linear filtering problem is that of showing that it is dual to a control problem with duality exhibited through a Riccati type equation ([10], Section 9).

In Section 1 it was pointed out that if ξ is a $[0, \infty)$-valued observable and m is a ξ-martingale, then the process

$$B_m = \{B_m(t) = A_m(t) + A_m^+(t) \mid t \geqslant 0\}$$

is a standard Brownian motion written in the language of *quantum probability.* If μ is another ξ-martingale such that, in the notation of Section 1, $\langle\langle m, \mu\rangle\rangle = 0$, then we say that the Brownian motions B_m and B_μ are *orthogonal.* If $\langle\langle m, \mu\rangle\rangle = 0$ and for every $t \geqslant 0$

$$d\langle\langle m, m\rangle\rangle [0, t] = d\langle\langle \mu, \mu\rangle\rangle [0, t],$$

then we say that the Brownian motions B_m and B_μ are *compatible.* The existence of at least two nontrivial ξ-martingales m and μ with the above compatibility property is implied by the following

LEMMA 4.1. *Let β denote the Borel σ-algebra of $[0, \infty)$, let $H = L^2(\boldsymbol{R}, \boldsymbol{C})$, and let $P(H)$ denote the set of projection operators from H to itself. Define $\xi\colon \beta \to P(H)$ by $\xi(E) = \chi_{E\cup(-E)}\cdot I$, where $-E = \{-x \mid x \in E\}$, χ denotes characteristic function, and I is the identity on H. Define also $m\colon t \in [0, \infty) \to m_t \in H$ and $\mu\colon t \in [0, \infty] \to \mu_t \in H$ by $m_t = \chi_{[-t,0]}$ and $\mu_t = \chi_{[0,t]}$. Then $\langle\langle m, \mu\rangle\rangle = 0$ and, for every $t \geqslant 0$, $d\langle\langle m, m\rangle\rangle [0, t] = d\langle\langle \mu, \mu\rangle\rangle [0, t]$.*

Proof. It is easy to see that ξ is a $[0, \infty)$-observable. Moreover, for each $t \geqslant 0$

$$\langle m_t, \mu_t\rangle = \int_{[-t,0]\cap[0,t]} ds = 0, \qquad \langle m_t, m_t\rangle = \int_{-t}^{0} ds = t = \int_{0}^{t} ds = \langle \mu_t, \mu_t\rangle,$$

and for $0 \leqslant s \leqslant t$

$$\xi([0, s])\, m_t = \chi_{[-s,s]}\chi_{[-t,0]} I = \chi_{[-s,0]} I = m_s,$$

$$\xi([0, s])\, \mu_t = \chi_{[-s,s]}\chi_{[0,t]} I = \chi_{[0,s]} I = \mu_s.$$

Thus $\langle\langle m, \mu\rangle\rangle = 0$ and $d\langle\langle m, m\rangle\rangle [0, t] = d\langle\langle \mu, \mu\rangle\rangle [0, t] = dt$, where dt is the usual time differential. ∎

Generalizing [11], Section 5, we assume that all processes appearing in this section are adapted with respect to the triple (ξ, D_0, Λ), where ξ is a $[0, \infty)$-observable, $D_0 \subset H_0$, where H_0 is a separable Hilbert space and where, given two compatible ξ-martingales m, μ, a manifold $\Lambda \subset H$ is such that for every $0 \leqslant s < t < \infty$ and $f \in \Lambda$ we have $\xi([s, t]) f \in \Lambda$ and $\operatorname{Re}\langle\langle m, f\rangle\rangle = \operatorname{Re}\langle\langle \mu, f\rangle\rangle = 0$. For example, if ξ, m, μ are as in Lemma 4.1, we can take Λ to be the real manifold of all bounded purely imaginary functions in $L^2(\boldsymbol{R}, \boldsymbol{C})$. This requirement implies by (1.1) and (1.2) that the noise processes B_m and B_μ are *uncorrelated* in the sense that, for all $t \geqslant 0$, $u \in D_0$ and $f \in \Lambda$,

$$\Big\langle \int_0^t X(s)\, dB_\mu(s)\, u \otimes y(f), \int_0^t Y^*(s)\, dB_m(s)\, u \otimes y(f)\Big\rangle = 0$$

and have *zero expectation* in the sense that, for all $t \geqslant 0$, $u \in D_0$ and $f \in \Lambda$,

$$\Big\langle u \otimes y(f), \int_0^t X(s)\, dB_\mu(s)\, u \otimes y(f)\Big\rangle = \Big\langle u \otimes y(f), \int_0^t Z(s)\, dB_m(s)\, u \otimes y(f)\Big\rangle = 0$$

for all processes X, Y, Z for which $\int_0^t X(s)\,dB_\mu(s)$, $\int_0^t Z(s)\,dB_m(s)$ and $\int_0^t Y^*(s)\,dB_m(s)$ make sense. The Fock space analogue of the classical linear filtering problem is formulated and solved in the following

THEOREM 4.1 (Bucy–Kalman). *Let $T \in [0, \infty]$ and suppose that the state $\Phi = \{\Phi(t) \mid 0 \leqslant t \leqslant T\}$ of a system is described by the Hudson–Parthasarathy equation*

$$d\Phi(t) = \Phi(t)A(t)\,d\tau_\mu(t) + \sigma(t)\,dB_\mu(t), \qquad \Phi(0) = \Phi_0, \qquad 0 \leqslant t \leqslant T, \tag{4.1}$$

and is available only through an observation process $\Pi = \{\Pi(t) \mid 0 \leqslant t \leqslant T\}$ defined by

$$d\Pi(t) = \Phi(t)K(t)\,d\tau_m(t) + \varrho(t)\,dB_m(t), \qquad \Pi(0) = 0, \tag{4.2}$$

where B_μ and B_m are compatible Brownian motions and Ψ, A, σ, K, ϱ are adapted processes with $\Psi(t)$, $A(t)$, $\sigma(t)$, $K(t)$, $\varrho(t)$, $\Phi_0 \in B(H_0 \otimes \Gamma)$.

Then the filtering process $\gamma = \{\gamma(t) \mid 0 \leqslant t \leqslant T\}$ which is defined linearly in terms of the observation process Π by

$$\gamma(t) = \int_0^t d\Pi(s) \cdot u(t, s) \tag{4.3}$$

(where, for each t, $u(t, \cdot) = \{u(t, s) \mid 0 \leqslant s \leqslant t\}$ is adapted and such that, for every $h \in B(H_0 \otimes \Gamma)$, $\int_0^t \|u(t, s)h\|^2\,d\tau_m(s) < \infty$) and which minimizes at each $t \in [0, T]$ the filtering error

$$\|[\Phi(t) - \gamma(t)]\,v \otimes y(f)\| \tag{4.4}$$

for every $v \in D_0$, $f \in \Lambda$, is the solution of the equation

$$\begin{aligned} d\gamma(t) = {} & \big[\big(A(t) - K(t)(\varrho^*\varrho)^{-1}(t)K^*(t)P(t)\big)\gamma(t) \\ & + \Phi(t)K(t)(\varrho^*\varrho)^{-1}(t)K^*(t)P(t)\big]\,d\tau_m(t) \\ & + \varrho(t)(\varrho^*\varrho)^{-1}(t)K^*(t)P(t)\,dB_m(t), \\ \gamma(0) = {} & 0, \end{aligned} \tag{4.5}$$

where $P = \{P(t) \mid 0 \leqslant t \leqslant T\}$ is the solution of the deterministic Riccati equation

$$\begin{aligned} dP(t) = {} & [A^*(t)P(t) + P(t)A(t) + (\sigma^*\sigma)(t) \\ & - P(t)K(t)(\varrho^*\varrho)^{-1}(t)K^*(t)P(t)]\,d\tau_m(t), \\ P(0) = {} & \Phi_0^*\Phi_0. \end{aligned} \tag{4.6}$$

Remark. Equations (4.1), (4.2), and (4.3) are interpreted as the duals of the Hudson-Parthasarathy equations

$$d\Phi^*(t) = A^*(t)\,\Phi^*(t)\,d\tau_m(t) + \sigma^*(t)\,dB_\mu(t), \qquad \Phi^*(0) = \Phi_0^*,$$

$$d\Pi^*(t) = K^*(t)\,\Phi^*(t)\,d\tau_m(t) + \varrho^*(t)\,dB_m(t), \qquad \Pi^*(0) = 0,$$

and

$$\gamma^*(t) = \int_0^t u^*(t, s)\,d\Pi^*(s) = \int_0^t u^*(t, s)\,K^*(s)\,\Phi^*(s)\,d\tau_m(s) + u^*(t, s)\,\varrho^*(s)\,dB_m(s),$$

respectively.

Proof of Theorem 4.1. For each $t \in [0, T]$ let $X = \{X(s) \mid 0 \leqslant s \leqslant t\}$ be the solution of the equation

$$(4.7) \qquad dX(s) = [-A(s)\,X(s) + K(s)\,u(t, s)]\,d\tau_m(s), \qquad X(t) = I.$$

By (1.7) and (1.8) we have

$$\int_0^t \Phi(s)\cdot dX(s) = \Phi(s)\cdot X(s)|_0^t - \int_0^t d\Phi(s)\cdot X(s),$$

which by (4.7) and (4.1) implies that

$$\int_0^t \Phi(s)[-A(s)\,X(s) + K(s)\,u(t, s)]\,d\tau_m(s)$$

$$= \Phi(s)\,X(s)|_0^t - \int_0^t [\Phi(s)\cdot A(s)\,d\tau_\mu(s) + \sigma(s)\,dB_\mu(s)]\,X(s)$$

from which we obtain

$$(4.8) \qquad \Phi(t) = \Phi_0\,X(0) + \int_0^t \Phi(s)\,K(s)\,u(t, s)\,d\tau_\mu(s) + \int_0^t \sigma(s)\,X(s)\,dB_\mu(s).$$

By (4.8), (4.3) and (4.2) we obtain

$$(4.9) \qquad \Phi(t) - \gamma(t) = \Phi_0\,X(0) + \int_0^t \sigma(s)\,X(s)\,dB_\mu(s) - \int_0^t \varrho(s)\,u(t, s)\,dB_m(s),$$

and so for each $h = v \otimes y(f)$ with $v \in D_0$, $f \in \Lambda$, by (1.1), (1.2) and the compatibility of B_m and B_μ we obtain

$$(4.10) \qquad \|[\Phi(t) - \gamma(t)]\,h\|^2 = \langle[\Phi(t) - \gamma(t)]\,h,\ [\Phi(t) - \gamma(t)]\,h\rangle$$

$$= \langle\Phi_0\,X(0)\,h,\ \Phi_0\,X(0)\,h\rangle + \int_0^t \langle\sigma(s)\,X(s)\,h,\ \sigma(s)\,X(s)\,h\rangle\,d\tau_m(s)$$

$$+ \int_0^t \langle\varrho(s)\,u(t, s)\,h,\ \varrho(s)\,u(t, s)\,h\rangle\,d\tau_m(s).$$

In view of (4.10) the filtering problem reduces to the control problem of finding the process $u = \{u(s) \stackrel{\text{def}}{=} u(s, t) \mid 0 \leqslant s \leqslant t\}$ that minimizes the performance criterion

$$J(u) = \int_0^t [\langle X(s)h, (\sigma^* \sigma)(s) X(s) h\rangle + \langle u(s)h, (\varrho^* \varrho)(s) u(s) h\rangle] \, d\tau_m(s) \tag{4.11}$$

$$+ \langle X(0)h, (\Phi_0^* \Phi_0) \cdot X(0) h\rangle,$$

where X is defined by (4.7). By Theorem 2.1 we obtain

$$u(s, t) = (\varrho^* \varrho)^{-1}(s) K^*(s) P(s) X(s), \tag{4.12}$$

where $P = \{P(s) \mid 0 \leqslant s \leqslant t\}$ is the solution of the deterministic Riccati equation

$$\begin{aligned} dP(s) &= [A^*(s) P(s) + P(s) A(s) + (\sigma^* \sigma)(s) \\ &\quad - P(s) K(s) (\varrho^* \varrho)^{-1}(s) K^*(s) P(s)] \, d\tau_m(s), \\ P(0) &= \Phi_0^* \Phi_0, \qquad 0 \leqslant s \leqslant t. \end{aligned} \tag{4.13}$$

In view of (4.12), equation (4.7) can be written as

$$dX(s) = [-A(s) + K(s) (\varrho^* \varrho)^{-1}(s) K^*(s) P(s)] X(s) \, d\tau_m(s), \qquad X(t) = I. \tag{4.14}$$

As a function of t the solution $X = \{X(s) \stackrel{\text{def}}{=} X(t, s) \mid 0 \leqslant s \leqslant t\}$ of (4.14) satisfies the adjoint equation (see [2] and [10])

$$\begin{aligned} dX(t, s) &= [A(t) - K(t) (\varrho^* \varrho)^{-1}(t) K^*(t) P(t)] X(t, s) \, d\tau_m(t), \\ X(s, s) &= I \end{aligned} \tag{4.15}$$

or in the integral form

$$X(t, s) = \int_s^t [A(\theta) - K(\theta) (\varrho^* \varrho)^{-1}(\theta) K^*(\theta) P(\theta)] X(\theta, s) \, d\tau_m(\theta) + I, \tag{4.16}$$

which yields

$$\begin{aligned} \int_0^t d\Pi(s) \cdot (\varrho^* \varrho)^{-1}(s) K^*(s) P(s) X(t, s) &= \int_0^t d\Pi(s) \cdot (\varrho^* \varrho)^{-1}(s) K^*(s) P(s) \\ &\quad \times \int_s^t [A(\theta) - K(\theta) (\varrho^* \varrho)^{-1}(\theta) K^*(\theta) P(\theta)] X(\theta, s) \, d\tau_m(\theta) \\ &\quad + \int_0^t d\Pi(s) \cdot (\varrho^* \varrho)^{-1}(s) K^*(s) P(s). \end{aligned} \tag{4.17}$$

Replacing $d\Pi(s)$ by (4.2) and then using (1.1) and (1.4) we can apply Fubini's theorem to the double integral on the right-hand side of (4.17) to switch the order of integration. By (4.12) and (4.3), equation (4.17) implies

$$\gamma(t) = \int_0^t [A(\theta) - K(\theta)(\varrho^* \varrho)^{-1}(\theta) K^*(\theta) P(\theta)] \times \int_0^\theta d\Pi(s)(\varrho^* \varrho)^{-1}(s) K^*(s) P(s) X(\theta, s) d\tau_m(\theta) + \int_0^t d\Pi(s) \cdot (\varrho^* \varrho)^{-1}(s) K^*(s) P(s). \tag{4.18}$$

Consequently, we obtain

$$d\gamma(t) = [A(t) - K(t)(\varrho^* \varrho)^{-1}(t) K^*(t) P(t)] \gamma(t) d\tau_m(t) + d\Pi(t) \cdot (\varrho^* \varrho)^{-1}(t) K^*(t) P(t), \qquad \gamma(0) = 0, \tag{4.19}$$

which using (4.2) becomes

$$d\gamma(t) = \big[\big(A(t) - K(t)(\varrho^* \varrho)^{-1}(t) K^*(t) P(t)\big) \gamma(t) + \Phi(t) K(t) \cdot (\varrho^* \varrho)^{-1}(t) K^*(t) P(t)\big] d\tau_m(t) + \varrho(t) \cdot (\varrho^* \varrho)^{-1}(t) K^*(t) P(t) dB_m(t), \qquad \gamma(0) = 0. \tag{4.20}$$ ∎

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Department of Mathematics
American College of Greece
6 Gravias Str., Aghia Paraskevi, Athens
Greece, 15342

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