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AUTOREGRESSIVE LAPLACE FUNCTIONALS ON STOCHASTIC PROCESSES

BY

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Abstract. The paper deals with non-negative stochastic processes $X(t, \omega)$ $(t \geqslant 0)$ with stationary and independent increments, continuous on the right sample functions, non-degenerate to 0, and fulfilling the initial condition $X(0, \omega) = 0$. The main aim is to study the probability distribution ν_t of the random Laplace functional $\int_0^\infty \exp(-tX(\tau, \omega))\, d\tau$ for $t > 0$. In particular, a necessary and sufficient condition in terms of corresponding representing measures for ν_t to be multiplicatively autoregressive is established.

1. Preliminaries and notation. This paper* is organized as follows. Section 1 collects together some basic facts and notation concerning infinitely divisible probability distributions needed in the sequel. In Section 2 a class $\mathscr{B}_0$ of Bernstein functions is discussed. In the last section this class $\mathscr{B}_0$ is applied to study the random Laplace functionals for some stochastic processes.

We denote by $\mathscr{M}$ the set of all non-negative bounded countably additive measures defined on Borel subsets of the real line $R = (-\infty, \infty)$. Given $M \in \mathscr{M}$ we denote by $\operatorname{supp} M$ the support of M and we put

$$b_1(M) = \inf \operatorname{supp} M, \qquad b_2(M) = \sup \operatorname{supp} M.$$

By $\mathscr{P}$ we denote the subset of $\mathscr{M}$ consisting of probability measures. By δ_c we denote the probability measure concentrated at the point c. For $M, N \in \mathscr{M}$ we write $M \leqslant N$ whenever $M(A) \leqslant N(A)$ for all Borel subsets A of R. Further, by $M * N$ we denote the convolution of M and N. Given $M \in \mathscr{M}$, by $\tilde{M}$ we denote the Fourier transform of M, i.e.

$$\tilde{M}(s) = \int_{-\infty}^{\infty} e^{isx} M(dx) \qquad (s \in R).$$

By the Lévy–Khinchin Representation Theorem ([3], Chapter XVII,2), infinitely divisible probability measures on R are of the form $e(a, N)$, where

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$a \in R$, $N \in \mathcal{M}$ and

$$(1.1)\quad \tilde{e}(a, N)(s) = \exp\left[ias + \int_{-\infty}^{\infty}\left(e^{isx} - 1 - \frac{isx}{1+x^2}\right)(1-e^{-|x|})^{-2} N(dx)\right],$$

where for $x = 0$ the integrand is assumed to be $-s^2/2$. The following statement was proved by Esseen in [2]:

PROPOSITION 1.1. *Let* $\mu = e(a, N)$. *Then* $b_1(\mu) > -\infty$ *if and only if* $N((-\infty, 0]) = 0$ *and* $\int_0^1 x^{-1} N(dx) < \infty$. *Moreover,*

$$(1.2)\qquad b_1(\mu) = a - \int_0^{\infty} x(1+x^2)^{-1}(1-e^{-x})^{-2} N(dx).$$

The class $\mathcal{L}$ consists of self-decomposable probability measures, i.e. the limit distributions of normed sums $a_n(X_1 + X_2 + \ldots + X_n) + b_n$, where $a_n > 0$, $b_n \in R$, and $X_1, X_2, \ldots$ are independent real-valued random variables fulfilling the condition

$$\lim_{n\to\infty} \max_{1\leqslant k\leqslant n} P(a_n|X_k| > \varepsilon) = 0 \quad \text{for every } \varepsilon > 0.$$

A representation of the class $\mathcal{L}$ was found by P. Lévy (see [4], formula (1.1)). Namely, $\mu \in \mathcal{L}$ if and only if $\mu = e(a, N)$ with $a \in R$ and

$$(1.3)\qquad N(dx) = c\delta_0(dx) + x^{-1}(1-e^{-|x|})^2 k(x)\,dx,$$

where $c \geqslant 0$, $k(x)$ is non-positive on $(-\infty, 0)$ and non-negative on $(0, \infty)$, $k(x)$ is non-increasing on each of the intervals $(-\infty, 0)$ and $(0, \infty)$, and

$$(1.4)\qquad \int_{|u|\leqslant 1} uk(u)\,du + \int_{|u|>1} u^{-1}k(u)\,du < \infty.$$

We say that a measure μ from $\mathcal{P}$ is *strictly unimodal at the point* q if μ is absolutely continuous on $(-\infty, q) \cup (q, \infty)$ and has a density increasing on $(b_1(\mu), q)$ and decreasing on $(q, b_2(\mu))$.

We denote by $\mathcal{M}_+$ the subset of $\mathcal{M}$ consisting of measures M with $\operatorname{supp} M \subset R_+ = [0, \infty)$ and $M(R_+) > 0$. Further, we put $\mathcal{P}_+ = \mathcal{P} \cap \mathcal{M}_+$. Given $M \in \mathcal{M}_+$ we denote by $\hat{M}$ the Laplace transform of M, i.e.

$$\hat{M}(z) = \int_0^{\infty} e^{-zx} M(dx) \qquad (z \in R_+).$$

PROPOSITION 1.2. *Let* $\mu \in \mathcal{P}_+$ *and* $\mu \neq \delta_0$. *Then* $\mu \in \mathcal{L}$ *and* μ *is strictly unimodal at* 0 *if and only if*

$$(1.5)\qquad \hat{\mu}(z) = \exp\int_0^{\infty} x^{-1}(e^{-zx}-1)k(x)\,dx,$$

where $k(x)$ *is non-increasing on* $(0, \infty)$, $0 < k(0+) \leqslant 1$, $0 \leqslant k(x) < 1$ *for* $x \in (0, \infty)$ *and* $\int_1^{\infty} x^{-1}k(x)\,dx < \infty$. *For* $x = 0$ *the integrand is assumed to be* $-k(0+)z$.

Proof. Necessity. Suppose that μ belongs to $\mathscr{L} \cap \mathscr{P}_+$ and is strictly unimodal at 0. Then

$$b_1(\mu) = 0 \tag{1.6}$$

and $\mu = e(a, N)$, where $a \in R$ and N is given by formula (1.3). Applying Proposition 1.1 we conclude that $N((-\infty, 0]) = 0$ or, equivalently, $k(x) = 0$ for $x \in (-\infty, 0)$ and $c = 0$ in (1.3). Moreover,

$$\int_0^1 x^{-1} N(dx) = \int_0^1 x^{-2}(1-e^{-x})^2 k(x)\, dx < \infty,$$

which yields the inequality

$$\int_0^1 k(x)\, dx < \infty. \tag{1.7}$$

Further, by (1.4),

$$\int_1^\infty x^{-1} k(x)\, dx < \infty \tag{1.8}$$

and, by (1.2), (1.3) and (1.6),

$$a = \int_0^\infty (1+x^2)^{-1} k(x)\, dx.$$

Consequently, formula (1.1) can be written in the form

$$\tilde{\mu}(s) = \exp \int_0^\infty x^{-1}(e^{isx}-1) k(x)\, dx \qquad (s \in R),$$

which yields representation (1.5). If $\mu \neq \delta_0$, then $k(0+) > 0$. By (1.8) it remains to prove the inequalities

$$k(0+) \leqslant 1 \tag{1.9}$$

and

$$k(x) < 1 \qquad \text{for } x \in (0, \infty). \tag{1.10}$$

Suppose the contrary $k(0+) > 1$. Then, by (1.7), the measure μ fulfils the conditions of Theorem 1.3, part (vii), in [4]. Consequently, μ is not unimodal at 0. This proves inequality (1.9).

If $k(0+) = 1 = k(u)$ for some $u \in (0, \infty)$, then, by Theorem 1.4 in [4], the measure μ is not strictly unimodal. This proves inequality (1.10). The necessity of our conditions is thus proved.

Sufficiency. Suppose that $\mu \in \mathscr{P}_+$ and its Laplace transform is given by formula (1.5). It is clear that $\mu \in \mathscr{L}$. Moreover, the measure μ fulfils the conditions of Theorem 1.3, parts (ii) and (iii), in [4]. Consequently, the measure μ is absolutely continuous on $[0, \infty)$ with a density decreasing on $(0, \infty)$. Thus μ is strictly unimodal at 0, which completes the proof.

As an immediate consequence of our Proposition 1.2, Theorem 1.3, parts (ii) and (iii), and Theorem 1.6 in [4] we get the following statement:

PROPOSITION 1.3. *Suppose that* $\mu \in \mathscr{P}_+ \cap \mathscr{L}$, $\mu \neq \delta_0$ *and* μ *is strictly unimodal at* 0. *Then*

$$b_2(\mu) = \infty \tag{1.11}$$

and μ *is absolutely continuous on* R_+ *with absolutely continuous positive density* $g(x)$ *fulfilling for some* $a \in [0, 1)$ *the condition*

$$g(x) = x^{-a} f(x), \tag{1.12}$$

where the function f *is slowly varying at* 0.

2. A class of Bernstein functions. Given $M \in \mathscr{M}_+$ we denote by $\langle M \rangle$ the Bernstein transform of M, i.e.

$$\langle M \rangle (z) = \int_0^\infty \frac{1-e^{-zx}}{1-e^{-x}} M(dx) \qquad (z \in R_+).$$

Here for $x = 0$ the integrand is assumed to be z. It is clear that the measure M is uniquely determined by its Bernstein transform. Moreover, it is easy to check the inequalities

$$z \frac{d}{dz} \langle M \rangle (z) \leqslant \langle M \rangle (z) \qquad (z \in (0, \infty)), \tag{2.1}$$

$$\langle M \rangle (az) \leqslant a \langle M \rangle (z) \qquad (a \geqslant 1, z \in R_+). \tag{2.2}$$

Denote by $\mathscr{C}$ the set of all completely monotone functions on the open half-line $(0, \infty)$. Let $\mathscr{B}$ be the set of all functions $\langle M \rangle$ with $M \in \mathscr{M}_+$. It is well known that $F \in \mathscr{B}$ if and only if F is continuous on $[0, \infty)$, $F(0) = 0$, F does not vanish identically, and F is differentiable on $(0, \infty)$ with $(d/dz) F \in \mathscr{C}$.

Given $M \in \mathscr{M}_+$ and $a > 0$ we define a probability measure $\gamma_a(M)$ on R_+ by setting

$$\gamma_a(M)(dx) = c_a(M\{0\}) \delta_0(dx) + e^{-ax} \int_x^\infty (1-e^{-y})^{-1} M(dy)\, dx, \tag{2.3}$$

where $c = a/\langle M \rangle (a)$. It is clear that for $a, b > 0$ the formula

$$\gamma_{a+b}(M)(dx) = c_{a,b} e^{-bx} \gamma_a(M)(dx) \tag{2.4}$$

holds with $c_{a,b} = c_{a+b}/c_a$. Further, by a simple calculation we get the formulae

$$\hat{\gamma}_a(z) = c_a (z+a)^{-1} \langle M \rangle (z+a) \tag{2.5}$$

and for $u > 0$

$$\gamma_a(M)([u, \infty)) = a^{-1} c_a \int_u^\infty (1-e^{-x})^{-1} (e^{-au} - e^{-ax}) M(dx).$$

Hence

$$b_2(M) = b_2(\gamma_a(M)) \quad (a > 0). \tag{2.6}$$

Moreover, from formula (2.3) we get the following statement:

PROPOSITION 2.1. *For every $M \in \mathscr{M}_+$ and $a > 0$ the probability measure $\gamma_a(M)$ is strictly unimodal at* 0.

Put for $x \in R_+$

$$\Pi(dx) = x^{-1}(1-e^{-x})e^{-x}\,dx. \tag{2.7}$$

It is evident that $\Pi \in \mathscr{M}_+$ and

$$\langle \Pi \rangle(z) = \log(1+z). \tag{2.8}$$

Let $M \in \mathscr{M}_+$. It is clear that for every $b > 0$ the function

$$\frac{\langle M \rangle(bz+b)}{\langle M \rangle(b)} - 1 \tag{2.9}$$

belongs to $\mathscr{B}$. Consequently, the superposition of functions (2.8) and (2.9) also belongs to $\mathscr{B}$ ([3], Chapter XIII,4). In other words, for every $M \in \mathscr{M}_+$ and $b > 0$ there exists a measure $\tau_b(M) \in \mathscr{M}_+$ fulfilling the condition

$$\langle \tau_b(M) \rangle(z) = \log \frac{\langle M \rangle(bz+b)}{\langle M \rangle(b)}. \tag{2.10}$$

Applying inequality (2.2) we have

$$\tau_b(M)(R_+) = \langle \tau_b(M) \rangle(1) \leqslant \log 2. \tag{2.11}$$

Denote by $\mathscr{D}$ the set of all continuous real-valued functions F on $[0, \infty)$, positive and differentiable on $(0, \infty)$, and fulfilling the initial condition $F(0) = 0$. Given $F \in \mathscr{D}$ we put

$$l(F)(z) = z\frac{d}{dz}F(z)/F(z) \quad \text{for } z \in (0, \infty).$$

In the sequel, $\mathscr{A}$ will denote the set of all functions F on R_+ of the form

$$F(z) = c\exp\int_0^\infty K(x, z)\,Q(dx),$$

where $c > 0$,

$$K(x, z) = \int_0^\infty y^{-1}(e^{-y} - e^{-zy})\,dy,$$

Q is a non-negative countably additive measure on R_+ fulfilling the condition

$$0 < \int_0^\infty e^{-zx} Q(dx) < \infty$$

for $z > 0$. It is easy to check the formula

$$F(z) = c \exp \int_0^\infty x^{-1}(e^{-x} - e^{-zx}) Q([0, x))\, dx, \tag{2.12}$$

where for $x = 0$ the integrand is assumed to be $Q(\{0\})z$. Observe that $\mathscr{A} \subset \mathscr{D}$ and the pair c, Q is uniquely determined by F. In the sequel we shall use the notation $F = [c, Q]$.

LEMMA 2.1. *$F \in \mathscr{A}$ if and only if $F \in \mathscr{D}$ and $l(F) \in \mathscr{C}$.*

Proof. The necessity follows from the inclusion $\mathscr{A} \subset \mathscr{D}$ and the formula $l([c, Q])(z) = \hat{Q}(z)$. To prove the sufficiency observe that the representation of completely monotone functions

$$l(F)(z) = \hat{Q}(z)$$

yields

$$\log F(z) - \log F(1) = \int_1^z u^{-1} \hat{Q}(u)\, du = \int_0^\infty K(x, z) Q(dx).$$

Consequently, $F = [F(1), Q]$, which completes the proof.

LEMMA 2.2. *If $F = [c, Q]$ and $Q(R_+) \leqslant 1$, then $F = \langle M \rangle$ for some $M \in \mathscr{M}_+$ and $\gamma_a(M) \in \mathscr{L}$ for all $a \in (0, \infty)$.*

Proof. Put for $a > 0$ and $z \geqslant 0$

$$G_a(z) = \frac{aF(z+a)}{(z+a)F(a)}. \tag{2.13}$$

By standard calculations we get the formula

$$G_a(z) = \exp \int_0^\infty x^{-1}(e^{-zx} - 1) k_a(x)\, dx,$$

where $k_a(x) = e^{-ax}\big(1 - Q([0, x))\big)$. First we shall prove that there exist probability measures μ_a $(a > 0)$ belonging to $\mathscr{L}$ with the property

$$\hat{\mu}_a(z) = G_a(z). \tag{2.14}$$

If $Q(\{0\}) = 1$, then $G_a(z) = 1$ and we put $\mu_a = \delta_0$. In the remaining case we have $k_a(0+) > 0$ and the functions k_a fulfil the conditions of Proposition 1.2, which yields the existence of probability measures μ_a with desired properties. Starting from the formula $l(F)(z) = \hat{Q}(z)$ and taking into account (2.13) and

(2.14) we have

$$\frac{d}{dz}F(z+a) = a^{-1}F(a)\hat{Q}(z+a)\hat{\mu}_a(z),$$

which yields $(d/dz)F(z+a)\in\mathscr{C}$ for $a>0$. Consequently, $F(z+a)-F(a)\in\mathscr{B}$ for $a>0$. Passing $a\to 0$ and taking into account the continuity of F on $[0, \infty)$ and the initial condition $F(0)=0$ we get the relation $F\in\mathscr{B}$. Thus the function F has a representation $F=\langle M\rangle$ for some $M\in\mathscr{M}_+$. By (2.5) and (2.13), $\hat{\gamma}_a(M)=G_a$, which, by (2.14), yields the equality $\gamma_a(M)=\mu_a$. This shows that $\gamma_a(M)\in\mathscr{L}$ for all $a>0$, which completes the proof.

Introduce the notation $\mathscr{B}_0=\mathscr{A}\cap\mathscr{B}$. The set $\mathscr{B}_0$ of Bernstein functions will play a crucial role in our considerations.

THEOREM 2.1. *The following conditions are equivalent:*
(i) $F\in\mathscr{B}_0$.
(ii) $F\in\mathscr{A}$ *and* $l(F)\leqslant 1$.
(iii) $F=[c, Q]$ *and* $Q(R_+)\leqslant 1$.
(iv) $F\in\mathscr{D}$, $l(F)\in\mathscr{C}$ *and* $l(F)\leqslant 1$.

Proof. By (2.1), $l(F)\leqslant 1$ for $F\in\mathscr{B}$, which yields the implication (i) $\Rightarrow$ (ii). The implication (ii) $\Rightarrow$ (iii) is an immediate consequence of the formulae $l([c, Q])=\hat{Q}$ and $\hat{Q}(0+)=Q(R_+)$. The above formulae and the inequality $\hat{Q}(z)\leqslant Q(R_+)$ yield the implication (iii) $\Rightarrow$ (iv). Finally, the implication (iv) $\Rightarrow$ (i) is an immediate consequence of Lemmas 2.1 and 2.2. This completes the proof.

THEOREM 2.2. *The set $\mathscr{B}_0$ is closed under superposition.*

Proof. Suppose that $F, G\in\mathscr{B}_0$ and put $H(z)=F(G(z))$. It is evident that $H\in\mathscr{D}$. Since $G\in\mathscr{B}$ and, by Theorem 2.1, part (iv), $l(F)\in\mathscr{C}$, we conclude that the superposition $l(F)(G(z))$ also belongs to $\mathscr{C}$ ([3], Chapter XIII,4). Taking into account the relation $l(G)\in\mathscr{C}$ and the formula

$$l(H)(z) = l(G)(z)\,l(F)(G(z))$$

we have $l(H)\in\mathscr{C}$. The above formula and the inequalities $l(F)\leqslant 1$ and $l(G)\leqslant 1$ (Theorem 2.1, part (iv)) show the inequality $l(H)\leqslant 1$, which, by Theorem 2.1, part (iv), implies the relation $H\in\mathscr{B}_0$. The theorem is thus proved.

Denote by $\mathscr{M}_0$ the subset of $\mathscr{M}_+$ consisting of all measures M with $\langle M\rangle\in\mathscr{B}_0$.

LEMMA 2.3. *Suppose that $M\in\mathscr{M}_0$ and $\langle M\rangle=[c, Q]$. Then*

(2.15) $$\tau_a(M)(dx) = x^{-1}(1-e^{-x})e^{-x}Q([0, a^{-1}x))\,dx \quad \textit{for } a>0,$$

(2.16) $$\tau_a(M)\geqslant\tau_b(M) \quad \textit{for } 0<a\leqslant b$$

and

$$\tau_a(M) \leqslant \Pi \quad \textit{for } a > 0. \tag{2.17}$$

Proof. By (2.10) and (2.12) we get the formula

$$\langle \tau_a(M) \rangle (z) = \int_0^\infty x^{-1}(1-e^{-zx})e^{-x} Q([0, a^{-1}x))\, dx,$$

which yields (2.15). Since the function $Q([0, x))$ is non-decreasing, we have also inequality (2.16). By Theorem 2.1, part (iii), the inequality $Q(R_+) \leqslant 1$ is true. Comparing this with (2.7) and (2.15) we get inequality (2.17), which completes the proof.

THEOREM 2.3. *The following conditions are equivalent*:

(i) $M \in \mathcal{M}_0$.

(ii) $M \in \mathcal{M}_+$ *and* $\gamma_a(M) \in \mathcal{L}$ *for all* $a > 0$.

(iii) $M \in \mathcal{M}_+$ *and* $\tau_b(M) \geqslant \tau_1(M)$ *for* $b \in (0, 1]$.

Proof. The implication (i) $\Rightarrow$ (ii) is a direct consequence of Theorem 2.1, part (iii), and Lemma 2.2.

To prove the implication (ii) $\Rightarrow$ (iii) assume that $\gamma_a(M) \in \mathcal{L}$ for all $a > 0$. First suppose that $\gamma_a(M) = \delta_0$ for some $a > 0$. Then, by (2.6), $b_2(M) = 0$ and, consequently, $M = c\delta_0$ with some $c > 0$. Thus $\langle M \rangle (z) = cz$ and, by (2.10), $\langle \tau_b(M) \rangle (z) = \log(1+z)$ for all $b > 0$. Thus $\tau_b(M) = \tau_1(M)$ for all $b > 0$, which yields condition (iii). Suppose now that $\gamma_a(M) \neq \delta_0$ for all $a > 0$. Taking into account Propositions 1.2 and 2.1 we have the representation

$$\hat{\gamma}_a(M)(z) = \exp \int_0^\infty x^{-1}(e^{-zx}-1)\, k_a(x)\, dx, \tag{2.18}$$

where $k_a(x)$ is non-increasing on $(0, \infty)$, $0 < k_a(0+) \leqslant 1$, $0 \leqslant k_a(x) < 1$ for $x \in (0, \infty)$ and $\int_1^\infty x^{-1} k_a(x)\, dx < \infty$. Without loss of generality we may assume that the functions k_a are continuous on the right. Taking into account (2.4) we get, by standard calculations,

$$k_b(x) = e^{-(b-a)x} k_a(x) \quad \text{for } 0 < a \leqslant b \text{ and } x \in R_+.$$

Hence it follows that the limit

$$\lim_{a \to 0} k_a(x) = k(x)$$

exists and the function $k(x)$ is non-negative, non-increasing and fulfils the equation

$$k_b(x) = e^{-bx} k(x) \quad \text{for all } b > 0 \text{ and } x \in R_+.$$

Substituting this into (2.18) and applying formulae (2.5) and (2.10) we get

$$\langle \tau_b(M) \rangle (z) = \log(1+z) + \int_0^\infty x^{-1}(e^{-zx}-1)\, e^{-x} k(b^{-1}x)\, dx. \tag{2.19}$$

Since the function $k(x)$ is non-increasing, the formula

$$N_b(dx) = x^{-1}(1-e^{-x})e^{-x}\big(k(x)-k(b^{-1}x)\big)\,dx$$

defines a non-negative measure on R_+ for $b\in(0, 1)$. Moreover, by (2.19),

$$\langle\tau_b(M)\rangle(z) = \langle\tau_1(M)\rangle(z)+\langle N_b\rangle(z).$$

Thus $\tau_b(M) = \tau_1(M)+N_b$, which yields condition (iii).

It remains to prove the implication (iii) $\Rightarrow$ (i). Suppose that $M\in\mathcal{M}_+$ and

$$\tau_b(M) \geqslant \tau_1(M) \quad \text{for } b\in(0, 1]. \tag{2.20}$$

From (2.10) we get the formula

$$\langle\tau_b(M)\rangle(z) = \langle\tau_1(M)\rangle(bz+b-1)+\log\langle M\rangle(1)-\log\langle M\rangle(b),$$

which yields

$$\tau_b(M)(dx) = h(x, b)\,\tau_1(M)(b^{-1}\,dx), \tag{2.21}$$

where $h(0, b) = b$ and

$$h(x, b) = (1-e^{-x})e^{-x}(1-e^{-x/b})^{-1}e^{x/b}$$

for $x\in(0, \infty)$. In particular, we have $\tau_b(M)(\{0\}) = b\tau_1(M)(\{0\})$, which, by (2.20), yields $\tau_1(M)(\{0\}) = 0$. Suppose that a Borel subset E of the half-line $(0, \infty)$ has the Lebesgue measure 0. Consequently, setting

$$g(E, x) = \int_0^1 1_E(bx)\,h(bx, b)\,db,$$

we have $g(E, x) = 0$ for $x\geqslant 0$. By (2.21) we have the formula

$$\int_0^1 \tau_b(M)(E)\,db = \int_0^\infty g(E, x)\,\tau_1(M)(dx) = 0,$$

which, by (2.20), yields $\tau_1(M)(E) = 0$. In other words, the measure $\tau_1(M)$ is absolutely continuous on R_+ and, consequently, can be written in the form

$$\tau_1(M)(dx) = x^{-1}(1-e^{-x})e^{-x}q(x)\,dx,$$

where $q(x)$ is a non-negative function. Moreover, by (2.21),

$$\tau_b(M)(dx) = x^{-1}(1-e^{-x})e^{-x}q(b^{-1}x)\,dx, \tag{2.22}$$

which, by (2.20), yields the inequality $q(b^{-1}x)\geqslant q(x)$ for all $b\in(0, 1]$ and almost all $x\in R_+$. Consequently, without loss of generality we may assume that the function q is non-decreasing and continuous on the left. By (2.11) and (2.22),

$$\int_0^\infty x^{-1}(1-e^{-x})e^{-x}q(b^{-1}x)\,dx \leqslant \log 2 \quad \text{for all } b\in(0, \infty).$$

Passing $b \to 0$ we get the inequality

$$q(\infty) \leqslant 1. \tag{2.23}$$

Since $\tau_1(M) \in \mathscr{M}_+$, we have also the inequality

$$q(\infty) > 0. \tag{2.24}$$

Starting from (2.10) and (2.22) we get, by standard calculations,

$$\begin{aligned}\langle M\rangle(z+b) &= \langle M\rangle(1+b)\exp\big(\langle\tau_b(M)\rangle(b^{-1}z) - \langle\tau_b(M)\rangle(b^{-1})\big)\\ &= \langle M\rangle(1+b)\exp\int_0^\infty x^{-1}(e^{-x}-e^{-zx})e^{-bx}q(x)\,dx.\end{aligned}$$

Passing $b \to 0$ we obtain the formula

$$\langle M\rangle(z) = \langle M\rangle(1)\exp\int_0^\infty x^{-1}(e^{-x}-e^{-zx})q(x)\,dx.$$

Setting $Q([0, x)) = q(x)$ for $x > 0$ we define a measure Q which, by (2.23) and (2.24) belongs to $\mathscr{M}_+$ and fulfils the inequality $Q(R_+) \leqslant 1$. Moreover, $\langle M\rangle = [\langle M\rangle(1), Q]$ which, by Theorem 2.1, part (iii), shows that $M \in \mathscr{M}_0$. The theorem is thus proved.

THEOREM 2.4. *Suppose that $M \in \mathscr{M}_0$. Then either $M = c\delta_0$ for some $c > 0$ or $b_2(M) = \infty$, the measure M is absolutely continuous on R_+ and for some $a, b \in [0, 1)$*

$$\int_x^\infty y^{-1} M(dy) = x^{-a} f_0(x) \tag{2.25}$$

and

$$\int_0^x yM(dy) = x^b f_\infty(x), \tag{2.26}$$

where f_0 and f_∞ are slowly varying at 0 and ∞, respectively.

Proof. Suppose that $M \in \mathscr{M}_0$. By Proposition 2.1 and Theorem 2.3, part (ii), the probability measure $\gamma_1(M)$ is strictly unimodal at 0 and belongs to $\mathscr{L}$. If $\gamma_1(M) = \delta_0$, then, by (2.6), $b_2(M) = 0$, which yields the equality $M = c\delta_0$ for some $c > 0$. Suppose now that $\gamma_1(M) \neq \delta_0$. Then, by Proposition 1.3, the measure $\gamma_1(M)$ is absolutely continuous on R_+ with absolutely continuous positive density $g(x)$ fulfilling for some $a \in [0, 1)$ condition (1.12) and $b_2(\gamma_1(M)) = \infty$. The last equality and (2.6) yield $b_2(M) = \infty$. Denoting by $g^*(x)$ the almost everywhere derivative of $g(x)$ and taking into account (2.3) we get, by standard calculations,

$$M(dx) = c_1^{-1}(1-e^x)\big(g(x)+g^*(x)\big)\,dx,$$

which shows that the measure M is absolutely continuous on R_+. Since, by (2.3), $\int_x^\infty y^{-1} M(dy)/g(x)$ tends to a positive limit as $x \to 0$, relation (2.25) is an immediate consequence of (1.12).

To prove relation (2.26) observe that, by Theorem 2.1,

$$\langle M\rangle(z) = c\exp\int_0^\infty K(x, z)\,Q(dx)$$

with some $c > 0$ and $0 < Q(R_+) \leqslant 1$. Define the auxiliary measure U by setting

$$U(dx) = x(1-e^{-x})^{-1} M(dx).$$

Then

$$\hat{U}(z) = \frac{d}{dz}\langle M\rangle(z) = z^{-1}\hat{Q}(z)\langle M\rangle(z). \tag{2.27}$$

Since for every $z > 0$

$$K(x, zt) - K(x, t) = K(xt, z) \to \log z \quad \text{as } t \to 0$$

and

$$\langle M\rangle(zt)/\langle M\rangle(t) = \exp\int_0^\infty K(xt, z)\,Q(dx),$$

we have, by (2.27),

$$\hat{U}(zt)/\hat{U}(t) \to z^{-b} \quad \text{as } t \to 0$$

with $b = 1 - Q(R_+) \in [0, 1)$. Applying the classical Tauberian Theorem ([3], Chapter XIII,5) we get the formula

$$U([0, x)) = x^b f(x), \tag{2.28}$$

where f is slowly varying at ∞. Observe that $\int_0^x yM(dy)/U([0, x))$ tends to a positive limit as $x \to \infty$. Thus relation (2.26) is an immediate consequence of (2.28), which completes the proof.

3. Autoregressive Laplace functionals. A family λ_t $(t > 0)$ of probability measures on R is called *autoregressive* if the mapping

$$(0, \infty) \ni t \to \lambda_t \in \mathscr{P}$$

is continuous in the topology of weak convergence in $\mathscr{P}$ and for every pair u, t with $u \in (0, t]$ there exists a measure $\varrho \in \mathscr{P}$ such that

$$\lambda_t = \lambda_u * \varrho.$$

The set of all such factors ϱ will be denoted by $\mathscr{D}(u, t)$. It is easy to check that the set $\mathscr{D}(u, t)$ is compact,

$$\mathscr{D}(t, t) = \{\delta_0\}, \tag{3.1}$$

$$\overline{\lim_{n\to\infty}}\, \mathscr{D}(u_n, t_n) \subset \mathscr{D}(u, t), \tag{3.2}$$

where $\overline{\lim}$ denotes the upper topological limit, $u_n \to u > 0$ and $t_n \to t$. Moreover, for $0 < t_0 < t_1 < \ldots < t_n$ the inclusion

$$\mathop{*}_{k=1}^{n} \mathscr{D}(t_{k-1}, t_k) \subset \mathscr{D}(t_0, t_n) \tag{3.3}$$

is true. Here $\mathscr{A}_1 * \mathscr{A}_2$ denotes the set of all measures $\mu * \nu$ with $\mu \in \mathscr{A}_1$ and $\nu \in \mathscr{A}_2$.

LEMMA 3.1. *For autoregressive families and every pair u, t with $u \in (0, t]$ the set $\mathscr{D}(u, t)$ contains an infinitely divisible probability measure.*

Proof. By (3.1) we may restrict ourselves to the case $0 < u < t$. For every integer n we put

$$t_{k,n} = u(t/u)^{k/n} \qquad (k = 0, 1, \ldots, n).$$

Let $\varrho_{k,n} \in \mathscr{D}(t_{k-1,n}, t_{k,n})$ $(k = 1, 2, \ldots, n)$. By (3.3) the relation

$$\mathop{*}_{k=1}^{n} \varrho_{k,n} \in \mathscr{D}(u, t) \tag{3.4}$$

is true. Moreover, by (3.1) and (3.2), the triangular array $\varrho_{k,n}$ $(k = 1, 2, \ldots, n;$ $n = 1, 2, \ldots)$ consists of asymptotically negligible measures, i.e. for every sequence k_n of indices fulfilling the inequality $1 \leqslant k_n \leqslant n$

$$\lim_{n \to \infty} \varrho_{k_n,n} = \delta_0.$$

By the compactness of $\mathscr{D}(u, t)$ and (3.4) the sequence $*_{k=1}^{n} \varrho_{k,n}$ contains a convergent subsequence. Its limit belongs to $\mathscr{D}(u, t)$ and is infinitely divisible ([3], Chapter XVII,7), which completes the proof.

An autoregressive family λ_t $(t > 0)$ is said to be *strictly autoregressive* if for every pair u, t with $u \in (0, t]$ the set $\mathscr{D}(u, t)$ consists of probability measures concentrated at single points.

Given a random variable X we denote by $\operatorname{distr} X$ its probability distribution. For two independent random variables X and Y with $\operatorname{distr} X = \mu$ and $\operatorname{distr} Y = \nu$ we put $\mu \circ \nu = \operatorname{distr} XY$.

A family λ_t $(t > 0)$ of probability measures on R is called *multiplicatively autoregressive* if the mapping

$$(0, \infty) \ni t \to \lambda_t \in \mathscr{P}$$

is continuous and for every pair u, t with $u \in (0, t]$ there exists a measure $\nu \in \mathscr{P}$ such that $\lambda_t = \lambda_u \circ \nu$. If in addition ν is concentrated at a single point, then the family in question is called *multiplicatively strictly autoregressive.*

Throughout this paper $X(t, \omega)$ $(t \geqslant 0)$ will denote a non-negative stochastic process with stationary and independent increments, continuous on the right sample functions, non-degenerate to 0 and fulfilling the initial condition

$X(0, \omega) = 0$. Setting $p_t = \operatorname{distr} X(t, \omega)$ $(t \geqslant 0)$ we have the formula

$$\hat{p}_t(z) = \exp\big(-t\langle M\rangle(z)\big) \quad \text{for some } M \in \mathcal{M}_+.$$

This uniquely determined measure M is called the *representing measure* for the process in question. We denote by $\mathscr{X}$ the class of all processes $X(t, \omega)$ with the above properties.

A process $X(t, \omega)$ from $\mathscr{X}$ with the representing measure M is called *stable* if $\langle M\rangle(z) = cz^p$ for some $c > 0$ and $p \in (0, 1]$. For $p = 1$ the process is deterministic, i.e. $X(t, \omega) = ct$ with probability 1.

Let $X(t, \omega)$ be a process from $\mathscr{X}$ with the representing measure M. It is evident that for every $a > 0$ the process $aX(t, \omega)$ also belongs to $\mathscr{X}$. Denote by M_a its representing measure. Since $\langle M_a\rangle(z) = \langle M\rangle(az)$, we have the formula

$$M_a(dx) = (1-e^{-x})(1-e^{-x/a})^{-1} M(a^{-1}dx). \tag{3.5}$$

Two processes $X(t, \omega)$ and $Y(t, \omega)$ from $\mathscr{X}$ are said to be *independent* if for all finite collections $t_1, t_2, \ldots, t_n$ and $u_1, u_2, \ldots, u_n$ of non-negative numbers the vector-valued random variables

$$\big(X(t_1, \omega), X(t_2, \omega), \ldots, X(t_n, \omega)\big) \quad \text{and} \quad \big(Y(u_1, \omega), Y(u_2, \omega), \ldots, Y(u_n, \omega)\big)$$

are independent. One can easily check that for independent processes $X(t, \omega)$ and $Y(t, \omega)$ from $\mathscr{X}$ the composition $Z(t, \omega) = Y\big(X(t, \omega), \omega\big)$ also belongs to $\mathscr{X}$. Moreover, denoting by M, N and S the representing measures for $X(t, \omega)$, $Y(t, \omega)$ and $Z(t, \omega)$, respectively, we have the formula

$$\langle S\rangle(z) = \langle M\rangle\big(\langle N\rangle(z)\big). \tag{3.6}$$

It was proved in [5], Example 3.4, that for every process $X(t, \omega) \in \mathscr{X}$ and $t > 0$ the random Laplace functional $\int_0^\infty e^{-tX(\tau,\omega)}\,d\tau$ is finite and positive with probability 1. Introduce the notation

$$\nu_t = \operatorname{distr} \int_0^\infty e^{-tX(\tau,\omega)}\,d\tau \qquad (t > 0). \tag{3.7}$$

The aim of this section is to describe processes $X(t, \omega)$ from $\mathscr{X}$ in terms of their representing measures for which the family ν_t $(t > 0)$ is multiplicatively autoregressive.

EXAMPLE 3.1. Let $X(t, \omega)$ be a stable process from $\mathscr{X}$ with $\langle M\rangle(z) = cz^p$ for some $c > 0$ and $p \in (0, 1]$. It was shown in [5], Example 4.1, that the probability distribution (3.7) is given by the formula

$$\nu_t = \sigma_p \circ \delta_{a(t)},$$

where $a(t) = (ct)^{-p}$ and σ_p is the probability distribution with the Laplace transform

$$\hat{\sigma}_p(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{(n!)^p}.$$

It is clear that the family ν_t is multiplicatively strictly autoregressive.

THEOREM 3.1. *Let $X(t, \omega)$ be a process from $\mathscr{X}$ with the representing measure M. Then the family of probability distributions of the random Laplace functional $\int_0^\infty e^{-tX(\tau,\omega)} d\tau$ $(t > 0)$ is multiplicatively autoregressive if and only if $M \in \mathscr{M}_0$. If it is the case the probability distribution*

$$\mu_t = \operatorname{distr}\Big(-\log \int_0^\infty e^{tX(\tau,\omega)} d\tau\Big) \qquad (t > 0)$$

belongs to the class $\mathscr{L}$ and for non-deterministic processes is absolutely continuous on R with continuously differentiable density. Moreover, if $\langle M \rangle = [c, Q]$, then

$$\mu_t = e(b_t, N_t) \tag{3.8}$$

for some $b_t \in R$ and

$$N_t(dx) = x^{-1}(1-e^{-x})e^{-x}\big(1-Q([0, t^{-1}x))\big)\,dx. \tag{3.9}$$

Proof. Let $X(t, \omega)$ be a process from $\mathscr{X}$ with the representing measure M. Given $a > 0$ we denote by M_a the representing measure for the process $aX(t, \omega)$. By (2.10) and (3.5) we have

$$\tau_1(M_a) = \tau_a(M).$$

Taking into account the above formula and applying Theorem 3.1 of [6] we get the equation

$$\mu_t * e\big(a_t, \tau_t(M)\big) = e(0, \Pi) \tag{3.10}$$

with some $a_t \in R$ and the measure Π defined by (2.7). Since, by (1.1), the Fourier transform of the right-hand side of the above equation is everywhere different from 0, we conclude that

$$\tilde{\mu}_t(s) \neq 0 \quad \text{for all } s \in R \text{ and } t > 0. \tag{3.11}$$

First we shall prove the sufficiency of our condition. Suppose that $M \in \mathscr{M}_0$ and $\langle M \rangle = [c, Q]$. Put $N_t = \Pi - \tau_t(M)$ for $t > 0$. By Lemma 2.3, inequality (2.17), N_t is a non-negative measure on R_+ and, by (2.15), formula (3.9) is true. Now, by (3.10) and (3.11), we get formula (3.8) with $b_t = -a_t$. Given a pair u, t satisfying the condition $u \in (0, t]$ we put $S_{u,t} = N_u - N_t$. By Lemma 2.3, inequality (2.16), we infer that $S_{u,t}$ are non-negative measures on R_+. Moreover, by (3.8),

$$\mu_t = \mu_u * e(b_t - b_u, S_{u,t}),$$

which shows that the family μ_t is autoregressive. Hence it follows that the family ν_t is multiplicatively autoregressive. Observe that formula (3.9) can be rewritten in the form

$$N_t(dx) = x^{-1}(1-e^{-x})^2 k_t(x)\,dx,$$

where

$$k_t(x) = (e^x-1)^{-1}\big(1-Q([0, t^{-1}x))\big).$$

It is clear that the function k_t is non-negative non-increasing on R_+ and fulfils the condition

$$\int_0^1 uk_t(u)\,du + \int_1^\infty u^{-1} k_t(u)\,du < \infty.$$

Consequently, by (1.3), $\mu_t \in \mathscr{L}$. If the process in question is non-deterministic, then $Q(\{0\}) < 1$. Consequently, $k_t(0+) = \infty$ and $\int_0^1 k_t(u)\,du = \infty$. Applying Theorem 1.3, part (xi), of [4] we conclude that the measure μ_t is absolutely continuous on R with continuously differentiable density.

To prove the necessity of our condition suppose that the family ν_t defined by (3.7) is multiplicatively autoregressive. Consequently, the family μ_t is autoregressive. Applying Lemma 3.1 we conclude that for every $u \in (0, 1]$ there exist $c_u \in R$ and $S_u \in \mathscr{M}$ such that

$$\mu_1 = \mu_u * e(c_u, S_u).$$

Setting the above expression into (3.10) for $t = 1$ we get the equality

$$\mu_u * e\big(a_1 + c_u, \tau_1(M) + S_u\big) = e(0, \Pi).$$

Comparing this with equality (3.10) for $t = u$ and taking into account (3.11) we get the formula

$$e\big(a_1 + c_u, \tau_1(M) + S_u\big) = e\big(a_u, \tau_u(M)\big),$$

which yields $\tau_1(M) + S_u = \tau_u(M)$. Thus $\tau_1(M) \leqslant \tau_u(M)$ for $u \in (0, 1]$, which, by Theorem 2.3, part (iii), shows that $M \in \mathscr{M}_0$. The theorem is thus proved.

THEOREM 3.2. *The family of probability distributions of the random Laplace functional on a process $X(t, \omega)$ from $\mathscr{X}$ is multiplicatively strictly autoregressive if and only if $X(t, \omega)$ is stable.*

Proof. The sufficiency of our condition is established in Example 3.1. To prove the necessity we assume that the probability distributions of the random Laplace functional on $X(t, \omega)$ form a multiplicatively strictly autoregressive family. Hence it follows that the family of probability distributions $\operatorname{distr}\int_0^\infty e^{-tX(\tau,\omega)}\,d\tau$ $(t > 0)$ is strictly autoregressive, which, by Theorem 3.1, formulae (3.7) and (3.8), yields the equality $Q([0, t^{-1}x)) = Q([0, u^{-1}x))$ for all $x > 0$ and $0 < u \leqslant t$. Hence $Q = p\delta_0$. Since $0 < Q(R_+) \leqslant 1$, we have $0 < p \leqslant 1$. Thus $\langle M\rangle(z) = [c, Q](z) = cz^p$, which shows that the process in question is stable. The theorem is thus proved.

As an immediate consequence of Theorems 2.2 and 3.1 and formula (3.6) we get the following statement:

THEOREM 3.3. *The composition of two independent processes from $\mathscr{X}$ with multiplicatively autoregressive random Laplace functionals also has multiplicatively autoregressive Laplace functional.*

We conclude this section with some examples.

EXAMPLE 3.2. *Poisson process.* The probability distributions p_t are given by the formula

$$p_t = e^{-ct}\left(\sum_{n=0}^{\infty} \frac{c^n t^n}{n!}\delta_n\right)$$

with some $c > 0$. Here we have $M = c(1-e^{-1})\delta_1$ which, by Theorem 3.1, shows that family (3.7) is not multiplicatively autoregressive.

EXAMPLE 3.3. *Gamma process.* The probability distributions p_t $(t > 0)$ are given by the formula

$$p_t(dx) = \frac{e^{-x} x^{t-1}}{\Gamma(t)} dx.$$

Thus $\hat{p}_t(z) = (1+z)^{-t}$ and $\langle M\rangle(z) = \log(1+z)$. Setting $Q = \int_0^1 p_t\,dt$ we have

$$\hat{Q}(z) = \int_0^1 \hat{p}_t(z)\,dt = \frac{z}{(1+z)\log(1+z)} = l(\langle M\rangle)(z),$$

which shows that $l(\langle M\rangle)\in\mathscr{C}$. Since $l(\langle M\rangle)(0+) = 1$, we conclude, by Theorem 2.1, part (iv), that $\langle M\rangle\in\mathscr{B}_0$ or, equivalently, $M\in\mathscr{M}_0$. Thus, by Theorem 3.1, family (3.7) is multiplicatively autoregressive.

EXAMPLE 3.4. *Bessel process.* The probability distributions p_t and the representing measure M are given by the formulae

$$p_t(dx) = x^{-1} te^{-x} I_t(x)\,dx, \qquad M(dx) = x^{-1}(1-e^{-x})e^{-x} I_0(x)\,dx \qquad (t > 0),$$

where I_t denotes the modified Bessel function of the first kind ([3], Chapter XIII,7). Setting $s(z) = 1+z+(z^2+2z)^{1/2}$ we have $\hat{p}_t(z) = s(z)^{-t}$ and $\langle M\rangle(z) = \log s(z)$. Let K_p denote the modified Heinkel function defined by the formula

$$K_p(x) = (2\sin p\pi)^{-1}\pi\left(I_{-p}(x) - I_p(x)\right) \tag{3.12}$$

for p being not an integer ([1], Chapter 7.2, formula (13)) or by the integral representation

$$\Gamma(p+\tfrac{1}{2})K_p(x) = \pi^{1/2}(x/2)^p \int_1^{\infty} e^{-xy}(y^2-1)^{p-1/2}\,dy$$

for $p > -1/2$ and $x \in (0, \infty)$ ([1], Chapter 7.3, formula (15)). It is clear that $K_p(x) > 0$ for $x \in (0, \infty)$ and $p > -1/2$. Put for $x > 0$

$$q(x) = \pi^{-1} e^{-x} \int_0^1 K_p(x)(\sin p\pi)\, dp$$

and $Q(dx) = q(x)\,dx$. From formula (18) in [1], Chapter 7.7, we get

$$\int_0^\infty e^{-zx} e^{-x} I_p(x)\, dx = (z^2+2z)^{-1/2} s(z)^{-p} \quad \text{for } p > -1,$$

which together with (3.12) yields

$$\hat{Q}(z) = z^{1/2}(z+2)^{-1/2}\big(\log s(z)\big)^{-1} = l(\langle M\rangle)(z).$$

Thus $l(\langle M\rangle) \in \mathscr{C}$. Since $l(\langle M\rangle)(0+) = \frac{1}{2}$, we conclude, by Theorem 2.1, part (iv), that $\langle M\rangle \in \mathscr{B}_0$ or, equivalently, $M \in \mathscr{M}_0$. Thus, by Theorem 3.1, family (3.6) is multiplicatively autoregressive.

EXAMPLE 3.5. *Gamma Poisson process.* The probability distributions p_t depend upon a positive parameter s and are given by the formula

$$p_t = e^{-t}\left(\delta_0 + \sum_{n=1}^{\infty} \frac{t^n}{n!} G_s^{*n}\right), \quad \text{where } G_s(dx) = \frac{e^{-x} x^{s-1}}{\Gamma(s)}\, dx.$$

Consequently, the representing measure M_s is given by the formula

$$M_s(dx) = \big(\Gamma(s)\big)^{-1}(1-e^{-x}) e^{-x} x^{s-1}\, dx,$$

which yields $\langle M\rangle(z) = 1-(1+z)^{-s}$.

First we shall prove the relation

$$M_4 \in \mathscr{M}_0. \tag{3.13}$$

It is easy to verify that the function

$$h(x) = e^{-x}(4-2\sin x - 2\cos x - 2e^{-x})$$

is non-negative on R_+ and

$$\int_0^\infty e^{-zx} h(x)\, dx = 4z(1+z)^{-1}\big((1+z)^4-1\big)^{-1} = l(\langle M_4\rangle)(z).$$

Consequently, $l(\langle M_4\rangle) \in \mathscr{C}_0$. Since $l(\langle M_4\rangle)(0+) = 1$, we infer, by Theorem 2.1, part (iv), that $\langle M_4\rangle \in \mathscr{B}_0$. Relation (3.13) is thus proved.

Given $r \in (0, 1]$ we put

$$F_r(z) = (1+z)^r - 1, \qquad U_r(z) = \big(\log(1+z)\big)^{-1}\big(1-(1+z)^{r-1}\big),$$

$$V_r(z) = (r+z)^{-1} \quad \text{and} \quad W_r(z) = \big(\log(1+z)\big)^{-1}\big((1+z)^r - 1\big) - r.$$

It is clear that $V_r \in \mathscr{C}$. Taking into account the formulae

$$U_r(z) = \int_0^{1-r} (1+z)^{-y}\,dy, \qquad W_r(z) = \int_0^r (1+z)^y\,dy - r$$

we conclude that $U_r \in \mathscr{C}$ and $W_r \in \mathscr{B}$. Consequently, the superposition

$$G_r(z) = r\big(1 + U_r(z)\,V_r(W_r(z))\big)$$

belongs to $\mathscr{C}$ ([3], Chapter XIII,4). Since

$$G_r(z) = rz(1+z)^{r-1}\big((1+z)^r - 1\big)^{-1} = l(F_r)(z) \quad \text{and} \quad l(F_r)(0+) = 1,$$

we infer, by Theorem 2.1, part (iv), that $F_r \in \mathscr{B}_0$. Taking into account (3.13) and applying Theorem 2.2 we conclude that the superposition $\langle M_4 \rangle (F_r(z))$ belongs to $\mathscr{B}_0$ for every $r \in (0, 1]$. Observe that

$$\langle M_{4r} \rangle (z) = \langle M_4 \rangle (F_r(z))$$

and, consequently, $M_s \in \mathscr{M}_0$ for $s \in (0, 4]$.

Now we shall prove the converse implication. Suppose that $M_s \in \mathscr{M}_0$. Then, by Theorem 3.1, the probability distributions

$$\mu_t = \operatorname{distr}\big(-\log \int_0^\infty e^{-tX(\tau,\omega)}\,d\tau\big) \qquad (t > 0)$$

are infinitely divisible, which, by Proposition 4.1 of [6], yields the relation $s \in (0, 4]$. Consequently, by Theorem 3.1, family (3.7) is multiplicatively autoregressive if and only if $s \in (0, 4]$.

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