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EXISTENCE THEOREM AND WONG–ZAKAI APPROXIMATIONS FOR MULTIVALUED STOCHASTIC DIFFERENTIAL EQUATIONS*

BY

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Abstract. We consider finite-dimensional multivalued stochastic differential equations where the drift has a multivalued and monotone term. Existence and approximation results are obtained by an existence theorem for deterministic differential inclusions and a polygonal approximation of the Brownian motion. The dispersion matrix is assumed to be state space independent.

Applications are given for Coulomb damping and hysteretic systems.

1. Introduction. Differential inclusions

$$x'(t) \in b\big(t,\, x(t)\big) - Ax(t) + f(t),$$

where A is a (multivalued) maximal monotone map, b is a Lipschitz continuous map, and f is a locally integrable function, have been studied by several authors, see e.g. Brézis [3], Benilan and Brézis [2], Aubin and Cellina [1], Chapter 3, Krée [6] and Krée and Soize [7], Chapter XIV, Lakshmikantham and Leela [8], Chapter 3, and Miyadera [9]. The state spaces in cited papers were general Hilbert spaces. In this paper we let the state space be finite dimensional.

A generalization of differential inclusions to 'stochastic differential inclusions,' called *multivalued stochastic differential equations*, is obtained by replacing f by a fixed matrix σ times the generalized derivative of the Brownian motion, i.e. Gaussian white noise. In this case it is convenient to write, analogously to stochastic differential equations, as follows:

$$d\xi(t) \in \big\{b\big(t,\, \xi(t)\big) - A\xi(t)\big\}\, dt + \sigma dB(t).$$

Krée [6] and the author [11] showed, for $\sigma = \sigma(t, x)$, the existence of a solution to multivalued stochastic differential equations in two different ways. The former used a fixed point argument and the latter a Yosida approximation technique.

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Here we take another approach. We consider the case when the dispersion σ is fixed, i.e. time and state independent. The idea is, for a given sample path of the Brownian motion B, to study deterministic differential inclusions where the driving force f is the right derivative of a polygonal approximation of the Brownian motion. By letting the polygon train get close to the Brownian motion, we obtain, under a suitable integrability condition, convergence of the solutions to the related differential inclusions, and the limit is shown to satisfy a multivalued stochastic differential equation.

Hence we obtain, in the spirit of Wong and Zakai [13], [14], a solution to a multivalued stochastic differential equation (with fixed dispersion σ) as the limit of a sequence of solutions to differential inclusions. This means that a solution to a multivalued stochastic differential equation can be approximated by a numerical method for a differential inclusion. It is also close to intuition in models where the driving force is Gaussian white noise.

In Section 2 results for differential inclusions are recalled. In Section 3 a solution to a multivalued stochastic differential equation is defined, and in Section 4, the main section in this note, the existence of a solution is shown by considering a sequence of differential inclusions. Section 5 contains convergence in mean square, Section 6 is devoted to applications, and in the last section a Wong–Zakai result is given for a case with space dependent dispersion.

2. Differential inclusions. Throughout, let $\mathfrak{R}^d$ be equipped with the usual norm $|\cdot|$ and inner product $\langle\cdot, \cdot\rangle$. A set-valued (or multivalued) map A from $\mathfrak{R}^d$ on $\mathfrak{R}^d$ is a map that to any x in $\mathfrak{R}^d$ associates a subset $A(x)$ (possibly the empty set) of $\mathfrak{R}^d$. For a set-valued map A on $\mathfrak{R}^d$ let $\mathscr{D}(A) = \{x \in \mathfrak{R}^d\colon\ A(x) \neq \emptyset\}$ be the domain of A.

Definition 2.1. A set-valued map A from $\mathfrak{R}^d$ into $\mathfrak{R}^d$ is called *monotone* if

$$\langle u_1 - u_2, x_1 - x_2\rangle \geqslant 0 \tag{1}$$

for all $x_i \in \mathscr{D}(A)$ and $u_i \in A(x_i)$, $i = 1, 2$. ▪

A monotone set-valued map A is said to be *maximal* if there is no other monotone set-valued map whose graph strictly contains the graph of A. We write throughout Ax instead of $A(x)$. For a maximal monotone map A and a fixed point $x = \mathscr{D}(A)$, the set Ax is closed and convex (see e.g. Brézis [3], chapitre II.4). Hence there exists a unique point $(Ax)^0$ such that $|(Ax)^0| = \min\{|y|\colon\ y \in Ax\}$. Let A^0 be the map from $\mathscr{D}(A)$ into $\mathfrak{R}^d$ defined by $A^0x = (Ax)^0$. For $x \in \mathscr{D}(A)$ and $y \in \mathfrak{R}^d$ we denote by $(Ax - y)^0$ the point in $Ax - y$ with smallest norm. Throughout, let $[0, T]$ be a finite closed interval in $\mathfrak{R}$. We recall the definition of a solution to a differential inclusion (cf. Krée [7], Definition XIV.1.8).

Definition 2.2. Let A be a set-valued map on $\mathfrak{R}^d$, $b = b(t, x)$ a map from $[0, T] \times \mathfrak{R}^d$ into $\mathfrak{R}^d$, $f \in L^1([0, T]; H)$, and u_0 a given point in $\overline{\mathscr{D}(A)}$.

A *solution to the differential inclusion*

$$u'(t) \in b(t, u(t)) - Au(t) + f(t), \qquad u(0) = u_0 \tag{2}$$

is any $u \in C([0, T]; H)$ such that
- $u(t) \in \mathscr{D}(A)$ for all t in $[0, T]$,
- the distribution derivative u' is in $L^1([0, T], \mathfrak{R}^d)$,
- $u'(t) \in b(t, u(t)) - Au(t) + f(t)$ for almost all t in $[0, T]$. ∎

Some ordinary differential equations, where the right-hand side satisfies a monotonicity condition, can be rewritten as differential inclusions. One of the simplest examples is perhaps the following

EXAMPLE 2.3. Consider the one-dimensional differential equation

$$x'(t) = -\operatorname{sign} x(t), \qquad x(0) = 1,$$

where $\operatorname{sign} 0 = 1$. This equation has obviously no solution in the usual sense for $t \geq 1$. However, a related differential inclusion $x'(t) \in -A(x(t))$, where A is defined by $A(x) = \operatorname{sign} x$ for $x \neq 0$ and $A(0) = [-1, 1]$, admits a solution and the solution $x(\cdot)$ has a right derivative $D^+x(\cdot)$ satisfying $D^+x(t) = -A^0 x(t)$, where, in this case, $A^0(x) = \operatorname{sign} x$ for $x \neq 0$ and $A^0(0) = 0$. ∎

Uniqueness of a solution to (2) is elementary to show by the Bellman–Gronwall inequality if e.g. b satisfies a Lipschitz condition in x. Krée showed an existence result for (2) which holds if $f \in C([0, T], \mathfrak{R}^d)$. However, since in Section 4 we will let f be the right derivative of a polygonal approximation of a Brownian motion, we have to allow f to have jumps. We therefore need the following theorem which is obtained by Benilan and Brézis [2], Corollaire 1.3, or a slight modification of the proof of Theorem 1.3 in Krée [6] (see also Krée and Soize [7], Theorem XIV.1.10).

THEOREM 2.4. *Let A be a maximal monotone map on H such that $\mathscr{D}(A)$ is closed and A^0 is bounded on compact subsets of $\mathscr{D}(A)$. Let b: $[0, T] \times \mathfrak{R}^d \mapsto \mathfrak{R}^d$ be a continuous map such that*

$$|b(t, x) - b(t, y)| \leq L|x-y| \quad \text{for } t \in [0, T],\ x, y \in \mathfrak{R}^d.$$

Further, let f: $[0, T] \mapsto \mathfrak{R}^d$ be piecewise continuous and $u_0 \in \mathscr{D}(A)$. Then there is a solution to $u'(t) \in b(t, u(t)) - Au(t) + f(t)$, $u(0) = u_0$. Furthermore, for every $t \in [0, T)$, u has a right derivative d^+u/dt, explicitly given by

$$\frac{d^+u}{dt} = \big(b(t, u(t)) - Au(t) + f(t+0)\big)^0, \tag{3}$$

where $f(t+0) = \lim_{h \downarrow 0, h>0} f(t+h)$. ∎

Remark 2.5. The condition that $\mathscr{D}(A)$ is closed and A^0 is bounded on compact subsets of $\mathscr{D}(A)$ is valid if for example $\mathscr{D}(A) = \mathfrak{R}^d$ or if A describes the outwards directed normal cone at the boundary of a closed convex set G

in $\Re^d$, i.e.

$$Ax = \{p \in \Re^d: \; -\langle x-y, p\rangle \leqslant 0, \; \forall y \in G\}$$

for $x \in G = \mathscr{D}(A)$ (see e.g. Benilan and Brézis [2], remarque 1.5). ∎

For existence results of solutions to the differential inclusions where $f \in L^1([0, T]; \Re^d)$, see e.g. Benilan and Brézis [2].

The aim in this note is to construct a solution to a differential inclusion if the driving force f is Gaussian white noise. For this case we use the concept of multivalued stochastic differential equations, defined in the following section.

3. Multivalued stochastic differential equations. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with a right continuous and complete filtration $\{\mathscr{F}_t\}_{t \geqslant 0}$. Let $\{B(t)\}_{t \geqslant 0}$ be an $\{\mathscr{F}_t\}$-adapted m-dimensional Brownian motion. Suppose A is a maximal monotone map on $\Re^d$, and let x_0 be a fixed point in $\mathscr{D}(A)$, and C be a universal constant. For a function g: $[0, T] \to \Re^d$ let $\|g\|_T = \sup_{0 \leqslant t \leqslant T} |g(t)|$.

Definition 3.1. Let b and σ be Borel measurable maps,

$$b: [0, T] \times \Re^d \mapsto \Re^d, \qquad \sigma: [0, T] \times \Re^d \mapsto \Re^d \times \Re^m,$$

and let A be a maximal monotone set valued map on $\Re^d$. By a *solution to the multivalued stochastic differential equation*

$$d\xi(t) \in b(t, \xi(t))\,dt - A\xi(t)\,dt + \sigma(t, \xi(t))\,dB, \qquad \xi(0) = x_0 \tag{4}$$

we mean a couple (ξ, η) on $\{\mathscr{F}_t\}$-adapted processes such that ξ is continuous almost surely and

(i) $\xi(t) \in \mathscr{D}(A)$ for all $t \in [0, T]$,

(ii) η is absolutely continuous where $\eta'(t) \in A\xi(t)$ for almost all t in $[0, T]$, and $\int_0^T |\eta'(t)|^2\,dt$ is finite,

(iii) $\xi(t) = x_0 + \int_0^t b(s, \xi(s))\,ds - \eta(t) + \int_0^t \sigma(s, \xi(s))\,dB(s)$ interpreted in the sense of Itô. ∎

We assume throughout that σ is constant and that b satisfies the usual Lipschitz and linear growth conditions

$$|b(t, x) - b(t, y)| \leqslant L|x-y|, \qquad |b(t, x)| \leqslant L(1+|x|) \tag{5}$$

for $0 \leqslant t \leqslant T$, $x, y \in \Re^d$.

Note that if (i)–(iii) in Definition 3.1 are satisfied together with (5), then

$$E\,\|\xi\|_T^2 < \infty \tag{6}$$

(see [11]).

Uniqueness (pathwise) of a solution to (4) is easy to verify if (5) is valid (see Krée [6] or Krée and Soize [7], Chapter XIV).

In this note we prove, by considering differential inclusions, the existence of a solution under a convenient integrability condition. The integrability condition is shown to hold in a product situation and if A satisfies a linear growth condition (Chapter 4). We also give an example when the integrability condition is not satisfied.

4. Wong–Zakai approximations. This section is the main part in this note. It is disposed as follows. First we introduce solutions ξ_δ of differential inclusions where the driving force $f = \sigma B'_\delta$ is a *constant* matrix σ times the right derivative of a polygonal approximation of B. Then we give, under a suitable integrability condition, an existence theorem for solutions of multivalued stochastic differential equations. Finally, we show that this condition holds if A satisfies a linear growth condition or if we have the mentioned product situation.

For $\delta > 0$ let $0 = t_0 < t_1 < \ldots < t_{c_\delta} = T$ be a partition of $[0, T]$ with

$$\text{mesh}\,\delta = \max\{\Delta t_k \colon 1 \leqslant k \leqslant c_\delta\}, \qquad \text{where } \Delta t_k = t_k - t_{k-1}.$$

Define B_δ as follows: $B_\delta(0) = 0$ and

$$B_\delta(t) = B(t_{k-1}) + (t - t_{k-1})\frac{\Delta B(t_k)}{\Delta t_k}, \qquad t_{k-1} \leqslant t \leqslant t_k, \tag{7}$$

where $\Delta B(t_k) = B(t_k) - B(t_{k-1})$. Let $B'_\delta(t) = \Delta B(t_k)/\Delta t_k$ for $t_{k-1} \leqslant t < t_k$, i.e. the right derivative of B_δ. By Theorem 2.4 it follows that if $\mathscr{D}(A)$ is closed and A^0 is bounded on compact subsets of $\mathscr{D}(A)$, and b is continuous and satisfies the usual Lipschitz condition, then there exists a unique solution ξ_δ to the following differential inclusion:

$$\xi'_\delta(t) \in b(t, \xi_\delta(t)) - A\xi_\delta(t) + \sigma B'_\delta(t). \tag{8}$$

We have thus, loosely speaking, replaced dB in the multivalued stochastic differential equation (4) by $B'_\delta(t)\,dt$. We assume throughout that the conditions for the existence of a solution to (8) are satisfied. Note that, by Definition 2.2, we can identify a solution to (8) by a couple $(\xi_\delta, \eta_\delta)$ of absolutely continuous components where $\xi_\delta(t) \in \mathscr{D}(A)$ for all t in $[0, T]$ and $\eta'_\delta(t) \in A\xi_\delta(t)$ for almost all t and

$$\xi_\delta(t) = x_0 + \int_0^t b(s, \xi_\delta(s))\,ds - \eta_\delta(t) + \int_0^t \sigma B'_\delta(s)\,ds$$

or, equivalently,

$$d\xi_\delta(t) = b(t, \xi_\delta(t))\,dt - d\eta_\delta(t) + \sigma B'_\delta(t)\,dt, \qquad \xi_\delta(0) = x_0. \tag{9}$$

By letting $\delta \downarrow 0$, we infer, under the condition that

$$\sup_{0<\delta\leqslant T} \int_0^T |\eta'_\delta(t)|^2\,dt < \infty \ \text{(a.s.)}, \tag{10}$$

that ξ_δ converges almost surely in supremum norm to some process ξ which is the first component of a couple (ξ, η) shown to be a solution to the corresponding multivalued stochastic differential equation (4).

THEOREM 4.1. *For $\delta > 0$, let ξ_δ and η_δ be given by* (9). *Further, assume that* (5) *and* (10) *are satisfied and that $\mathcal{D}(A)$ is closed. Then there exists a unique solution (ξ, η) to the multivalued stochastic differential equation* (4). *Furthermore, $\xi_\delta \to \xi$ uniformly on $[0, T]$ (almost surely) as $\delta \downarrow 0$.*

In order to obtain the convergence of $\{\xi_\delta\}_{\delta>0}$ we first need an important lemma.

LEMMA 4.2. *For $\delta > 0$ let $(\xi_\delta, \eta_\delta)$ be given by* (9). *Then, for $\delta, \varrho > 0$,*

$$
\begin{aligned}
(11)\quad & |\xi_\delta(t)-\xi_\varrho(t)|^2 \\
& \leqslant |\sigma B_\delta(t)-\sigma B_\varrho(t)|^2+2\int_0^t \langle \xi_\delta(s)-\xi_\varrho(s),\, b(s, \xi_\delta(s))-b(s, \xi_\varrho(s))\rangle\, ds \\
& \quad +2\int_0^t \langle(\sigma B_\delta(t)-\sigma B_\varrho(t))-(\sigma B_\delta(s)-\sigma B_\varrho(s)),\, b(s, \xi_\delta(s))-b(s, \xi_\varrho(s))\rangle\, ds \\
& \quad +2\int_0^t \langle(\sigma B_\delta(t)-\sigma B_\varrho(t))-(\sigma B_\delta(s)-\sigma B_\varrho(s)),\, d(\eta_\delta-\eta_\varrho)(s)\rangle.
\end{aligned}
$$

Further,

$$
\begin{aligned}
(12)\quad |\xi_\delta(t)-x_0|^2 = {} & \Big|\int_0^t \sigma B'_\delta(s)\, ds\Big|^2+2\int_0^t \langle \xi_\delta(u)-x_0,\, b(u, \xi_\delta(u))\, du-d\eta_\delta(u)\rangle \\
& +2\int_0^t \Big\langle \int_u^t \sigma B'_\delta(s)\, ds,\, b(u, \xi_\delta(u))\, du-d\eta_\delta(u)\Big\rangle.
\end{aligned}
$$

Proof of Lemma 4.2. Let us show (11); (12) is proved similarly. We have

$$
(13)\quad \xi_\delta(t)-\xi_\varrho(t) = \sigma B_\delta(t)-\sigma B_\varrho(t)+\int_0^t b(s, \xi_\delta(s))-b(s, \xi_\varrho(s))\, ds-\int_0^t d(\eta_\delta-\eta_\varrho)(s);
$$

hence

$$
\begin{aligned}
|\xi_\delta(t)-\xi_\varrho(t)|^2 = {} & |\sigma B_\delta(t)-\sigma B_\varrho(t)|^2 \\
& +\Big|\int_0^t b(s, \xi_\delta(s))-b(s, \xi_\varrho(s))\, ds-\int_0^t d(\eta_\delta-\eta_\varrho)(s)\Big|^2 \\
& +2\Big\langle \sigma B_\delta(t)-\sigma B_\varrho(t),\, \int_0^t b(s, \xi_\delta(s))-b(s, \xi_\varrho(s))\, ds-\int_0^t d(\eta_\delta-\eta_\varrho)(s)\Big\rangle.
\end{aligned}
$$

We observe that

$$\left|\int_0^t b(s,\xi_\delta(s))-b(s,\xi_\varrho(s))\,ds-\int_0^t d(\eta_\delta-\eta_\varrho)(s)\right|^2 = 2\int_0^t \Big\langle \int_0^s b(u,\xi_\delta(u))-b(u,\xi_\varrho(u))\,du$$

$$-\int_0^s d(\eta_\delta-\eta_\varrho)(u),\ \big(b(s,\xi_\delta(s))-b(s,\xi_\varrho(s))\big)ds-d(\eta_\delta-\eta_\varrho)(s)\Big\rangle$$

and

$$2\Big\langle \sigma B_\delta(t)-\sigma B_\varrho(t),\ \int_0^t b(s,\xi_\delta(s))-b(s,\xi_\varrho(s))\,ds-\int_0^t d(\eta_\delta-\eta_\varrho)(s)\Big\rangle$$

$$= 2\int_0^t \Big\langle \big(\sigma B_\delta(s)-\sigma B_\varrho(s)\big)+\big(\sigma B_\delta(t)-\sigma B_\varrho(t)\big)-\big(\sigma B_\delta(s)-\sigma B_\varrho(s)\big),$$

$$\big(b(s,\xi_\delta(s))-b(s,\xi_\varrho(s))\big)ds-d(\eta_\delta-\eta_\varrho)(s)\Big\rangle.$$

Hence, by (13),

$$|\xi_\delta(t)-\xi_\varrho(t)|^2 = |\sigma B_\delta(t)-\sigma B_\varrho(t)|^2$$

$$+2\int_0^t \Big\langle \xi_\delta(s)-\xi_\varrho(s),\ \big(b(s,\xi_\delta(s))-b(s,\xi_\varrho(s))\big)ds-d(\eta_\delta-\eta_\varrho)(s)\Big\rangle$$

$$+2\int_0^t \Big\langle \big(\sigma B_\delta(t)-\sigma B_\varrho(t)\big)-\big(\sigma B_\delta(s)-\sigma B_\varrho(s)\big),\ b(s,\xi_\delta(s))-b(s,\xi_\varrho(s))\Big\rangle ds$$

$$-2\int_0^t \Big\langle \big(\sigma B_\delta(t)-\sigma B_\varrho(t)\big)-\big(\sigma B_\delta(s)-\sigma B_\varrho(s)\big),\ d(\eta_\delta-\eta_\varrho)(s)\Big\rangle.$$

Since A is a monotone map and, for almost all s in $[0, T]$, $\eta'_\delta(s)\in A\xi_\delta(s)$ and $\eta'_\varrho(s)\in A\xi_\varrho(s)$, the inequality (11) holds. ■

By Lemma 4.2 and the boundedness assumption (10) of $\{\eta'_\delta\}_{\delta>0}$ in $L^2([0, T]; \mathfrak{R}^d)$, we infer that $\{\xi_\delta\}_{\delta>0}$ is a Cauchy sequence in supremum norm (almost surely). In fact, it is sufficient that $\{\eta'_\delta\}_{0<\delta\leqslant T}$ is bounded in $L^1([0, T]; \mathfrak{R}^d)$.

LEMMA 4.3. *Let $(\xi_\delta, \eta_\delta)$ be the solution to* (9) *and let* $\sup_{0<\delta\leqslant T}\int_0^T|\eta'_\delta(s)|\,ds$ *be finite almost surely. Then*

$$\|\xi_\delta-\xi_\varrho\|_T\to 0 \quad (a.s.)$$

as δ and ϱ tend to zero.

Proof of Lemma 4.3. By Lemma 4.2 we have

$$|\xi_\delta(t)-\xi_\varrho(t)|^2 \leqslant \|\sigma B_\delta-\sigma B_\varrho\|_T^2+2\int_0^t \langle \xi_\delta(s)-\xi_\varrho(s),\ b(s,\xi_\delta(s))-b(s,\xi_\varrho(s))\rangle ds$$

$$+4\,\|\sigma B_\delta-\sigma B_\varrho\|_T\int_0^t\big|b(s,\,\xi_\delta(s))-b(s,\,\xi_\varrho(s))\big|\,ds$$

$$+4\,\|\sigma B_\delta-\sigma B_\varrho\|_T\Big(\int_0^T|\eta'_\delta(s)|\,ds+\int_0^T|\eta'_\varrho(s)|\,ds\Big),$$

where, by assumption, $\int_0^T|\eta'_\delta(s)|\,ds+\int_0^T|\eta'_\varrho(s)|\,ds$ is almost surely bounded. By using the Lipschitz condition on b we obtain

$$\|\xi_\delta-\xi_\varrho\|_t^2\leqslant(1+2T)\,\|\sigma B_\delta-\sigma B_\varrho\|_T^2+4\,\|\sigma B_\delta-\sigma B_\varrho\|_T\Big(\int_0^T|\eta'_\delta(s)|\,ds+\int_0^T|\eta'_\delta(s)|\,ds\Big)$$

$$+2(L+L^2)\int_0^t\|\xi_\delta-\xi_\varrho\|_s^2\,ds.$$

Since $\|\sigma B_\delta-\sigma B_\varrho\|_T\to0$ almost surely as δ and ϱ tend to zero and $\int_0^t\|\xi_\delta-\xi_\varrho\|_s^2\,ds$ is almost surely finite (ξ_δ and ξ_ϱ are continuous), Bellman–Gronwall's inequality completes the proof. ■

Using Lemma 4.3 and the property that η'_δ is almost surely bounded in $L^2([0,\,T];\,\mathfrak{R}^d)$ we are able to show Theorem 4.1.

Proof of Theorem 4.1. It remains to show existence. For convenience we omit the words 'almost surely.' By Lemma 4.3 there exists a continuous process ξ such that

$$\|\xi-\xi_\delta\|_T\to0,\quad \delta\downarrow0, \tag{14}$$

and consequently, by Schwarz' inequality and the Lipschitz assumption on b,

$$\Big\|\int_0^\cdot b(s,\,\xi_\delta(s))\,ds-\int_0^\cdot b(s,\,\xi(s))\,ds\Big\|_T\to0,\qquad \delta\downarrow0,$$

and since also

$$\|\sigma B_\delta-\sigma B\|_T\to0,\qquad \delta\downarrow0,$$

we obtain with $\eta(t)=x_0+\int_0^t b(s,\,\xi(s))\,ds+\sigma B(t)-\xi(t)$ the relation

$$\|\eta_\delta-\eta\|_T\to0,\quad \delta\downarrow0. \tag{15}$$

We now show that $(\xi,\,\eta)$ is indeed a solution to (4). We need to verify that $\xi(t)\in\mathscr{D}(A)$ for all t in $[0,\,T]$ and that η is absolutely continuous with $\eta'(t)\in A\xi(t)$ for almost all t in $[0,\,T]$.

Since, by assumption, $\sup\big\{\int_0^T|\eta'_\delta(t)|^2\,dt\colon\ 0<\delta\leqslant T\big\}<\infty$ we obtain by Banach–Alaoglu's theorem that there exists a subsequence $\{\eta_{\delta_n}\}_{n\geqslant1}$ such that η'_{δ_n} converges weakly in $L^2([0,\,T];\,\mathfrak{R}^d)$ to some v in $L^2([0,\,T];\,\mathfrak{R}^d)$ as $\delta_n\downarrow0$.

Now we use theorems for maximal monotone maps to show that $\xi(t)\in\mathscr{D}(A)$ and $v(t)\in A\xi(t)$ for almost all points t in $[0,\,T]$. Let $\mathscr{A}$ be the set--valued map on $L^2([0,\,T];\,\mathfrak{R}^d)$ defined by $(\mathscr{A}x)(t)=Ax(t)$ a.e. in $[0,\,T]$ for

$x \in L^2([0, T]; \mathfrak{R}^d)$. Then, by Aubin and Cellina [1], Chapter 3, $\mathscr{A}$ is a maximal monotone map on $L^2([0, T]; \mathfrak{R}^d)$.

We have thus as follows: $\xi_{\delta_n} \to \xi$ in $L^2([0, T]; \mathfrak{R}^d)$ by (14), η'_{δ_n} tends weakly in $L^2([0, T]; \mathfrak{R}^d)$ to v as $\delta_n \downarrow 0$ and $\eta'_\delta \in \mathscr{A}\xi_\delta$. This implies, by Aubin and Cellina [1], Proposition 3.1.2, that $v \in \mathscr{A}\xi$ in $L^2([0, T]; \mathfrak{R}^d)$ which means that $v(t) \in A\xi(t)$ and, in particular, $\xi(t) \in \mathscr{D}(A)$ for almost all t in $[0, T]$.

Since ξ_δ is continuous and ξ_δ converges uniformly on $[0, T]$ to ξ, and $\mathscr{D}(A)$ by assumption is closed, we infer that $\xi(t) \in \mathscr{D}(A)$ for all t in $[0, T]$.

Next we show that η is absolutely continuous with derivative $\eta' = v$. For $0 \leqslant s \leqslant t \leqslant T$ we have

$$\eta_{\delta_k}(t) - \eta_{\delta_k}(s) = \int_s^t \eta'_{\delta_k}(u)\,du,$$

where the left-hand side by (15) converges to $\eta(t) - \eta(s)$ and the right-hand side, by the weak convergence in $L^2([0, T]; \mathfrak{R}^d)$, converges to $\int_s^t v(u)\,du$. Since $v \in L^2([0, T]; \mathfrak{R}^d)$, this means that η is absolutely continuous with $\eta' = v$. This also implies that $\eta'(t) \in A\xi(t)$ for almost all t in $[0, T]$.

Finally, $\xi(t)$ and $\eta(t)$ are, for fixed t, $\mathscr{F}_t$-adapted. This follows since $\xi_\delta(t)$ and $\eta_\delta(t)$ are $\mathscr{F}_{\beta_\delta(t)}$-adapted, where $\beta_\delta(t) = \min\{t_k^\delta\colon t_k^\delta > t\}$, the convergences (14) and (15) hold true, and $\{\mathscr{F}_t\}$ is a right-continuous filtration. ∎

Theorem 4.1 can easily be extended to the case when $\sigma = \sigma(t)$.

R e m a r k 4.4. There is one important case when the integrability condition (10) is *not* satisfied. Let A be the outwards directed normal cone for the set $G = [0, \infty)$ in $\mathfrak{R}$ (recall Remark 2.5). For simplicity, assume $x_0 = 0$ and $b \equiv 0$. In that case, from the theory of the Skorohod equation (see e.g. Ikeda and Watanabe [4]),

$$\xi_\delta(t) = B_\delta(t) - \eta_\delta(t), \qquad \eta_\delta(t) = \min_{0 \leqslant s \leqslant t} B_\delta(s).$$

If (10) were true, then by the Banach–Alaoglu theorem, there would be a subsequence $\{\eta_{\delta_n}\}_{n \geqslant 1}$ such that, almost surely, η'_{δ_n} would converge weakly to some $v \in L^2(0, T)$. In particular, we would especially get, for any $0 \leqslant t_1 < t_2 \leqslant T$,

$$\int_{t_1}^{t_2} v(t)\,dt = \min_{0 \leqslant s \leqslant t_2} B(s) - \min_{0 \leqslant s \leqslant t_1} B(s),$$

which would imply that $\min_{0 \leqslant s \leqslant t} B(s)$ is absolutely continuous, which, as is well known, is not true ([4], p. 122). ∎

In the rest of this section we show that the integrability condition (10) is satisfied in two important cases.

(i) L i n e a r g r o w t h c o n d i t i o n of A:

(16) $$\sup\{|y|\colon y \in Ax\} \leqslant L(1 + |x|), \qquad x \in \mathscr{D}(A);$$

this condition may be useful for stochastic differential equations with discontinuous drift and for application in mechanics, see Example 6.1.

(ii) Product situation: for some p, $1 < p < d$, the first p components of Ax are zero for $x \in \mathscr{D}(A)$ and the last $d-p$ rows of σ are zero, i.e.

(17) $$(Ax)_i = 0,\ i = 1, \ldots, p, \quad \text{and} \quad \sigma_{ij} = 0,\ i = p+1, \ldots, d,\ j = 1, \ldots, m.$$

This means that, in this case, (4) is a coupled system where the first p rows describe a stochastic differential equation and the last $d-p$ rows describe a (deterministic) differential inclusion. One application in seismic reliability analysis is given by Example 6.2. Conditions (16) and (17) are somewhat more general than in Krée [6] and Pettersson [11].

First we show that the sequence $\{\xi_\delta\}_{0<\delta\leqslant T}$ is almost surely uniformly bounded on $[0, T]$ under conditions (16) and (17).

Proposition 4.5. *Let ξ_δ be given by the differential inclusion* (8), *where b satisfies the Lipschitz and linear growth conditions* (5). *Assume either A satisfies the linear growth condition* (16) *or we have the product situation* (17). *Then $\sup_{0<\delta\leqslant T} \|\xi_\delta\|_T$ is finite almost surely.*

Proof. For convenience we suppress the words 'almost surely.' Consider first the case when (16) is satisfied. By Schwarz' inequality, for $0 \leqslant t \leqslant T$,

$$|\xi_\delta(t)|^2 \leqslant 4|x_0|^2 + 4T\int_0^t |b(s, \xi_\delta)|^2 ds + 4T\int_0^T |\eta'_\delta(t)|^2 dt + 4\left|\int_0^t \sigma B'_\delta(s)ds\right|^2,$$

where, for almost all t in $[0, T]$, $\eta'_\delta(t) \in A\xi_\delta(t)$; hence, by (16), $|\eta'_\delta(t)| \leqslant L(1+|\xi_\delta(t)|)$, t-a.e. in $[0, T]$. Further, $\left|\int_0^t \sigma B'_\delta(s)ds\right| \leqslant \|\sigma B\|_T$ is bounded. Bellman–Gronwall's inequality together with the linear growth condition of b then gives $\sup\{\|\xi_\delta\|_T\colon 0 < \delta \leqslant T\} < \infty$.

Now let us consider the product situation. For this case, (12) implies

$$|\xi_\delta(t) - x_0|^2 = \left|\int_0^t \sigma B'_\delta(s)ds\right|^2 + 2\int_0^t \langle \xi_\delta(s) - x_0, b(s, \xi_\delta(s))ds - d\eta_\delta(s)\rangle$$
$$+ 2\int_0^t \left\langle \int_u^t \sigma B'_\delta(s)ds, b(u, \xi_\delta(u))\right\rangle du$$

(the term $\int_0^t \langle \int_u^t \sigma B'_\delta(s)ds, d\eta_\delta(u)\rangle$ vanishes by (17)). Observe that, for almost all s in $[0, T]$, $\eta'(s) \in A\xi_\delta(s)$, $A^0x_0 \in Ax_0$ and A is a monotone map; hence

$$-\int_0^t \langle \xi_\delta(s) - x_0, d\eta_\delta(s)\rangle = -\int_0^t \langle \xi_\delta(s) - x_0, \eta'_\delta(s) - A^0x_0 + A^0x_0\rangle ds$$
$$\leqslant -\int_0^t \langle \xi_\delta(s) - x_0, A^0x_0\rangle ds.$$

Further,

$$2\int_0^t \langle \int_u^t \sigma B'_\delta(s)ds,\, b(u,\, \xi_\delta(u))\rangle\, du$$

$$\leqslant 2 \sup_{s\in[0,t]} |\sigma B_\delta(t)-\sigma B_\delta(s)| \int_0^t |b(u,\, \xi_\delta(u))|\, du \leqslant 4\, \|\sigma B\|_T \int_0^t |b(u,\, \xi_\delta(u))|\, du,$$

then use the same arguments as for the linear growth case (16). ∎

Now we show that, in the linear growth condition (16) or the product situation (17), $\{\eta'_\delta\}_{0<\delta\leqslant T}$ is almost surely bounded in $L^2([0,\, T];\, \mathfrak{R}^d)$. In fact, $\{\eta'_\delta\}_{0<\delta\leqslant T}$ is (almost surely) bounded in $L^\infty([0,\, T];\, \mathfrak{R}^d)$. For a function $g\colon [0,\, T] \to \mathfrak{R}^d$, let

$$|g|_{L^\infty} = \operatorname{ess\,sup}\{|g(t)|\colon\; 0 \leqslant t \leqslant T\}.$$

PROPOSITION 4.6. *Let ξ_δ be the solution to* (8), *where either A satisfies the linear growth condition* (16) *or we have the product situation* (17) *with A^0 bounded on compact subsets of $\mathscr{D}(A)$ and $\mathscr{D}(A)$ closed. Then* $\sup_{0<\delta\leqslant T}|\eta'_\delta|_{L^\infty}$ *is finite almost surely.*

Proof. Suppress the words 'almost surely.'

For the linear growth case (16), the result follows by the fact that $\eta'_\delta(t) \in A\xi_\delta(t)$ for almost all t in $[0,\, T]$ and by Proposition 4.5.

Now consider the product situation. We can write $\xi_\delta = (\xi_{1,\delta},\, \xi_{2,\delta}) \in \mathfrak{R}^p \times \mathfrak{R}^{d-p}$ given by the differential inclusion

(18) $$\xi'_{1,\delta}(t) = b_1(t,\, \xi_\delta(t)) + \sigma_1 B'_\delta(t), \qquad \xi'_{2,\delta}(t) \in b_2(t,\, \xi_\delta(t)) - A_2\, \xi_{2,\delta}(t),$$

where b_1 and b_2 are the first p and the last $d-p$ rows of b, respectively. Similarly, the dispersion σ_1 represents the p first rows of σ (the rest are by assumption zero) and A_2 denotes the $d-p$ last components of A (the p first are zero). The second expression in (18) can be rewritten as $\xi'_{2,\delta}(t) = b_2(t,\, \xi_\delta(t)) - \eta'_{2,\delta}(t)$, where $\eta'_{2,\delta}(t) \in A_2\, \xi_{2,\delta}(t)$ for almost all t in $[0,\, T]$. By the linear growth condition of b (hence also of b_2) and by Proposition 4.5, $b_2(t,\, \xi_\delta(t))$ is almost surely uniformly bounded for $0 \leqslant t \leqslant T$ and $0 < \delta \leqslant T$. Hence, to conclude the proposition we only need to show that $\sup_{0<\delta\leqslant T}|\xi'_{2,\delta}|_{L^\infty}$ is finite. By (18) we get

$$|\xi_{2,\delta}(t) - \xi_{2,\delta}(s)|^2 = 2\int_s^t \langle \xi_{2,\delta}(u) - \xi_{2,\delta}(s),\, b_2(u,\, \xi_\delta(u)) - \eta'_{2,\delta}(u)\rangle\, du, \qquad s<t$$

(here, the norm $|\cdot|$ and inner product $\langle\cdot,\,\cdot\rangle$ represent the usual norm and inner product in $\mathfrak{R}^{d-p}$, respectively). Since $b_2(s,\, \xi_\delta(s)) - \big(b_2(s,\, \xi_\delta(s)) - A_2(\xi_\delta)_2(s)\big)^0 \in A_2\xi_{2,\delta}(s)$ for almost all s in $[0,\, T]$, we obtain, by adding and subtracting terms,

$$|\xi_{2,\delta}(t)-\xi_{2,\delta}(s)|^2 \leqslant 2\int_s^t \langle \xi_{2,\delta}(u)-\xi_{2,\delta}(s),\, b_2(u,\, \xi_\delta(u))-b_2(s,\, \xi_\delta(s))\rangle\, du$$

$$+2\int_s^t \langle \xi_{2,\delta}(u)-\xi_{2,\delta}(s),\, b_2(s,\, \xi_\delta(s))-A_2(\xi_{2,\delta}(s))^0\rangle\, du.$$

Proposition 4.5 of boundedness of ξ_δ and the assumption that $\mathscr{D}(A)$ is closed yield that $\{\xi_{2,\delta}(s)\}_{0\leqslant s\leqslant T,\, 0<\delta\leqslant T}$ belongs to a compact subset of $\mathscr{D}(A_2)$; consequently, by the assumption on A^0, $\{(A_2\xi_{2,\delta}(s))^0\}_{0\leqslant s\leqslant T,\delta>0}$ is bounded. Finally, by using the inequality $|(y-A_2x)^0| \leqslant |y|+|(A_2x)^0|$, the linear growth assumption of b and Proposition 4.5, we obtain

$$|\xi_{2,\delta}(t)-\xi_{2,\delta}(s)|^2 \leqslant C\int_s^t |\xi_{2,\delta}(u)-\xi_{2,\delta}(s)|\, du$$

which by Brézis [3], lemme A.5, gives $|\xi_{2,\delta}(t)-\xi_{2,\delta}(s)| \leqslant C(t-s)$, and hence $\sup_{0<\delta\leqslant T}|\eta'_{2,\delta}|_{L^\infty}$ is finite. ∎

5. Convergence in mean square. In this section we show that, in the linear growth case or in the product situation, the convergence of ξ_δ to ξ is also, in mean square, uniform on compacts. First we need a boundedness condition for ξ_δ similar to (6).

PROPOSITION 5.1. *Let ξ_δ be given by the differential inclusion* (8). *Assume either A satisfies the linear growth condition* (16) *or we have the product situation* (17). *Then* $\sup_{0<\delta\leqslant T} E\,\|\xi_\delta\|_T^2$ *is finite.*

Proof. Introduce stopping times $\tau_\delta^N = \inf\{0 \leqslant t \leqslant T\colon\ |\xi_\delta(t)| \geqslant N\}$ (equal to T if the corresponding set is empty) and use similar arguments to those in the proof of Proposition 4.5 applied to the processes $\xi_\delta^N(\cdot) = \xi_\delta(\cdot\wedge\tau_\delta^N)$ and $\eta_\delta^N(\cdot) = \eta_\delta(\cdot\wedge\tau_\delta^N)$. Then we find that $E\,\|\xi_\delta^N\|_T^2$ is bounded by a constant independent of N and δ. Since ξ_δ is continuous almost surely, $\|\xi_\delta^N\|_T \to \|\xi_\delta\|_T$ as $N\uparrow\infty$ almost surely, and hence we can deduce by Fatou's lemma that also $E\,\|\xi_\delta\|_T^2$ is bounded by a constant. ∎

PROPOSITION 5.2. *Assume that ξ is a solution to* (4) *and let ξ_δ be given by* (8). *Then under the linear growth condition* (16) *of A,*

$$E\,\|\xi_\delta-\xi\|_T^2 = O\big((\delta\log\delta^{-1})^{1/2}\big) \qquad (19)$$

and for the product situation (17),

$$E\,\|\xi_\delta-\xi\|_T^2 = O(\delta\log\delta^{-1}) \qquad (20)$$

for small $\delta > 0$.

Proof. By a modification of the proof of Lemma 4.2 we have

$$|\xi_\delta(t)-\xi(t)|^2 \leqslant |\sigma B_\delta(t)-\sigma B(t)|^2+2\int_0^t \langle \xi_\delta(s)-\xi(s),\, b(s,\, \xi_\delta(s))-b(s,\, \xi(s))\rangle\, ds$$

$$+2\int_0^t \langle (\sigma B_\delta(t)-\sigma B(t))-(\sigma B_\delta(s)-\sigma B(s)),\, b(s,\xi_\delta(s))-b(s,\xi(s))\rangle\, ds$$

$$+2\int_0^t \langle (\sigma B_\delta(t)-\sigma B(t))-(\sigma B_\delta(s)-\sigma B(s)),\, d(\eta_\delta-\eta)(s)\rangle.$$

Consider first the linear growth case. By arguments as in the proof of Lemma 4.3,

$$\|\xi_\delta-\xi\|_t^2 \leqslant (1+2T)\,\|\sigma B_\delta-\sigma B\|_T^2$$
$$+4\,\|\sigma B_\delta-\sigma B\|_T\Big(\int_0^T |\eta'_\delta(s)|\,ds+\int_0^T |\eta'(s)|\,ds\Big)+2(L+L^2)\int_0^t \|\xi_\delta-\xi\|_s^2\,ds.$$

For small δ,

$$E\,\|\sigma B_\delta-\sigma B\|_T^2 \leqslant C\delta\log\delta^{-1}$$

(see e.g. Pettersson [10]) and by the linear growth assumption of A, the boundedness (6) of $\xi(\cdot)$ and Proposition 5.1,

$$\sup_{0<\delta\leqslant T}\int_0^T E\,|\eta'_\delta(s)|\,ds+\int_0^T E\,|\eta'(s)|\,ds$$

is finite. Cauchy–Schwarz's inequality and Bellman–Gronwall's inequality then give (19).

For the product situation we have

$$\|\xi_\delta-\xi\|_t^2 \leqslant (1+2T)\,\|\sigma B_\delta-\sigma B\|_T^2+2(L+L^2)\int_0^t \|\xi_\delta-\xi\|_s^2\,ds,$$

which, by arguments as in the linear growth case, gives (20). ∎

Remark 5.3. If (16) or (17) is satisfied, then $\|\xi_\delta-\xi\|_T^2$ is almost surely $O((\delta\log 1/\delta)^{1/2})$ or $O(\delta\log 1/\delta)$, respectively. This is easily seen by a trivial modification of the proof of Proposition 5.2 and by using the well-known modulus of continuity for the Brownian motion. ∎

6. Applications. In this section we give some examples where it may be useful to consider multivalued stochastic differential equations by using approximative differential inclusions. In the first example, A satisfies a linear growth condition, and in the second one, a product situation is described.

EXAMPLE 6.1 (Coulomb damping). An equation for describing a mechanical system with both linear viscous damping and friction is as follows:

(21) $$m\ddot{x}+\beta\dot{x}+r\eta_2(\dot{x})+kx=f,\qquad m,r,\beta,k>0,$$

where f is an excitation force instantly assumed to be piecewise continuous. Further, it is assumed that $\eta_2(\dot{x})\in A_2(\dot{x})$ for the maximal monotone set-valued map A_2 from $\mathfrak{R}$ to $\mathfrak{R}$ defined by

$$A_2(z) = \begin{cases} \operatorname{sign} z & \text{if } z \neq 0, \\ [-1, 1] & \text{if } z = 0. \end{cases}$$

For simplicity, we assume the mass m is equal to 1. We rewrite the second degree system (21) as a first degree system: for $u = (u_1, u_2)$ in $\mathfrak{R}^2$ let

$$b(t, u) = \begin{pmatrix} u_2 \\ -ku_1 - \beta u_2 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and let A be the maximal monotone map given by $Au = \{(0, y)\colon\ y \in rA_2(u_2)\}$. Then, with $u = (x, \dot{x})$, the second order equation (21) may be reformulated as a differential inclusion

$$u'(t) \in b(u(t)) - Au(t) - \sigma f(t). \tag{22}$$

Since A^0 is bounded on compacts (we even have $|A^0 u| \leqslant 1$ for all u in $\mathfrak{R}^2$), it follows by Remark 2.5 that there exists a unique solution $u(\cdot)$ to (22) such that

$$\frac{d^+u}{dt} = \{b(u(t)) - Au(t) + \sigma f(t+0)\}^0. \tag{23}$$

For given u, $\{b(u) - A(u) + \sigma f(t+0)\}^0$ can be written in the form

$$(u_2, \{f(t+0) - ku_1 - \beta u_2 - rA_2(u_2)\}^0).$$

For $u_2 \neq 0$,

$$\{f(t+0) - ku_1 - \beta u_2 - rA_2(u_2)\}^0 = f(t+0) - ku_1 - \beta u_2 - r \operatorname{sign} u_2$$

and for $u_2 = 0$,

$$\{f(t+0) - ku_1 - rA_2(u_2)\}^0 = \{f(t+0) - ku_1 - r[-1, 1]\}^0,$$

i.e.

$$\{f(t+0) - ku_1 - rA_2(u_2)\}^0 = \begin{cases} f(t+0) - ku_1 + r & \text{if } f(t+0) - ku_1 < -r, \\ 0 & \text{if } |f(t+0) - ku_1| \leqslant r, \\ f(t+0) - ku_1 - r & \text{if } f(t+0) - ku_1 > r. \end{cases} \tag{24}$$

In fact, it is (23) and (24) which usually characterize the Coulomb damping (cf. Krée [7], Chapter XIV.2). If the piecewise continuous f is replaced by Gaussian white noise $\dot{B}$, it seems useful to consider a solution to (21) as the limit as $\delta \downarrow 0$ of solutions to (21) with $f = B'_\delta$. For more details about this particular example see e.g. Jogréus [5], Example 1.5. ∎

Example 6.2 (bilinear hysteresis and earthquakes). Let $f\colon [0, T] \mapsto \mathfrak{R}^d$, as in the previous example, be piecewise continuous. Assume $-f$ describes an acceleration in a given direction generated by an earthquake. Suppose this acceleration influences a structure to be deformed x units in this direction.

Let $-R$ be the restoring force produced by the deformation. If we take the mass equal one, the equation of motion is usually written as

(25) $$\ddot{x}+2h\dot{x}+R=f, \quad x(0)=\dot{x}(0)=f(0)=0,$$

where the constant $h>0$ characterizes the structural damping. In this example we assume bilinear hysteresis. Then R may be written in the form

(26) $$R=\alpha x+(1-\alpha)z,$$

where α is a fixed constant in $[0, 1]$ and $z(\cdot)$ is an almost surely absolutely continuous process with

(27) $$z'\in\dot{x}-A_3z,$$

where A_3 is the outwards directed normal cone for the set $[-1, 1]$ explicitly written as

(28) $$A_3z=\begin{cases}[0,\infty) & \text{if } z=1,\\ \{0\} & \text{if } -1<z<1,\\ (-\infty,0] & \text{if } z=-1.\end{cases}$$

Heuristically we may think as follows: if $\alpha=1$, the structure is elastic, and if $0\leqslant\alpha<1$, we have an elasto-plastic system. In the latter case the structure is elastic when $|z|<1$ (i.e. $-(1-\alpha)+\alpha x<R<1-\alpha+\alpha x$) and permanently deformed when $z=1$ and $\dot{x}>0$ or $z=-1$ and $\dot{x}<0$ (i.e. when $R=1-\alpha+\alpha x$ and $\dot{x}>0$ or $R=-(1-\alpha)+\alpha x$ and $\dot{x}<0$).

It may be more convenient to rewrite the equation (25) with the conditions (26)–(28) into a first degree system. Put $u=(x, \dot{x}, z)$,

$$b(u)=\begin{bmatrix}u_2\\ -\alpha u_1-2hu_2-(1-\alpha)u_3\\ u_2\end{bmatrix} \quad\text{and}\quad \sigma=\begin{bmatrix}0\\1\\0\end{bmatrix}.$$

Let A be the outwards directed normal on the boundary of the set $G=\mathfrak{R}^2\times[-1, 1]$, i.e.

$$Au=\{(0,0,y)\colon y\in A_3z\}.$$

Then

(29) $$u'(t)\in b(u(t))+\sigma f(t)-Au(t).$$

Remark 2.5 gives $d^+u/dt=\{\sigma f(t+0)-Au(t)\}^0$. This means, in particular, that

$$\frac{d^+z}{dt}=\{\dot{x}-A_3z\}^0,$$

i.e.

$$\frac{d^+z}{dt}(t)=\begin{cases}\dot{x}(t)\,I_{\{\dot{x}(t)<0\}} & \text{if } z(t)=1,\\ \dot{x}(t) & \text{if } -1<z(t)<1,\\ \dot{x}(t)\,I_{\{\dot{x}(t)>0\}} & \text{if } z(t)=-1.\end{cases} \tag{30}$$

If f is replaced by a Gaussian white noise $\dot{B}$, it seems again plausible to consider a solution to (25)–(28) as the limit (as $\delta\downarrow 0$) of solutions ξ_0 to (29) with $f=B'_\delta$. For more details about bilinear hysteresis and earthquakes see e.g. Krée [6] or Krée and Soize [7], Chapter XIV. ■

7. Product situation and space dependent dispersion. We consider a Wong–Zakai result for a product situation where $\sigma=\sigma(x)$. We use ideas from Wong and Zakai's original papers [13] and [14].

EXAMPLE 7.1. With notations as those to the proof of Proposition 4.6, we consider the differential inclusion

$$\xi'_{1,\delta}(t)=b_1(\xi_\delta(t))+\sigma_1(\xi_{1,\delta}(t))B'_\delta(t),\qquad \xi'_{2,\delta}(t)\in b_2(\xi_\delta(t))-A_2\,\xi_{2,\delta}(t), \tag{31}$$

$\xi_\delta(0)=x_0$, where $p=m=1$. Here, $\sigma=(\sigma_1,\sigma_2)$ as well as $b=(b_1,b_2)$ are assumed to be Lipschitz continuous with Lipschitz constant L. Note that we furthermore assume that σ_1 depends on ξ_δ only through $\xi_{1,\delta}$. A solution to (31) can be constructed inductively by [2], corollaire 1.3, since on each interval $[t_{i-1},t_i]$, $A-b-\sigma B_\delta+L(1+|\sigma B'_\delta|)I$ is maximal monotone, where $Ix=x$ for $x\in\mathfrak{R}^d$. We assume there exists some $\varepsilon>0$ such that $\varepsilon\leqslant\sigma_1(x)\leqslant 1/\varepsilon$. Let

$$F(x)=\int_0^x\frac{dy}{\sigma_1(y)},\qquad x\in\mathfrak{R},$$

and $\tilde{\xi}_\delta=(\tilde{\xi}_{1,\delta},\tilde{\xi}_{2,\delta})$, where $\tilde{\xi}_{1,\delta}=F(\xi_{1,\delta})$ and $\tilde{\xi}_{2,\delta}=\xi_{2,\delta}$. Then

$$\tilde{\xi}'_{1,\delta}(t)=\tilde{b}_1(\tilde{\xi}_\delta(t))+B'_\delta(t),\qquad \tilde{\xi}'_{2,\delta}(t)\in\tilde{b}_2(\tilde{\xi}_\delta(t))-A_2\tilde{\xi}_{2,\delta}(t),$$

$$\tilde{\xi}_\delta(0)=\left(F^{-1}(x_{1,0}),\,x_{2,0}\right),\qquad x_0=(x_{1,0},x_{2,0}),$$

where $\tilde{b}_1(x,y)=b_1(F^{-1}(x),y)/\sigma(F^{-1}(x))$ and $\tilde{b}_2(x,y)=b_2(F^{-1}(x),y)$ for $(x,y)\in\mathfrak{R}\times\mathfrak{R}^{d-1}$. Since also the $\tilde{b}_i$'s are Lipschitz continuous, by Proposition 4.6, $\tilde{\xi}_\delta\to\tilde{\xi}$ uniformly on $[0,T]$ almost surely, where

$$d\tilde{\xi}_1(t)=\tilde{b}_1(\tilde{\xi}(t))\,dt+dB(t),\qquad d\tilde{\xi}_2(t)\in\tilde{b}_2(\tilde{\xi}(t))\,dt-A_2\tilde{\xi}_2\,dt.$$

By the continuity of F^{-1}, $\xi_\delta=\left(F^{-1}(\tilde{\xi}_{1,\delta}),\,\xi_{2,\delta}\right)$ converges (almost surely) uniformly on $[0,T]$ to $\xi=(\xi_1,\xi_2)$, where $\xi_1=F^{-1}(\tilde{\xi}_1)$ and $\xi_2=\tilde{\xi}_2$. By Itô's formula,

$$d\xi_1(t)=b_1(\xi(t))dt+\sigma_1(\xi_1(t))\circ dB(t),$$

$$d\xi_2(t)\in b_2(\xi(t))dt-A_2\,\xi_2\,dt,$$

$\xi(0) = x_0$, where $\circ$ denotes Stratonovich integration. A similar convergence result can be obtained for Example 6.2, with space dependent σ. However, the above arguments do go through for one-dimensional multivalued stochastic differential equations with a linear growth condition on A. ■

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