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ASYMPTOTIC STATIONARITY OF TANDEM OF QUEUES

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Abstract. In [12] it was proved that the process of waiting times for single server queues is asymptotically stationary if

(1) a generic process $X = \{X_k, k \geqslant 1\}$ is asymptotically stationary,

(2) X satisfies condition AB and

(3) the two-sided stationary extension $X^* = \{X_k^*, -\infty < k < \infty\}$ of the stationary representation of X is such that $\sum_{j=-n}^{0} X_j^* \to -\infty$ a.e.

The paper gives a detailed characterization of condition AB. Next, the sufficient conditions on the input to a tandem of queues are given under which the inputs to the nodes in that tandem satisfy conditions (1)–(3).

1. Introduction. An important problem of queueing networks is their *stability* meant as an asymptotic stationarity (shortly, AS) of some processes related to those networks. This subject is considered here for single server queues (nodes) in series with unlimited interstage storage and FIFO discipline of service (see [5]). Units enter the node and after servicing there they immediately go to the second node where their service starts if the node is empty; otherwise they join the queue and wait for the service. After servicing they go to the next node, and so on. In such a case an output from a node is an input to the next one. Therefore the problem of an AS of these queueing networks leads to finding extra conditions on the output under which an AS of the input gives the AS of the output and holding those extra conditions for the output. Notice that it is enough to consider this problem for the tandem of two single server queues. To formulate this problem let $(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{u}_1) = \{(v_{1,k}, v_{2,k}, u_{1,k}), k \geqslant 1\}$ be a generic process for a tandem of two single server queues, where $v_{i,k}$ is the service time of the k-th unit in the i-th system, $i = 1, 2$, and $u_{1,k}$ is the interarrival time between the k-th and $(k+1)$-st units to the first system. Furthermore, let $w_{i,k}$ be the waiting time of the k-th unit in the i-th system, $i = 1, 2$, and $u_{2,k}$ the interarrival time between the k-th and $(k+1)$-st units to the second system. Of course, $u_{2,k}$ is also the interdeparture time between the k-th and $(k+1)$-st units from the first system and

$$u_{2,k} = u_{1,k} + w_{1,k+1} + v_{1,k+1} - w_{1,k} - v_{1,k}, \quad k \geqslant 1. \tag{1.1}$$

Therefore (v_1, v_2, u_1) is the generic process for the whole system and we call it the *input to the first node* while (v_2, u_2) is the generic process for the second node and we call it the *input to the second node*. Then the problem can be formulated as follows: which property of the input to the first node — (v_1, v_2, u_1), jointly with its AS, does imply holding this property for the input to the second node — (v_2, u_2), jointly with its AS? To indicate the main point in this consideration let us put

$$X_i = \{X_{i,k} = v_{i,k} - u_{i,k},\ k \geqslant 1\},$$

and let

$$X_i^* = \{X_{i,k}^*,\ -\infty < k < \infty\}$$

denote the two-sided stationary extension of a stationary representation of X_i (if X_i is asymptotically stationary) and $S_k^*(i) = \sum_{j=k+1}^{0} X_{i,j}^*$, $k \leqslant 0$, $i = 1, 2$. In [12] it was shown that if X_1 is AS and its stationary representation is such that $S_{-n}^*(1) \to -\infty$ as $n \to \infty$, then (w_1, X_1) is AS iff the following *condition* AB holds:

$$\lim_k \liminf_n P\big(\max_{k \leqslant j \leqslant n} (S_n(1) - S_{n-j}(1)) < 0\big) = 1, \tag{1.2}$$

where $S_0(i) = 0$, $S_n(i) = \sum_{j=1}^{n} X_{i,j}$ for $k \geqslant 1$. Hence and from (1.1) it follows that if (v_1, v_2, u_1) is AS and X_1 satisfies condition AB and $S_{-n}^*(1) \to -\infty$ a.e., then (v_2, u_2) is also AS (Lemma 5.1). Therefore the main problem is the following: which properties of (v_1, v_2, u_1) jointly with its AS do guarantee holding condition AB for X_1 and X_2 as well as holding $S_{-n}^*(i) \to -\infty$ a.e. as $n \to \infty$ for $i = 1, 2$?

The paper gives the answer to the last question. The main results are the following. In Section 3, a characterization of condition AB is given. Namely, the sufficient conditions for holding condition AB under different types of AS are given for any sequence of random variables $X = \{X_k,\ k \geqslant 1\}$, where the index i related to the label of the node is dropped. For example, X satisfies condition AB either if it is strongly AS, ergodic and $EX_i^* < 0$ (Corollary 3.2), where $X^* = \{X_k^*,\ -\infty < k < \infty\}$ is the two-sided stationary extension of a stationary representation of X or if it is AS in variation, $\sum_{j=1}^{n} X_j \xrightarrow{P} -\infty$ and $\sum_{j=-n}^{0} X_j^* \to -\infty$ a.e. as $n \to \infty$ (Corollary 3.3). In Section 4 it is proved that $n^{-1} w_{-n}^* \to 0$ a.e. if X^* satisfies the functional strong law of large numbers with $EX_1^* < 0$, where w^* is the two-sided stationary extension of the stationary representation of w. In Section 5 the sufficient conditions are given for (v_1, v_2, u_1) under which X_1 and X_2 satisfy condition AB. Those conditions guarantee the AS of the process of waiting time and a queue length for the tandem of queues (Theorem 1 and Corollary 5.1).

2. Notation. Most of the notation used here are from [12], and for completeness we recall some of them. For a Polish metric space S, let S^∞ denote the right-sided infinite product of S, and $S_{-\infty}^{\infty}$ the two-sided infinite product of S. Any metric space is considered with the Borel σ-field, and the finite or infinite products of metric spaces are considered with the product topology. On the metric spaces we consider 6 types of convergence of probability measures μ_n to a probability measure μ. Namely, putting $\bar{\mu}_n = n^{-1}\sum_{j=1}^{n}\mu_j$ we have:

c_1 — weak convergence;
c_2 — weak convergence in mean, when $\bar{\mu}_n$ weakly converges to μ;
c_3 — strong convergence, when $\mu_n(B) \to \mu(B)$ for all Borel sets B;
c_4 — strong convergence in mean, when $\bar{\mu}_n$ strongly converges to μ;
c_5 — convergence in variation, when $\|\mu_n - \mu\| \to 0$; and
c_6 — convergence in variation in mean, when $\|\bar{\mu}_n - \mu\| \to 0$.

The asymptotic stationarity is defined in terms of convergence of probability measures. Namely, an S-valued process $\xi = \{\xi_k, k \geqslant 1\}$, where S is a Polish metric space, is said to be *asymptotically stationary* (shortly, AS) in some sense of convergence of probability measures, say c, if the sequence of distributions $\{\mathscr{L}(T^n\xi)\}$ converges in the sense c. Here $\mathscr{L}(Z)$ denotes the probability distribution of a random element Z, T is the shift transformation on S^{∞}, i.e. $T(x) = \{x_{k+1}, k \geqslant 1\}$ for $x = \{x_k, k \geqslant 1\}$, and T^n is the n-th iteration of T. If $\{\mathscr{L}(T^n\xi)\}$ converges weakly or strongly or in variation, then we say that ξ is *weakly AS* or *strongly AS* or *AS in variation*, respectively.

Let $\xi^0 = \{\xi_k^0, k \geqslant 1\}$ be an S-valued process with the distribution being the limiting distribution of $\{\mathscr{L}(T^n\xi)\}$ (it is stationary and called the *stationary representation* of ξ), and let $\xi^* = \{\xi_k^*, -\infty < k < \infty\}$ be the two-sided infinite S-valued process, being the stationary extension of ξ^0.

Let $X = \{X_k, k \geqslant 1\}$ be an $\boldsymbol{R}$-valued stochastic process which generates the stochastic process $\boldsymbol{w} = \{w_k, k \geqslant 1\}$ via

$$w_{k+1} = \max(0, w_k + X_k), \quad k \geqslant 1, \tag{2.1}$$

with $w_1 \geqslant 0$ being the initial state of the process $\boldsymbol{w}$. In the sequel we consider also a pair $(\boldsymbol{w}, \boldsymbol{q})$ of processes $\boldsymbol{w}$ and $\boldsymbol{q} = \{q_k, k \geqslant 1\}$ defined by the process $(\boldsymbol{v}, \boldsymbol{u}) = \{(v_k, u_k), k \geqslant 1\}$. Then $\boldsymbol{w}$ is defined by (2.1) with $X = \boldsymbol{v} - \boldsymbol{u}$, and q_k is defined as

$$q_k = \sum_{l=1}^{k-1} I\Big(w_{k-l} + v_{k-l} \geqslant \sum_{j=k-l}^{k-1} u_j\Big), \quad k \geqslant 1. \tag{2.2}$$

In the sequel we make no notational distinction between transformation T on $\boldsymbol{R}^{\infty}$, S^{∞} or on other infinite products and we assume that $\sum_{j=n}^{k} = 0$ when $k < n$.

3. A characterization of condition AB. Let $X = \{X_k, k \geqslant 1\}$ be a random element of $\boldsymbol{R}^{\infty}$, $X^0 = \{X_k^0, k \geqslant 1\}$ its stationary representation (if it exists), and $X^* = \{X_k^*, -\infty < k < \infty\}$ a double infinite stationary extension of X^0. For X define a sequence of sums $S_0 = 0$, $S_n = \sum_{j=1}^{n} X_j$. For X^* define a sequence of sums $S_n^* = S_{n,0}^*$, $n \leqslant 0$, where $S_{n,k}^* = \sum_{j=n+1}^{k} X_j^*$ for $n < k$, and $S_{k,k}^* = 0$ for

$-\infty < k < \infty$. From (2.1) it follows that

$$w_{k+1} = \max(S_k + w_1,\, S_k - \min_{0 \leqslant j \leqslant k} S_j).$$

Here we will use also the following notation:

$$w_{k+1}(T^n X) \stackrel{\mathrm{df}}{=} S_{n+k} - S_n - \min_{0 \leqslant j \leqslant k} (S_{n+j} - S_n), \qquad n, k \geqslant 1. \tag{3.1}$$

We say that X satisfies *condition* AB or *condition* AB *in mean* if

$$\lim_{k\to\infty} \liminf_{n\to\infty} P\{\max_{k\leqslant j\leqslant n} (S_n - S_{n-j}) < 0\} = 1$$

or

$$\lim_{k\to\infty} \liminf_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} P\{\max_{k\leqslant j\leqslant l} (S_l - S_{l-j}) < 0\} = 1,$$

respectively.

The following lemma, being a little stronger version of Proposition 2 from [10], states that condition AB is necessary for the AS of w.

LEMMA 3.1. *Let X be such that either* (i) *the sequence $\{\mathscr{L}(T^n X)\}$ is tight or* (ii) *$\{n^{-1}\sum_{j=1}^{n} \mathscr{L}(T^j X)\}$ is tight and all limiting probability measures of all subsequences, say $\mathscr{L}(X^0)$ (they may be different), be such that $S^*_{-n} \to -\infty$ a.e. as $n \to \infty$. Furthermore, let $\{\mathscr{L}(w_n)\}$ and $\{n^{-1}\sum_{j=1}^{n} \mathscr{L}(w_j)\}$ be tight in the cases* (i) *and* (ii), *respectively. Then X satisfies condition* AB *in the case* (i) *and condition* AB *in mean in the case* (ii).

Proof. First notice that

$$\max_{k\leqslant j\leqslant n} (S_n - S_{n-j}) = S_n - S_{n-k} + w_{n-k+1}(X). \tag{3.2}$$

By the assumption dealing with tightness, we infer that for any $\varepsilon > 0$ there exists a positive number a such that for all n

$$P(w_n \leqslant a) \geqslant 1 - \varepsilon$$

in the case (i), and

$$\frac{1}{n} \sum_{j=1}^{n} P(w_j \leqslant a) \geqslant 1 - \varepsilon$$

in the case (ii). Hence for any $b > 0$ we get

$$\begin{aligned} P(S_n - S_{n-k} + w_{n-k+1}(X) < b) & \\ &\geqslant P(S_n - S_{n-k} + w_{n-k+1}(X) < b,\, w_{n-k+1}(X) \leqslant a) \\ &\geqslant P(S_n - S_{n-k} + a < b) - \varepsilon. \end{aligned} \tag{3.3}$$

Using (3.3) and (3.2) we have

$$(3.4)\quad \lim_k \liminf_n P\big(\max_{k\leqslant j\leqslant n}(S_n-S_{n-j})<b\big)$$

$$\geqslant \lim_k \liminf_n P(S_n-S_{n-k}+a<b)-\varepsilon \geqslant \lim_k P(S^*_{-k}+a<b)-\varepsilon = 1-\varepsilon.$$

Since ε and b were arbitrary, X satisfies condition AB in the case (i). In (3.4) we have used the Prokhorov theorem which states that for Polish metric spaces the tightness is equivalent to the relative compactness.

The proof of the second assertion runs in a similar way to the above one with the obvious modifications of considering convergence in mean instead of ordinary convergence in (3.3) and (3.4). ∎

In the sequel under different types of AS we will find sufficient conditions under which X satisfies condition AB.

LEMMA 3.2. *Let X be either* (i) *weakly AS or* (ii) *weakly AS in mean and $S^*_{-n}\to-\infty$ a.e. as $n\to\infty$. Then there exists a nondecreasing sequence $\{l_n\}$, $l_n\to\infty$, $l_n/n\to 0$, such that*

$$(3.5)\quad \lim_{k\to\infty}\liminf_{n\to\infty} P\big(\max_{k\leqslant j\leqslant l_n}(S_n-S_{n-j})<0\big)=1$$

in the case (i), *and*

$$(3.6)\quad \lim_{k\to\infty}\liminf_{n\to\infty}\frac{1}{n}\sum_{i=k}^{n} P\big(\max_{k\leqslant j\leqslant l_i}(S_i-S_{i-j})<0\big)=1$$

in the case (ii).

Proof. For any $k\leqslant l$, $n\geqslant 1$, let us put

$$a_k = P\big(\sup_{j\leqslant -k} S^*_j<0\big),\qquad a_{k,l}=P\big(\max_{k\leqslant j\leqslant l}\sum_{i=l-j+1}^{l} X^0_i<0\big),$$

$$a_{k,l,n}=P\big(\max_{k\leqslant j\leqslant l}(S_n-S_{n-j})<0\big)\equiv P\big(\sup_{k\leqslant j\leqslant l}\sum_{i=l-j+1}^{l} X_{n-l+i}<0\big).$$

Because X^* is a two-sided stationary extension of X^0, we get

$$(3.7)\quad a_{k,l}=P\big(\max_{k\leqslant j\leqslant l}\sum_{i=l-j+1}^{l} X^*_{-l+i}<0\big)=P\big(\sup_{-l\leqslant j\leqslant -k} S^*_j<0\big).$$

Since the function $x\mapsto \max_{k\leqslant j\leqslant l}x_j$ for $x=\{x_j, j\geqslant 1\}\in R^\infty$ is continuous on R^∞ and X is either (i) weakly AS or (ii) weakly AS in mean, then for any $k\leqslant l$ we have $\liminf_n a_{k,l,n}\geqslant a_{k,l}$ in the case (i), and $\liminf_n n^{-1}\sum_{i=1}^{n} a_{k,l,i}\geqslant a_{k,l}$ in the case (ii). Moreover, by (3.7), for any k, $\liminf_{l\to\infty} a_{k,l}\geqslant a_k$. Hence, by Lemma 5 in [8], there exists a nondecreasing sequence $l_n \to \infty$, $l_n/n \to 0$, such that $\liminf_n a_{k,l_n,n} \geqslant a_k$ in the case (i) and $\liminf_n n^{-1} \sum_{i=1}^{n} a_{k,l_n,i} \geqslant a_k$ in the case (ii). Since $S^*_{-n} \to -\infty$ a.e., we have $a_k \to 1$, which implies $\lim_k \lim_n a_{k,l_n,n} = 1$ and completes the proof in the case (i). To end the proof in the case (ii) notice that $a_{k,l_i,i} \geqslant a_{k,l_n,i}$ for $1 \leqslant i \leqslant n$, which together with the above gives

$$\lim_k \liminf_n \frac{1}{n} \sum_{i=1}^{n} a_{k,l_i,i} \geqslant \lim_k \liminf_n \frac{1}{n} \sum_{i=1}^{n} a_{k,l_n,i} = \lim_k a_k = 1.$$

This completes the proof. ∎

Now we prove the following technical lemma:

LEMMA 3.3. *Let $\{Z_{n,k},\, n,\, k \geqslant 1\}$ be a sequence of random variables defined on a common probability space such that, for each $n, k \geqslant 1$, $Z_{n,k} \leqslant Z_{n,k+1}$ and, for each k, $Z_{n,k} \xrightarrow{p} -\infty$ as $n \to \infty$. Then there exists a nondecreasing sequence $\{l_n\}$ tending to infinity such that $l_n/n \to 0$ and $Z_{n,l_n} \xrightarrow{p} -\infty$ as $n \to \infty$.*

Proof. Let $\{\varepsilon_i\}$ be a sequence of nonincreasing positive numbers tending to 0 and let $\{a_i\}$ be a nondecreasing sequence of numbers tending to ∞. Since, for each $k \geqslant 1$, $Z_{n,k} \xrightarrow{p} -\infty$ as $n \to \infty$, it follows that for each $k \geqslant 1$ there exists an increasing sequence $\{n_i(k),\, i \geqslant 1\}$ tending to ∞ such that $n_i(k) < n_{i+1}(k)$ and

$$\tilde{b}_{i,k} \overset{\text{df}}{=} \inf_{n \geqslant n_i(k)} P(Z_{n,k} \leqslant -a_i) \geqslant 1-\varepsilon_i.$$

Moreover, by $Z_{n,k} \leqslant Z_{n,k+1}$ we have $n_i(k) \leqslant n_i(k+1)$. Let $n_i = n_i(i)$, $i \geqslant 1$, and

$$b_{i,k} \overset{\text{df}}{=} \inf_{n \geqslant n_i} P(Z_{n,k} \leqslant -a_i).$$

Since for $i \geqslant k$ we have $n_i = n_i(i) \geqslant n_i(k)$, we obtain $b_{i,k} \geqslant \tilde{b}_{i,k} \geqslant 1-\varepsilon_i$. Hence, for each $k \geqslant 1$, $b_{i,k} \to 1$ as $i \to \infty$. Therefore, using Lemma 5 from [8], we infer that there exists a nondecreasing sequence $\{k_i\}$, tending to ∞ and such that $k_i/i \to 0$ and $b_{i,k_i} \to 1$. Define the sequence $\{l_n\}$ as follows: $l_n = k_i$ for $n_i \leqslant n < n_{i+1}$. Then $\{l_n\}$ is nondecreasing and $l_n/n \leqslant k_i/n_i \leqslant k_i/i \to 0$ as $n \to \infty$. Now for a given $a > 0$ there exists i such that $a_i > a$ and we have the following inequalities:

$$\begin{aligned}\inf_{n \geqslant n_i} P(Z_{n,l_n} \leqslant -a) &\geqslant \inf_{n \geqslant n_i} P(Z_{n,l_n} \leqslant -a_i) = \inf_{j \geqslant i} \inf_{n_j \leqslant n < n_{j+1}} P(Z_{n,l_n} \leqslant -a_i) \\ &\geqslant \inf_{j \geqslant i} \inf_{n_j \leqslant n < n_{j+1}} P(Z_{n,l_n} \leqslant -a_j) \\ &\geqslant \inf_{j \geqslant i} \inf_{n_j \leqslant n < n_{j+1}} P(Z_{n,k_j} \leqslant -a_j) \geqslant \inf_{j \geqslant i} (1-\varepsilon_j) = 1-\varepsilon_i.\end{aligned}$$

Hence $\inf_{n \geqslant n_i} P(Z_{n,l_n} \leqslant -a) \geqslant 1-\varepsilon_i \to 1$ as $i \to \infty$, which completes the proof. ∎

LEMMA 3.4. *Let X be such that $S_n \xrightarrow{p} -\infty$. Then there exists a nondecreasing sequence $\{k_n\}$ tending to ∞ such that $n-k_n \to \infty$, $k_n/n \to 1$, and*

$$\max_{k_n \leqslant j \leqslant n} (S_n - S_{n-j}) \xrightarrow{p} -\infty \quad \textit{as } n \to \infty.$$

Proof. The proof follows by using Lemma 3.3 with $Z_{n,k} = S_n - \inf_{0 \leqslant j \leqslant k} S_j$, $n, k \geqslant 1$. Then $k_n = n - l_n$. This completes the proof. ∎

In the sequel we will use the following remark, which is a slight modification of Lemma 2 from [4], p. 67.

Remark 3.1. *Let Z_n, $n \geqslant 1$, be random variables. Then $Z_n \xrightarrow{p} -\infty$ as $n \to \infty$ iff every subsequence of $\{Z_n\}$, say $\{Z_{n'}\}$, has itself a subsequence, say $\{Z_{n_k}\}$, converging to $-\infty$ with probability 1. Furthermore, $\{n_k\}$ can be chosen in such a way that $Z_{n_k+i} \to -\infty$, with probability 1, as $k \to \infty$ for all positive integers i.*

Proof. The convergence $Z_n \xrightarrow{p} -\infty$ is equivalent to the statement that for any $\varepsilon > 0$, $a > 0$, there exists n_0 such that

$$\sup_{n \geqslant n_0} (1 - P(Z_n \leqslant -a)) \leqslant \varepsilon.$$

Let $\varepsilon_k = 1/2^k$ and $a_k \to \infty$, $a_k < a_{k+1}$. Then for any $k \geqslant 1$ there exists n_k such that

$$\inf_{n \geqslant n_k} P(Z_n \leqslant -a_k) \geqslant 1 - 1/2^k.$$

Let A_k be the complement of the set $\{Z_{n_k} \leqslant -a_k\}$. Then

$$\sum_k P(A_k) \leqslant \sum_k 1/2^k < \infty,$$

so $P\{\limsup_k A_k\} = 0$. For a given $a > 0$, there exists k such that $a < a_k$ and we have the following inequalities:

$$P(\sup_{i \geqslant k} Z_{n_i} \leqslant -a) \geqslant P(\sup_{i \geqslant k} Z_{n_i} \leqslant -a_k) = P\Big(\bigcap_{i=k}^{\infty} \{Z_{n_i} \leqslant -a_k\}\Big)$$

$$\geqslant P\Big(\bigcap_{i=k}^{\infty} \{Z_{n_i} \leqslant -a_i\}\Big) = P\Big(\bigcap_{i=k}^{\infty} A_i^c\Big) \geqslant 1 - \sum_{i=k}^{\infty} P(A_i).$$

Hence and by the finiteness of $\sum_k P(A_k)$ we have $\lim_k P(\sup_{i \geqslant k} Z_{n_i} \leqslant -a) = 1$ for any $a > 0$, which means that $Z_{n_k} \to -\infty$ a.e. Notice that from the above consideration we also infer that $Z_{n_k+i} \to -\infty$ a.e. for any positive integer $i \geqslant 1$. This completes the proof. ∎

LEMMA 3.5. *Let X be either* (i) *weakly AS or* (ii) *weakly AS in mean and let $P(X \in B) = P(X^0 \in B)$ for all invariant sets B in R^∞. Furthermore, let $S^*_{-n} \to -\infty$ a.e. as $n \to \infty$. Then for any nondecreasing sequence l_n tending to infinity such that $l_n \leqslant n$ we have*

$$\max_{l_n \leqslant j \leqslant n} (S_n - S_{n-j}) \xrightarrow{p} -\infty \quad \text{as } n \to \infty \tag{3.8}$$

in the case (i), *and*

$$\frac{1}{n} \sum_{i=1}^{n} P\big(\max_{l_i \leqslant j \leqslant i} (S_i - S_{i-j}) < 0\big) \to 1 \quad \text{as } n \to \infty \tag{3.9}$$

in the case (ii).

Proof. For fixed nondecreasing subsequences $\{n_i\}$ and $\{l_n\}$ tending to infinity such that $l_n \leqslant n$, let us define $A \subset R^\infty$ as the set of elements $x \in R^\infty$ such that for any positive integer k the following holds:

$$\limsup_i \max_{l_{n_i} \leqslant j \leqslant n_i} \sum_{s=n_i-j+1}^{n_i} x_{s+k} = -\infty.$$

Of course, A is the invariant set in R^∞. Using definition (3.1) we get

$$\max_{l_n \leqslant j \leqslant n} (S_n - S_{n-j}) = S_n - S_{n-l_n} + w_{n-l_n+1}(X).$$

Since $S^*_{-n} \to -\infty$ a.e. as $n \to \infty$, by Borovkov's result the sequence $\{w_k(X^0), k \geqslant 1\}$ weakly converges. Furthermore, for any sequence $\{l_n\}$ described above we have $(S^0_n - S^0_{n-l_n}) \xrightarrow{p} -\infty$, which implies

$$\max_{l_n \leqslant j \leqslant n} (S^0_n - S^0_{n-j}) = S^0_n - S^0_{n-l_n} + w_{n-l_n+1}(X^0) \xrightarrow{p} -\infty. \tag{3.10}$$

Hence any sequence $\{n'\}$ contains a subsequence $\{n_i\}$, $n_i \to \infty$, such that

$$S^0_{n_i} - S^0_{n_i-l_{n_i}} + w_{n_i-l_{n_i}+1}(X^0) \to -\infty \text{ a.e.} \tag{3.11}$$

The sequence $\{n_i\}$ can be chosen in such a way that (3.11) holds also for the sequence $n'_i = n_i + k$ with any positive integer k (see Remark 3.1). This means that $P(X^0 \in A) = 1$, where A is defined for the sequence $\{n_i\}$ chosen in (3.11). Hence and by $P(X \in B) = P(X^0 \in B)$ for all invariant sets B in R^∞ we have $P(X \in A) = 1$. Now using Remark 3.1 we get (3.8).

In the case (ii) we also have $P(X^0 \in A) = 1$, so by the assumptions and Remark 3.1 we have $P(X \in A) = 1$, which implies (3.9). This completes the proof. ∎

Note 1. The assumption $P(X \in B) = P(X^0 \in B)$ for all invariant sets $B \subset R^\infty$ could be restricted to all invariant sets A defined in the proof of Lemma 3.5.

LEMMA 3.6. *Let X be either* (i) *AS in variation or* (ii) *AS in variation in mean, and let $S^*_{-n} \to -\infty$ a.e. as $n \to \infty$. Then for any sequence $\{l_n\}$ such that $l_n \to \infty$ and $n - l_n \to \infty$ we have*

$$(3.12)\qquad \lim_k \liminf_n P\big(\max_{k \leqslant j \leqslant l_n} (S_n - S_{n-j}) < 0\big) = 1$$

in the case (i), *and*

$$(3.13)\qquad \lim_k \liminf_n \frac{1}{n}\sum_{i=k}^{n} P\big(\max_{k \leqslant j \leqslant l_i} (S_i - S_{i-j}) < 0\big) = 1$$

in the case (ii).

Proof. By Theorems 4.3.2 and 4.3.3 from [1], p. 96, it follows that there exist random elements $\hat{X}$ and $\hat{X}^0$ defined on a common probability space such that $\hat{X} \stackrel{\mathscr{D}}{=} X$, $\hat{X}^0 \stackrel{\mathscr{D}}{=} X^0$, and

$$(3.14)\quad P(T^n \hat{X} \neq T^n \hat{X}^0) \to 0 \text{ in (i)} \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n} P(T^i \hat{X} \neq T^i \hat{X}^0) \to 0 \text{ in (ii)}.$$

Now notice that $\max_{k \leqslant j \leqslant l_n} (S_n - S_{n-j})$ depends only on $\{S_j, j \geqslant n - l_n + 1\}$. So putting $\hat{S}_n = \sum_{j=1}^{n} \hat{X}_j$, $\hat{S}^0_n = \sum_{j=1}^{n} \hat{X}^0_j$ and using $\hat{X} \stackrel{\mathscr{D}}{=} X$, $\hat{X}^0 \stackrel{\mathscr{D}}{=} X^0$, and (3.14) we get

$$\begin{aligned}(3.15)\quad P\big(\max_{k \leqslant j \leqslant l_n} (S_n - S_{n-j}) < 0\big) &= P\big(\max_{k \leqslant j \leqslant l_n} (\hat{S}_n - \hat{S}_{n-j}) < 0\big) \\ &\geqslant P\big(\max_{k \leqslant j \leqslant l_n} (\hat{S}_n - \hat{S}_{n-j}) < 0,\ T^{n-l_n} \hat{X} = T^{n-l_n} \hat{X}^0\big) \\ &= P\big(\max_{k \leqslant j \leqslant l_n} (\hat{S}^0_n - \hat{S}^0_{n-j}) < 0,\ T^{n-l_n} \hat{X} = T^{n-l_n} \hat{X}^0\big) \\ &\geqslant P\big(\max_{k \leqslant j \leqslant l_n} (\hat{S}^0_n - \hat{S}^0_{n-j}) < 0\big) - P(T^{n-l_n} \hat{X} \neq T^{n-l_n} \hat{X}^0) \\ &= P\big(\max_{-l_n \leqslant j \leqslant -k} S^*_j < 0\big) - o(1).\end{aligned}$$

Now passing to infinity in the above first with n and next with k and using $S^*_{-n} \to -\infty$ a.e. we get (3.12).

The proof of the convergence (3.13) runs in a similar way to that of (3.12) with the obvious modifications of considering convergence in mean instead of ordinary convergence in (3.15). This completes the proof. ∎

Now using

$$(3.16)\qquad \max_{k \leqslant j \leqslant n} (S_n - S_{n-j}) = \max\big(\max_{k \leqslant j \leqslant l_n} (S_n - S_{n-j}),\ \max_{l_n \leqslant j \leqslant n} (S_n - S_{n-j})\big)$$

and Lemmas 3.2–3.6 we get the following corollaries:

COROLLARY 3.1. *Let X be either* (i) *weakly AS or* (ii) *weakly AS in mean, and let $P(X \in B) = P(X^0 \in B)$ for all invariant sets B in R^∞. Furthermore, let $S^*_{-n} \to -\infty$ a.e. as $n \to \infty$. Then X satisfies condition* AB *in the case* (i) *and condition* AB *in mean in the case* (ii).

COROLLARY 3.2. *Let X be either* (i) *strongly AS or* (ii) *strongly AS in mean, and let X^* be ergodic and $a \overset{df}{=} \mathrm{E}X_1^0 < 0$. Then X satisfies condition* AB *in the case* (i) *and condition* AB *in mean in the case* (ii).

COROLLARY 3.3. *Let X be either* (i) *AS in variation or* (ii) *AS in variation in mean such that $S_n \xrightarrow{p} -\infty$, and $S^*_{-n} \to -\infty$ a.e. as $n \to \infty$. Then X satisfies condition* AB *in the case* (i) *and condition* AB *in mean in the case* (ii).

Corollary 3.1 follows from (3.16) by using Lemma 3.2 to the first term of (3.16) and Lemma 3.5 to the second term. Since the strong AS implies the weak AS, Corollary 3.2 follows from Corollary 3.1 by using Proposition 1 from [7]. Corollary 3.3 follows from (3.16) by using Lemma 3.4 to the second term of (3.16) with $l_n = k_n$, where $\{k_n\}$ is given in Lemma 3.4, and by using Lemma 3.6 to the first term of (3.16). In view of Note 1 the class of all invariant sets in Corollary 3.1 could be restricted.

Remark 3.2. *All the above lemmas and Corollaries* 3.1–3.3 *are true in the case when X is a random element of $R^{m,\infty} \equiv (R^m)^\infty$ and $w = \{w_k, k \geq 1\}$ is defined by X via* (2.1) *where all operations are meant in the coordinatewise sense.*

4. Almost sure convergence $n^{-1} w^*_{-n} \to 0$. To investigate an AS of the process of the interdeparture times it is needed to know an asymptote of $n^{-1} w_n$ and $n^{-1} w^*_{-n}$ as $n \to -\infty$. Here $w^*_{n+1} = \sup_{j \leq n} S^*_{j,n}$, where $S^*_{j,n} = \sum_{i=j+1}^{n} X^*_i$ for $-\infty < n < \infty$, $j \leq n$, and $S^*_n = S^*_{n,0}$ for $n \leq 0$. Because of the stationarity of $w^* = \{w^*_k, -\infty < k < \infty\}$ we have $n^{-1} w^*_{-n} \xrightarrow{p} 0$ as $n \to \infty$. In the following Lemma 4.1 we give the sufficient conditions for the almost sure convergences $n^{-1} w^*_{-n} \to 0$ and $n^{-1} w_n \to 0$ as $n \to \infty$. They are expressed in terms of the almost sure convergence in the functional space $D[0, \infty)$ (for the metric see [13]), precisely in terms of the functional strong law of large numbers for the sequences X and X^*. To state this let us define the processes ω_n, $\bar{S}_n$ and $\bar{S}^*_n$ as follows:

$$\omega_n(t) = n^{-1} w_{[nt]}, \quad \bar{S}_n(t) = n^{-1} S_{[nt]} \quad \text{and} \quad \bar{S}^*_n(t) = n^{-1} S^*_{-[nt]} \quad \text{for } t \geq 0.$$

Furthermore, let e denote the identity function $e(t) = t$, $t \geq 0$.

LEMMA 4.1. (i) *Let X be such that $\bar{S}_n \to ae$ in probability* (*or with probability* 1) *as $n \to \infty$, where $a < 0$. Then $\omega_n \to 0$ and $n^{-1} w_n \to 0$ in probability* (*or with probability* 1) *as $n \to \infty$.*

(ii) *Let X^* be such that $\bar{S}^*_n \to ae$ a.e. as $n \to \infty$. Then $n^{-1} w^*_{-n} \to 0$ a.e. as $n \to \infty$.*

Proof. To prove part (i) notice that by $w_{n+1} = S_n - \inf_{0 \leqslant j \leqslant n} S_j$ we have $\omega_{n+1} = f(\bar{S}_n)$, where $f(x)(t) = x(t) - \inf_{0 \leqslant s \leqslant t} x(s)$. Using the continuity of the mapping f in the Skorokhod topology (see [13]) and the convergence $\bar{S}_n \xrightarrow{p} ae$ (or with probability 1) we get $\omega_n \xrightarrow{p} f(ae)$ (or with probability 1). Because of $a < 0$ we have

$$f(ae)(t) = \sup_{0 \leqslant s \leqslant t} a(t-s) = 0, \qquad t \geqslant 0,$$

which completes the proof of part (i).

To prove part (ii) define for a fixed m, $m > 2$, and for any positive integers n, j, random variables $\gamma_{n,j} = n^{-1} S^*_{-nm-j,-n}$. Since $\bar{S}^*_n(1) = n^{-1} S^*_{-n} \to a$ a.e. as $n \to \infty$ and

$$\gamma_{n,j} = \frac{nm+j}{n} \frac{1}{nm+j} S^*_{-nm-j} - \frac{1}{n} S^*_{-n},$$

we have $\limsup_{n,j \to \infty} \gamma_{n,j} \leqslant (m-1)a$ with probability 1. Hence for any $\varepsilon > 0$ there exist $n_0 > 0$ and $j_0 > 0$ such that

$$P(\sup_{n \geqslant n_0} \sup_{j \geqslant j_0} \gamma_{n,j} > 0) \leqslant \varepsilon/2,$$

which implies

$$P(\sup_{n \geqslant n_0} \sup_{j \leqslant s} n^{-1} S^*_{j,-n} > 0) \leqslant \varepsilon/2, \qquad \text{where } s = -nm - j_0. \tag{4.1}$$

Using $S^*_{j,-n} = S^*_j - S^*_{-n}$ for $j \leqslant -n < 0$ we infer that for any $c > 0$ the following holds:

$$n^{-1} \sup_{-nm-c \leqslant j \leqslant -n} S^*_{j,-n} = \sup_{1 \leqslant t \leqslant m+c/n} (\bar{S}^*_n(t) - \bar{S}^*_n(1)) \leqslant \sup_{1 \leqslant t \leqslant m+c} (\bar{S}^*_n(t) - \bar{S}^*_n(1)).$$

Since the mapping $x \mapsto \sup_{1 \leqslant t \leqslant m+c} x(t)$ is continuous on the set of continuous functions and $\bar{S}^*_n \to ae$ a.e.,

$$\sup_{1 \leqslant t \leqslant m+c} (\bar{S}^*_n(t) - \bar{S}^*_n(1)) \to \sup_{1 \leqslant t \leqslant m+c} a(t-1) = 0 \text{ a.e.} \quad \text{as } n \to \infty.$$

Hence for any positive $b > 0$ we have

$$P\big(\sup_{n \geqslant k} \sup_{1 \leqslant t \leqslant m+c} (\bar{S}^*_n(t) - \bar{S}^*_n(1)) > b\big) \to 0 \quad \text{as } k \to \infty,$$

which implies that for any given $\varepsilon > 0$ and $b > 0$ there exists n_0 such that

$$P(\sup_{n \geqslant n_0} \sup_{-nm-c \leqslant j \leqslant -n} n^{-1} S^*_{j,-n} > b/2) \leqslant \varepsilon/2. \tag{4.2}$$

Using $w^*_{-n+1} = \max(\sup_{j \leqslant s} S^*_{j,-n}, \sup_{s \leqslant j \leqslant -n} S^*_{j,-n})$ with $s = -nm - j_0$ we get

$$P(\sup_{n \geqslant n_0} n^{-1} w^*_{-n+1} > b)$$
$$\leqslant P(\sup_{n \geqslant n_0} \sup_{j \leqslant s} n^{-1} S^*_{j,-n} > b/2) + P(\sup_{n \geqslant n_0} \sup_{s \leqslant j \leqslant -n} n^{-1} S^*_{j,-n} > b/2).$$

Now choosing n_0 and j_0 such that inequalities (4.1) and (4.2) hold we get

$$P(\sup_{n \geqslant n_0} n^{-1} w^*_{-n} > b) \leqslant \varepsilon.$$

Since ε and b were arbitrary, we obtain $n^{-1} w^*_{-n+1} \to 0$ a.e., which in turn implies $n^{-1} w^*_{-n} \to 0$ a.e. as $n \to \infty$. This completes the proof of part (ii) of the lemma. ∎

To verify the condition $\bar{S}_n \to ae$ a.e. or $\bar{S}^*_n \to ae$ a.e. as $n \to \infty$, we use the following remark in which $V_n = \sum_{j=1}^{n} v_j$, $U_n = \sum_{j=1}^{n} u_j$, $n \geqslant 1$.

Remark 4.1. *Let $n^{-1} V_n \to \bar{v}$ and $n^{-1} U_n \to \bar{u}$ in probability (or with probability 1) as $n \to \infty$, where $\bar{v}$ and $\bar{u}$ are nonrandom. Furthermore, if $X_k = v_k - u_k$, then $\bar{S}_n \to (\bar{v} - \bar{u})e$ in $D[0, \infty)$ in probability (or with probability 1).*

Proof. Defining processes $\bar{V}_n(t) = n^{-1} V_{[nt]}$ and $\bar{U}_n(t) = n^{-1} U_{[nt]}$ for $t \geqslant 0$, we see that their sample paths are nondecreasing and nonnegative. Hence by Proposition 2.1 from [11] it follows that $\bar{V}_n \to \bar{v}e$ and $\bar{U}_n \to \bar{u}e$ in probability (or with probability 1) as $n \to \infty$. The proof of Proposition 2.1 in [11] is complete in the case when the limiting process has continuous sample paths, which we have here. Now, because the operation of summation on the set of continuous functions is continuous (see [13]), we obtain $\bar{V}_n - \bar{U}_n \xrightarrow{p} (\bar{v} - \bar{u})e$ in $D[0, \infty)$ as $n \to \infty$, which completes the proof. ∎

To get the convergence $\bar{S}^*_n \to ae$ a.e. we use the above remark with $V_n = \sum_{j=1}^{n} v^*_{-j}$, $U_n = \sum_{j=1}^{n} u^*_{-j}$, $n \geqslant 1$.

5. Condition AB for the second node. Here we formulate the sufficient conditions under which the input to the second node satisfies the conditions of Corollaries 3.1–3.3, which imply that it satisfies condition AB. First notice that from formula (1.1) it follows that the process of the interdeparture times $\boldsymbol{u}_2$ is an image of $(\boldsymbol{w}_1, \boldsymbol{v}_1, \boldsymbol{u}_1)$ by a continuous mapping. Hence, defining

$$\zeta \stackrel{\mathrm{df}}{=} (\boldsymbol{w}_1, \boldsymbol{v}_1, \boldsymbol{u}_1, \boldsymbol{\eta}) = \{(w_{1,k}, v_{1,k}, u_{1,k}, \eta_k), k \geqslant 1\},$$

where $\boldsymbol{\eta}$ is an S-valued process, we get the following lemma:

LEMMA 5.1. *If ζ is AS in some sense c_j, $1 \leqslant j \leqslant 6$, then $(\zeta, \boldsymbol{u}_2)$ is AS in the same sense as ζ, and a double ended stationary extension of its stationary representation $(\zeta^0, \boldsymbol{u}_2^0)$ is*

$$(\zeta^*, \boldsymbol{u}_2^*) = \{(w^*_{1,k}, v^*_{1,k}, u^*_{1,k}, \eta^*_k, u^*_{2,k}), -\infty < k < \infty\},$$

where $(w_1^*, v_1^*, u_1^*, \eta^*)$ *is a double ended stationary extension of* $(w_1^0, v_1^0, u_1^0, \eta^0)$, *and*

$$u_{2,k}^* = u_{1,k}^* + w_{1,k+1}^* + v_{1,k+1}^* - w_{1,k}^* - v_{1,k}^*, \qquad -\infty < k < \infty. \tag{5.1}$$

Now we give the sufficient conditions under which $P(X_2 \in B) = P(X_2^0 \in B)$ for all invariant sets B in R^∞.

LEMMA 5.2. *Let* $\psi = (v_1, v_2, u_1)$ *be either* (i) *weakly AS or* (ii) *weakly AS in mean and let* $P(\psi \in B) = P(\psi^0 \in B)$ *for all invariant sets* B *in* $R^{3,\infty}$. *Then* $P(X_1 \in B) = P(X_1^0 \in B)$ *for all invariant sets* B *in* R^∞. *Furthermore, if* (ψ, w_1) *is either* (i) *weakly AS or* (ii) *weakly AS in mean, then* $P(X_2 \in B) = P(X_2^0 \in B)$ *for all invariant sets* B *in* R^∞.

To prove Lemma 5.2 we use the following

IDEA. *If* $G\colon S^\infty \mapsto (S_1)^\infty$ *is a measurable mapping such that* $TG(x) = G(Tx)$, *while* B *is an invariant set in* $(S_1)^\infty$, *then* $G^{-1}B$ *is an invariant set in* S^∞.

Indeed, if $x \in G^{-1}B$, then $G(x) \in B$. Since B is invariant, $TG(x) \in B$, which by $TG(x) = G(Tx)$ gives $G(Tx) \in B$, and this implies $Tx \in G^{-1}B$.

Proof of Lemma 5.2. Applying the Idea to $X_1 = v_1 - u_1$ with $G(x, y) = x - y$, $x, y \in R^\infty$ and using the assumptions of the lemma for all invariant sets B in R^∞ we have

$$P(X_1 \in B) = P\big((v_1, u_1) \in G^{-1}B\big) = P\big((v_1^0, u_1^0) \in G^{-1}B\big) = P(X_1^0 \in B). \tag{5.2}$$

To prove the second assertion notice that if B is an invariant set in $R^{4,\infty}$, then

$$A = \{(\psi, w_1) \in B\} = \{T^n(\psi, w_1) \in B\} \quad \text{for each } n \geqslant 1.$$

Hence A does not depend on a finite number of coordinates of the sequence (ψ, w_1). Therefore assume that $w_{1,1} = 0$, $v_{1,1} - u_{1,1} = v_{1,2} - u_{1,2} = 0$.

Now define mappings $F_k\colon R^{2,\infty} \mapsto R$ and $\bar{F}\colon R^{2,\infty} \mapsto R^\infty$ in $x, y \in R^\infty$ by

$$F_1(x, y) = 0, \qquad F_{k+1}(x, y) = \max_{0 \leqslant j \leqslant k} \sum_{i=j+1}^{k} (x_i - y_i), \qquad k \geqslant 1,$$

and $\bar{F}(x, y) = \{F_k(x, y), k \geqslant 1\}$. Furthermore, define a mapping $F\colon R^{3,\infty} \mapsto R^{4,\infty}$ by

$$F(x_1, x_2, y) = \big(x_1, x_2, y, \bar{F}(x_1, y)\big) \quad \text{for } x_1, x_2, y \in R^\infty.$$

Notice that

$$(v_1, v_2, u_1, w_1) = F(v_1, v_2, u_1). \tag{5.3}$$

Now we will show that for any invariant set B in $R^{4,\infty}$ the set $F^{-1}B$ is invariant in $R^{3,\infty}$. To do this we need to show that if $(x_1, x_2, y) \in F^{-1}B$, then

$T(x_1, x_2, y) \in F^{-1}B$. Here, without loss of generality, we assume that $x_{1,1} - y_1 = x_{1,2} - y_2 = 0$. First notice that

$$F(Tx_1, Tx_2, Ty) = \big(Tx_1, Tx_2, Ty, \bar{F}(Tx_1, Ty)\big), \tag{5.4}$$

while $\bar{F}(Tx_1, Ty) = \{z_k, k \geqslant 1\}$, where

$$z_1 = 0, \quad z_{k+1} = \max_{1 \leqslant j \leqslant k} \sum_{i=j+1}^{k} \gamma_{i+1} \quad \text{and} \quad \gamma_k = x_{1,k} - y_k \text{ for } k \geqslant 1.$$

Since $\gamma_1 = \gamma_2 = 0$, we have $z_1 = z_2 = 0$ and $z_{k+1} = \max_{0 \leqslant j \leqslant k+1} \sum_{i=j+1}^{k+1} \gamma_i$, $k \geqslant 2$. On the other hand, $\bar{F}(x_1, y) = \{\tilde{z}_k, k \geqslant 1\}$, where $\tilde{z}_1 = 0$, $\tilde{z}_{k+1} = \max_{0 \leqslant j \leqslant k} \sum_{i=j+1}^{k} \gamma_i$ for $k \geqslant 1$. Since $\gamma_1 = \gamma_2 = 0$, we obtain $\tilde{z}_1 = \tilde{z}_2 = \tilde{z}_3 = 0$ and $\tilde{z}_{k+1} = z_k$ for $k \geqslant 1$, which in turn implies $\bar{F}(Tx_1, Ty) = T\bar{F}(x_1, y)$. The last equality together with (5.4) gives

$$F(Tx_1, Tx_2, Ty) = TF(x_1, x_2, y) = T\big(x_1, x_2, y, \bar{F}(x_1, y)\big). \tag{5.5}$$

Assuming that $(x_1, x_2, y) \in F^{-1}B$, we get $F(x_1, x_2, y) \in B$, i.e. $\big(x_1, x_2, y, \bar{F}(x_1, y)\big) \in B$. Since B is invariant, $T\big(x_1, x_2, y, \bar{F}(x_1, y)\big) \in B$ which together with (5.5) gives $F(Tx_1, Tx_2, Ty) \in B$, so $T(x_1, x_2, y) \in F^{-1}B$. Hence $F^{-1}B$ is an invariant set. Now defining a mapping $G\colon R^{4,\infty} \mapsto R^{\infty}$ by

$$G(x_1, x_2, y, z) = x_2 - y - Tz - Tx_1 + z + x_1,$$

we see that

$$TG(x_1, x_2, y, z) = G(Tx_1, Tx_2, Ty, Tz).$$

Since $X_2 = v_2 - u_1 - Tw_1 - Tv_1 + w_1 + v_1$, we obtain $X_2 = G(v_1, v_2, u_1, w_1)$. Hence, using the Idea and the assumptions of the lemma, for any invariant set B in R^{∞} we have

$$\begin{aligned} P(X_2 \in B) &= P\big((v_1, v_2, u_1, w_1) \in G^{-1}B\big) = P\big((v_1, v_2, u_1) \in F^{-1}G^{-1}B\big) \\ &= P\big((v_1^0, v_2^0, u_1^0) \in F^{-1}G^{-1}B\big) \\ &= P\big((v_1^0, v_2^0, u_1^0, w_1^0) \in G^{-1}B\big) = P(X_2^0 \in B). \end{aligned} \tag{5.6}$$

The fourth equality follows by the fact that $G^{-1}B$ is invariant if B is invariant, and next by the fact that $G^{-1}B$ does not depend on a finite number of coordinates of the sequence $(v_1^0, v_2^0, u_1^0, w_1^0)$. This completes the proof. ∎

In the sequel we use the notation $a_j = \mathrm{E}v_{j,1}^0 - \mathrm{E}u_{1,1}^0$, $j = 1, 2$.

THEOREM 1. *Let $\psi = (v_1, v_2, u_1)$ be AS is some sense c_j, $1 \leqslant j \leqslant 6$, and let ψ^* be ergodic and $\mathrm{E}v_{j,1}^0 < \mathrm{E}u_{1,1}^0$ for $j = 1, 2$. Furthermore, in the cases c_1 and c_2 assume additionally that $P(\psi \in B) = P(\psi^0 \in B)$ for all invariant sets B in $R^{3,\infty}$. Then the following assertions hold*:

(i) *$n^{-1} S^*_{-n}(1) \to a_1$ a.e. and $n^{-1} S_n(1) \to a_1$ a.e. as $n \to \infty$ and X_1 satisfies condition* AB *in the cases c_1, c_3, c_5 and condition* AB *in mean in the cases c_2, c_4, c_6. Moreover, (w_1, u_2, ψ) is AS in the same sense as ψ.*

(ii) $n^{-1}S^*_{-n}(2) \to a_2$ *a.e. and* $n^{-1}S_n(2) \to a_2$ *a.e. as* $n \to \infty$ *and* X_2 *satisfies condition* AB *in the cases* c_1, c_3, c_5 *and condition* AB *in mean in the cases* c_2, c_4, c_6.

Proof. First we consider the cases c_1 and c_2. To prove the first assertion in these cases notice that by Lemma 5.2 we get $P(X_1 \in B) = P(X_1^0 \in B)$ for all invariant sets B in $\boldsymbol{R}^\infty$. This and the ergodicity of ψ^* imply the convergences $n^{-1}S_n(1) \to a_1$ a.e. and $n^{-1}S^*_{-n}(1) \to a_1$ a.e. as $n \to \infty$. The last convergence in view of $a_1 < 0$ gives $S^*_{-n}(1) \to -\infty$ a.e. as $n \to \infty$. Hence and by Corollary 3.1 we see that X_1 satisfies condition AB in the case c_1 and condition AB in mean in the case c_2. This together with Theorem 1 from [12] imply that (w_1, ψ) is AS in the same sense as ψ, and (w_1^*, ψ^*) is ergodic. This and the equality $u_2^* = u_1^* + Tw_1^* + Tv_1^* - w_1^* - v_1^*$ and next the Idea imply that (w_1^*, u_2^*, ψ^*) is ergodic. A similar argument gets the ergodicity of X_2^*.

To prove the second assertion notice that by the ergodicity of ψ^* we get the following convergences:

$$(5.7) \qquad \frac{1}{n}\sum_{j=-n}^{0} (v_{2,j}^* - u_{1,j}^*) \to a_2 \text{ a.e.} \quad \text{as } n \to \infty$$

and

$$(5.8) \qquad \frac{1}{n}v_{1,-n}^* = \frac{1}{n}\sum_{j=1}^{n} v_{1,-j}^* - \frac{1}{n}\sum_{j=1}^{n-1} v_{1,-j}^* \to \mathrm{E}v_{1,1}^0 - \mathrm{E}v_{1,1}^0 = 0 \text{ a.e.} \quad \text{as } n \to \infty.$$

Now putting $\bar{V}_n(t) = n^{-1}\sum_{j=1}^{[nt]} v_{1,-j}^*$ and $\bar{U}_n(t) = n^{-1}\sum_{j=1}^{[nt]} u_{1,-j}^*$, $t \geqslant 0$, and next using the ergodicity of ψ^* and Remark 4.1 we get $\bar{V}_n - \bar{U}_n \to a_1 e$ a.e. in $D[0, \infty)$ as $n \to \infty$. This in turn implies that the conditions of Lemma 4.1 in the case (ii) are satisfied. Therefore, by that lemma we get the convergence

$$(5.9) \qquad \frac{1}{n}w_{1,-n}^* \to 0 \text{ a.e.} \quad \text{as } n \to \infty.$$

Notice that

$$S_n^*(2) = \sum_{j=n+1}^{0} (v_{2,j}^* - u_{1,j}^*) - w_{1,1}^* - v_{1,1}^* + w_{1,n+2}^* + v_{1,n+2}^*, \qquad n < 0.$$

Therefore, by convergences (5.7)–(5.9), we get $n^{-1}S^*_{-n}(2) \to a_2$ a.e. as $n \to \infty$, which in view of $a_1 < 0$ gives $S^*_{-n}(2) \to -\infty$ a.e. as $n \to \infty$. Now notice that the weak AS of (w_1, ψ) in the case c_1 and the weak AS in mean in the case c_2 and next Lemma 5.2 imply that $P(X_2 \in B) = P(X_2^0 \in B)$ for all invariant sets B in $\boldsymbol{R}^\infty$. This and the ergodicity of X_2^* and the convergence $n^{-1}S^*_{-n}(2) \to a_2$ a.e. as $n \to \infty$ imply the convergence $n^{-1}S_n(2) \to a_2 = \mathrm{E}X_{1,1}^0$ a.e. as $n \to \infty$.

To complete the proof notice that the equality $P(X_2 \in B) = P(X_2^0 \in B)$ for all invariant sets B in $\boldsymbol{R}^\infty$ and Corollary 3.1 imply that X_2 satisfies condition AB in the case c_1 and condition AB in mean in the case c_2. To get the same for other cases we apply Corollaries 3.2 and 3.3. This completes the proof. ∎

Theorem 1 can be used to analyze an asymptotic stationarity of m single server queues in series. Namely, let

$$\xi = (v_1, v_2, \ldots, v_m, u_1) = \{(v_{1,k}, v_{2,k}, \ldots, v_{m,k}, u_{1,k}), k \geqslant 1\}$$

describe a series of m single server queues (see [5]), where $v_{i,k}$ is the service time of the k-th unit in the i-th queue while $u_{1,k}$ is the interarrival time between the k-th and $(k+1)$-st units to the first queue. Furthermore, let $w_{i,k}$ denote the waiting time of the k-th unit in the i-th queue, and $q_{i,k}$ the number of units at the i-th queue just before the k-th arrival to that queue. Then using Theorem 1 and next Theorems 1 and 2 from [12] we get the following corollary:

COROLLARY 5.1. *Let ξ be AS in some sense c_j, $1 \leqslant j \leqslant 6$, and let ξ^* be ergodic and* $\mathrm{E}v_{i,1}^0 < \mathrm{E}u_{1,1}^0$ *for $i = 1, 2, \ldots, m$. Furthermore, in the cases c_1 and c_2 assume additionally that $P(\xi \in B) = P(\xi^0 \in B)$ for all invariant sets B in $R^{m+1,\infty}$. Then (w, ξ) is AS in the same sense as ξ, where*

$$w = \{w_k = (w_{1,k}, w_{2,k}, \ldots, w_{m,k}), k \geqslant 1\}$$

and (w, q, ξ) is AS in the same sense as ξ in the cases c_j, $3 \leqslant j \leqslant 6$, where

$$q = \{q_k = (q_{1,k}, q_{2,k}, \ldots, q_{m,k}), k \geqslant 1\}.$$

For AS of (w, q, ξ) in the cases c_1 and c_2 we need the condition

$$P\Big(w_{i,1}^* + v_{i,1}^* - \sum_{j=1}^{k} u_{i,j}^* = 0\Big) = 0 \quad \text{for all } k \geqslant 1,\ 1 \leqslant i \leqslant m.$$

Theorem 1 and Corollary 5.1 complete the proof of Theorems 1 and 2 in [5] in the case of the strong AS.

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