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A NOTE ON DOMAINS OF ATTRACTION FOR q-TRANSFORMED RANDOM VARIABLES

BY

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Abstract. We show that a random variable X lies in the strict domain of attraction of a non-degenerate strictly stable random variable Z with exponent $\alpha \in]0, 2[$ iff the q-transform of X lies in the strict domain of attraction of mZ for some constant m depending on q and α with the same norming sequence.

1. Introduction. q-algebra and q-analysis $(0 < q < 1)$ on the real line may be interpreted as a generalization of ordinary addition, which (roughly speaking) corresponds to the case $q = 1$. However, it is not possible to define q-addition directly on the real line itself, but rather indirectly on the space of measures on $\boldsymbol{R}$ in the sense that the q-convolution of two Dirac measures is in general not a Dirac measure (in the ordinary sense). So the whole theory is somewhat similar to hypergroups (cf. Bloom and Heyer [1]), but it does not fit exactly into this context.

Feinsilver [3] began a probabilistic study on q-added random variables. In the last part of his paper, he initiated an investigation of limit theorems for q-sums of random variables. The purpose of this note is to give a further contribution to this subject. We will show that a random variable X lies in the strict domain of attraction of a non-degenerate strictly stable random variable Z with exponent $\alpha \in]0, 2[$ iff the q-transform of X lies in the strict domain of attraction of mZ for some constant m depending on q and α with the same norming sequence. The proof consists essentially of getting rid of the centering constants appearing in the case of ordinary addition and of a desintegration procedure for the "only if" direction.

2. q-addition. Let $0 < q < 1$. We first give some definitions on q-algebra (see e.g. Feinsilver and Schott [4] and Koornwinder [7]). The q-natural numbers q_k are given as

$$q_k := \sum_{i=0}^{k-1} q^i = \frac{1-q^k}{1-q}.$$

Consequently, one defines the q-factorial as

$$(k)! := \prod_{i=1}^{k} q_i$$

and the q-exponential function $e(x)$ is defined as

$$e(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k)!},$$

whereas the q-derivative is given by

$$D_q f(x) = \frac{f(x) - f(qx)}{1 - qx}.$$

Now one defines q-addition (indirectly) by defining a *Dirac measure* $\delta_{x \oplus y}$ by

$$\delta_{x \oplus y}(f) := (\delta_x * \delta_y)(f) := e\big(-(1-q)\, y D_q\big) f(x)$$

(cf. Feinsilver [3], IV). The q-convolution of measures $\mu \in M^b(\mathbb{R})$ is then extended from the q-convolution of Dirac probability measures as indicated above also in the natural way by linearity and weak continuity.

The q-characteristic function of $\mathscr{L}(X)$ for the random variable X is defined by

$$\psi_X(u) := E\big(e(iuX)\big) \qquad (u \in \mathbb{R}).$$

The symbol $\varphi_X(u)$ will be used for the ordinary characteristic function. It has been shown by Feinsilver [3] (Theorem 3) that $\mathscr{L}(X)$ is uniquely determined by its q-characteristic function ψ_X. Furthermore, if X_1 and X_2 are independent random variables, then the q-convolution of X_1 and X_2 is a random variable Z whose q-characteristic function is the product of the q-characteristic functions of X_1 and X_2: $\psi_Z = \psi_{X_1} \psi_{X_2}$ (cf. [3]). Define the random variable Y by

$$Y := \sum_{k=0}^{\infty} T_k, \tag{1}$$

where the T_k are independent random variables, T_k obeying to an exponential law with mean q^k. Assume X is any random variable on $\mathbb{R}$, independent of Y. Then the q-characteristic function of X is the ordinary characteristic function of XY: $\psi_X = \varphi_{XY}$ (cf. Feinsilver [3], Proposition 4). We will call XY the *q-transform* of X. If $F(x) = P(X \leqslant x)$ is the law of the random variable X, then the law of the q-transform is given by the mixture

$$G(x) = P(XY \leqslant x) = \int_0^{\infty} F(x/y)\, \mathscr{L}(Y)(dy). \tag{2}$$

So what we have to study are sums of the type

$$\sum_{k=1}^{n} X_k Y_k,$$

where $X_1, X_2, \ldots$ are any independent random variables and $Y_1, Y_2, \ldots$ are i.i.d., as in (1), and independent of $X_1, X_2, \ldots$

3. Domains of attraction. First, we recall some facts on stable laws with respect to ordinary addition. As references, see e.g. Gnedenko and Kolmogorov [6] or Breiman [2]. A random variable Z is called *stable* if for every $n \geqslant 1$ and i.i.d. copies $Z_1, Z_2, \ldots, Z_n$ of Z there are $c_n > 0$, $d_n \in \boldsymbol{R}$ such that

$$Z \stackrel{\mathscr{L}}{=} c_n \sum_{k=1}^{n} (Z_k + d_n).$$

Equivalently, Z is stable iff there are i.i.d. random variables $X_1, X_2, \ldots$ and $a_n > 0$, $b_n \in \boldsymbol{R}$ such that

$$a_n \sum_{k=1}^{n} (X_k + b_n) \xrightarrow{w} Z \qquad (n \to \infty) \tag{3}$$

(where $\xrightarrow{w}$ denotes weak convergence). If $X \stackrel{\mathscr{L}}{=} X_1$, then X is said to *lie in the domain of attraction* of Z. The sequence $\{(a_n, b_n)\}_{n \geqslant 1}$ is called a *norming sequence.* We will use the term *strictly stable* if $d_n = 0$ and the term *strict domain of attraction* if $b_n = 0$. It can be shown that there exists $\alpha \in]0, 2]$ such that

$$c_n = n^{-1/\alpha}.$$

The number α is called the *exponent of stability.* The case $\alpha = 2$ corresponds to the case where Z obeys to a normal distribution. Z is non-degenerate and stable with exponent $\alpha \in]0, 2[$ iff its characteristic function takes the form

$$\varphi_Z(u) = \exp\left\{i\gamma u + \left(v \int_{-\infty}^{0} + w \int_{0}^{\infty}\right)\left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)\frac{dx}{|x|^{1+\alpha}}\right\}$$

$$(\gamma \in \boldsymbol{R},\ v,\ w \geqslant 0,\ v + w > 0).$$

For short, we write $Z \stackrel{\mathscr{L}}{=} (\alpha, \gamma, v, w)$. It follows that for $k \in \boldsymbol{N}$, we have

$$(\alpha, k\gamma, kv, kw) \stackrel{\mathscr{L}}{=} k^{1/\alpha} Z.$$

For the following lemma see Le Page et al. [8], Remark 3 on p. 628.

LEMMA 1. *In the case* $0 < \alpha < 1$ *we have*

$$n a_n b_n \to b \ (n \to \infty) \quad \textit{for some } b \in \boldsymbol{R}.$$

The next lemma follows also from Le Page et al. [8], Remark 3 on p. 628 (see also Gnedenko and Kolmogorov [6], Theorem 35.3).

LEMMA 2. *In the case* $1 < \alpha < 2$ *we have*

$$b_n = -E(X_1) + \frac{b + o(1)}{na_n} \ (n \to \infty) \quad \textit{for some } b \in \mathbf{R}.$$

What remains, is the case $\alpha = 1$. This situation is somewhat special in the following sense. If $\alpha \in]0, 1[\cup]1, 2[$ and if Z is α-stable, then it is always possible to center Z so that it becomes strictly stable (see e.g. Sharpe [11], Theorem 6). However, the following property follows at once by considering the characteristic function for $\alpha = 1$ in the "explicit form"

$$\varphi_Z(u) = \exp\left\{i\beta u - \varrho|u|\left(1 + i\theta\frac{u}{|u|}\frac{2}{\pi}\log|u|\right)\right\} \quad \left(\beta \in \mathbf{R},\ \varrho > 0,\ \theta = \frac{v - w}{v + w}\right);$$

the centering statement then follows from Feller [5], Theorem XVII.5.3.

LEMMA 3. *The only non-degenerate strictly stable laws* μ *with exponent* $\alpha = 1$ *are the shifted* (*with shift* $b \in \mathbf{R}$) *symmetric* (*Cauchy*) *ones* (*i.e. those with* $v = w$). *In this case, we have*

$$a_n b_n = -E\left(\sin(a_n X_1)\right) + \frac{b + o(1)}{n} \quad (n \to \infty),$$

where $b \in \mathbf{R}$ *is the aforementioned shift of* μ.

The domains of attraction of stable laws with exponent $0 < \alpha < 2$ may be characterized as follows (cf. Meerschaert [10], p. 344):

PROPOSITION 1. *For a non-degenerate stable random variable* Z *with exponent* $0 < \alpha < 2$ *the relation* (3) *holds iff*

$$nP(a_n X_1 < x) \to v \int_{-\infty}^{x} \frac{dt}{|t|^{1+\alpha}} \ (n \to \infty) \quad (x < 0)$$

and

$$nP(a_n X_1 > x) \to w \int_{x}^{\infty} \frac{dt}{t^{1+\alpha}} \ (n \to \infty) \quad (x > 0).$$

Let X and Z be random variables. The q-transform of X lies in the strict domain of attraction of Z with norming sequence $\{a_n\}$ if for i.i.d. copies $X_1, X_2 \ldots$ of X and i.i.d. random variables $Y_1, Y_2, \ldots$ as in (1) and independent of $X_1, X_2, \ldots$ there exist $a_n > 0$ such that

$$a_n \sum_{k=1}^{n} X_k Y_k \xrightarrow{w} Z. \tag{4}$$

LEMMA 4. *The characteristic function of* $\log Y$ *is analytic in a neighborhood of the real axis.*

Proof. By Feinsilver [3], the density of Y is given by

$$g(x) = C\sum_{j=0}^{\infty}\frac{(-1)^j}{(j)!}q^{\binom{j}{2}}\exp\{-q^{-j}x\} \qquad (x \geqslant 0);$$

hence the density of $\log Y$ is

$$h(x) = e^x g(e^x) = C\sum_{j=0}^{\infty}\varrho_j \exp\{-q^{-j}e^x + x\}, \tag{5}$$

where

$$\varrho_j = \frac{(-1)^j}{(j)!}q^{\binom{j}{2}}.$$

Clearly,

$$\sum_{j=0}^{\infty}|\varrho_j| < \infty. \tag{6}$$

Since

$$-q^{-j}e^x + x \leqslant -e^x + x \leqslant K \qquad (x \in \boldsymbol{R}), \tag{7}$$

it follows from (6) and (7) that the series in (5) converges uniformly for $x \in \boldsymbol{R}$, so we get, by (5)–(7),

$$\begin{aligned} P(|\log Y| > x) &= \int_{\boldsymbol{R}\setminus[-x,x]} h(t)\,dt \leqslant C\Big(\sum_{j=0}^{\infty}|\varrho_j|\Big)\int_{\boldsymbol{R}\setminus[-x,x]}\exp\{-e^t + t\}\,dt \\ &= O(1 - \exp\{-e^{-x}\} + \exp\{-e^x\}) = O(e^{-x}) \qquad (x \to \infty). \end{aligned}$$

Now the assertion follows from Lukacs [9], Theorem 7.2.1. ∎

For fixed α and Y as in (1), define the constant $m := (EY^{\alpha})^{1/\alpha}$.

THEOREM 1. *Let Z be a non-degenerate strictly stable random variable with exponent $0 < \alpha < 2$ and let X be any random variable. Then the q-transform of X lies in the strict domain of attraction of mZ with norming sequence $\{a_n\}_{n \geqslant 1}$ iff X lies in the strict domain of attraction of Z with norming sequence $\{a_n\}_{n \geqslant 1}$.*

Proof. 1. "If" direction. Assume X lies in the strict domain of attraction of Z with norming sequence $\{a_n\}_{n \geqslant 1}$. Let $Y_1, Y_2, \ldots$ be as in (4). By Gnedenko and Kolmogorov [6], Theorem 25.1 and the Remark on p. 121, it follows that the conditions (i)–(iii) mentioned before Proposition 8 in Feinsilver [3] are indeed fulfilled. Hence, by [3], Proposition 8, the condition of our Proposition 1 carries over to the q-transforms (2) of $X_1, X_2, \ldots$ in the sense that XY lies in the domain of attraction of mZ with some norming sequence $\{a_n, b_n\}_{n \geqslant 1}$ for certain $b_n \in \boldsymbol{R}$.

1.1. Case $0 < \alpha < 1$. By Lemma 1 it follows that XY lies in the strict domain of attraction of $mZ - b$ for some $b \in \boldsymbol{R}$. By the convergence of types theorem (see e.g. Breiman [2], Theorem 8.32) it follows that $mZ-b$ is also strictly α-stable; hence $b=0$.

1.2. Case $1<\alpha<2$. By Lemma 2 it follows that

$$na_n E(X)\to b \qquad (n\to\infty)$$

for some $b\in\mathbf{R}$. Hence also

$$na_n E(XY)\to bE(Y) \qquad (n\to\infty).$$

So it follows from Lemma 2 that XY lies in the strict domain of attraction of $mZ-b'+bE(Y)$ for some $b, b'\in\mathbf{R}$ and the rest of the proof is as under 1.1.

1.3. Case $\alpha=1$. By Lemma 3 it follows that

$$nE\big(\sin(a_n X)\big)\to b \qquad (n\to\infty),$$

where $b\in\mathbf{R}$ is the shift of Z as in Lemma 3. Hence, by the dominated convergence theorem, Proposition 1, and the stability property, we get

$$\begin{aligned}\lim_{n\to\infty} nE\big(\sin(a_n XY)\big) &= \lim_{n\to\infty}\int_0^\infty nE\big(\sin((a_n y)X)\big)\,\mathscr{L}(Y)(dy)\\ &= \lim_{n\to\infty}\int_0^\infty y\frac{n}{y}E\big(\sin(a_{\lfloor n/y\rfloor}X)\big)\,\mathscr{L}(Y)(dy)\\ &= bE(Y)=mb.\end{aligned}$$

So by Lemma 3 it follows that XY lies in the strict domain of attraction of mZ.

"Only if" direction. By Proposition 1 it follows that

$$\begin{aligned}& n\int_0^\infty\int_J \mathscr{L}(a_n yX)(dt)\,\mathscr{L}(Y)(dy)\\ &\quad\to m^\alpha\int_0^\infty\int_J (v\cdot\mathbb{1}\{t<0\}+w\cdot\mathbb{1}\{t>0\})\frac{dt}{|t|^{1+\alpha}}\\ &\quad=\int_0^\infty\int_J (v\cdot\mathbb{1}\{t<0\}+w\cdot\mathbb{1}\{t>0\})\frac{dt}{|t/y|^{1+\alpha}}\,\mathscr{L}(Y)(dy) \qquad (n\to\infty),\end{aligned}$$

and thus

$$\begin{aligned}(8)\qquad & n\int_0^\infty\int_J \frac{(t/y)^2}{1+(t/y)^2}\,\mathscr{L}(a_n yX)(dt)\,\mathscr{L}(Y)(dy)\\ &\to\int_0^\infty\int_J \frac{(t/y)^2}{1+(t/y)^2}(v\cdot\mathbb{1}\{t<0\}+w\cdot\mathbb{1}\{t>0\})\frac{dt}{|t/y|^{1+\alpha}}\,\mathscr{L}(Y)(dy) \qquad (n\to\infty),\end{aligned}$$

where J is an interval of the form $]-\infty, x]$ $(x<0)$ or $[x, \infty[$ $(x>0)$. Let η and η_n denote the finite measures on $]0, \infty[$ given by

$$\eta(B) := w \int_B \frac{x^2}{1+x^2} \frac{dx}{x^{1/\alpha}}$$

and

$$\eta_n(B) := n \int_B \frac{x^2}{1+x^2} \mathscr{L}(a_n X)(dx)$$

for Borel subsets $B \subset]0, \infty[$. Define λ and λ_n to be the finite measures on $\boldsymbol{R}$ given by

$$\int_{-\infty}^{\infty} f(x)\lambda(dx) := \int_0^{\infty} f(\log y)\eta(dy)$$

and

$$\int_{-\infty}^{\infty} f(x)\lambda_n(dx) := \int_0^{\infty} f(\log y)\eta_n(dy)$$

for bounded real-valued continuous functions on $\boldsymbol{R}$. Let H be the distribution function of $\log Y$. By the "basic estimate" in Feinsilver [3] and (8) it follows (as in the proof of [3], Proposition 8) that

(9) $$\lambda_n * H \xrightarrow{w} \lambda * H \quad (n \to \infty).$$

Let $\zeta(u)$, $\zeta_n(u)$, and $\xi(u)$ be the characteristic function (in the ordinary sense) of λ, λ_n, and H, respectively. Then (9) may be rewritten as

(10) $$\zeta_n(u) \cdot \xi(u) \to \zeta(u) \cdot \xi(u) \ (n \to \infty) \quad (u \in \boldsymbol{R}).$$

By Lemma 4, ξ is analytic in a neighborhood of the real axis, and hence it has only isolated zeros there. So we may divide (10) by $\xi(u)$ for all real u with the exception of isolated points, and of course in a neighborhood of $u_0 = 0$. Thus it follows from the Lévy continuity theorem and the continuity of ζ that

$$\zeta_n(u) \to \zeta(u) \ (n \to \infty) \quad (u \in \boldsymbol{R}),$$

and hence

$$\lambda_n \xrightarrow{w} \lambda \quad (n \to \infty),$$

and thus

$$\eta_n \xrightarrow{w} \eta \quad (n \to \infty).$$

An analogous argument holds also for the negative real axis; hence by Proposition 1 it follows that X lies in the domain of attraction of Z with norming sequence $\{a_n, b_n\}_{n \geq 1}$ for certain $b_n \in \boldsymbol{R}$. Now the same type of argument as in the proof of the "if" direction shows that one may replace b_n by 0 by the strictness of the domain of attraction. ∎

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