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RANDOM STAIN*

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Abstract. A stochastic model of a stain of pollution is proposed. The asymptotic shape of the edges of a stain is examined. The description is kept at the elementary level of probability theory.

Introduction. We would like to start with not a very serious, but funny example, known to older mathematicians from their own childhood. This example, however, shows the essence of the phenomenon we would like to describe.

Let us imagine some ink flowing slowly onto a sheet of blotting paper. At first, the blot is only one point but, as the ink flows faster, it shortly grows bigger and bigger. We shall assume that the stain of ink which appears in this way is spreading out radially from the centre of its initial source. At each moment the shape of the stain is random. On the edge of the stain we can most often observe little 'springs' of its further expansion, appearing irregularly, with some or without any preference of specific direction.

The stain just described shows some similarities to less amusing but real events like an oil leak onto the surface of the ocean, caused by a broken, not moving tanker or the radioactive pollution caused by a damaged nuclear power station. It may happen that the stain grows unboundedly and rapidly resembling rather a kind of explosion. A disaster area caused by a dramatically growing population of animals is a good example of such a stain. Let us imagine that after having eaten everything around themselves the animals gather in groups in some points of the edge of devastated region. These points of the gathering of animals are just 'springs of further expansion' of the stain.

The same complexion may be worn by a stain of polution caused by a growing number of different negative effects of civilization.

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The aim of this work is an attempt to describe the above phenomenon mathematically. We keep the description at the level of standard stochastic notions. We treat the stain or, more specifically, its edges as a realization of a two-parameter stochastic process $\xi(t, \alpha)$ parameterized with time t and direction α of stain expansion.

We make a natural assumption that with probability one the trajectory $\xi(t, \alpha)$ is a nondecreasing continuous function of time t for every α. It is also natural to demand the continuity of $\xi(t, \alpha)$ with respect to α.

Obviously, we assume that, in some scale of time t, the increments of $\xi(t, \alpha)$ are chaotic. Those chaotic increments correspond to 'springs of expansion' mentioned above. Roughly speaking, in our rather elementary model we are not trying to describe the diffusion which is as a rule behind the evolution of the stain. We rather want to describe the final effects of this diffusion: growing the stain realized by randomly appearing (in a poissonian way) 'springs of expansion' with their shapes depending on the intensity of increase.

Of course, there are many ways of choosing the model for random factors influencing the evolution of the stain in such a way that the discussed postulates are satisfied.

We decide that the velocity

$$V(t, \alpha) = \frac{d}{dt}\xi(t, \alpha)$$

exists and, in fact, all our assumptions on ξ will concern V. We define V as a stochastic integral of the form

$$V(t, \alpha) = \int_0^t h(s, \alpha)\, d\pi(s), \tag{1}$$

where $\pi = (\pi(s), s \geqslant 0)$ is a positive increasing point process with independent integer increments, counting the springs of expansion. A nonnegative stochastic process $h(s, \alpha)$ should have a rather elementary structure describing randomness of expansion in the direction α.

We want to embrace both cases: of unbounded expansion (explosion) and stabilization (petrification) of the stain ξ. That is why it seems to be necessary to assume additionally that in (1) the integrand h also depends on t. Thus we set

$$V(t, \alpha) = \int_0^t h_t(s, \alpha)\, d\pi(s). \tag{1'}$$

Without more specific assumptions it seems to be rather hopeless to obtain some results concerning the asymptotic behaviour of the stain for large t we are interested in. That is why we confine ourselves to the process h of (elementary) form

$$h_t(s, \alpha) = \psi(s) f(\alpha - \alpha_{\pi(s)}, t-s), \tag{2}$$

where f is a continuous nonnegative function describing the shape of the 'springs of expansion' and a nonnegative monotone function ψ describing the intensity of stain expansion. The random directions α_n $(n = 1, 2, \ldots)$ are independent identically distributed, independent of the process π.

For the sake of simplicity, we shall assume that the process π is a homogeneous Poisson process with parameter $\lambda = 1$.

In spite of all its randomness, the stain often has a tendency to reach finally some specific shape after a long observation.

In the described model we are able to show a number of asymptotic properties of a function $\alpha \to \xi(t, \alpha)$ for large t. This results in specific limit theorems. In particular, we prove a kind of laws of large numbers about rounding out of a stain or its petrification which in turn are the main goal of the paper.

1. Notation and definitions. Let us begin with some notation and definitions. Some assumptions made here will be obligatory in all that follows and as a rule they will be omitted in the formulation of theorems and lemmas.

Let G be a metric abelian compact group (of directions). Let f be a continuous nonnegative function defined on $G \times [0, \infty)$. We put additionally $f(\alpha, t) = 0$ for $\alpha \in G$ and $t \in (-\infty, 0)$.

Let ψ be a continuous positive monotone function defined on $[0, \infty)$.

Let $\pi = (\pi(t)\colon t \geqslant 0)$ be a Poisson process with parameter $\lambda = 1$. Denote by τ_i the time of occurrence of the i-th event for the process π.

Moreover, let $(\alpha_i\colon i = 1, 2, \ldots)$ be a sequence of G-valued independent identically distributed random variables independent of the process $(\pi(t)\colon t \geqslant 0)$.

For $\alpha \in G$ let us put

$$V(t, \alpha) = \int_0^t \psi(s) f(\alpha - \alpha_{\pi(s)}, t-s)\, d\pi(s), \qquad t \geqslant 0, \tag{1.1}$$

and

$$\xi(t, \alpha) = \int_0^t V(\tau, \alpha)\, d\tau, \qquad t \geqslant 0. \tag{1.2}$$

A two-parameter stochastic process $\xi = (\xi(t, \alpha)\colon t \geqslant 0, \alpha \in G)$ is called a *stain* (with continuous time). $V(t, \alpha)$ is called a *velocity of expansion of the stain* ξ at the moment t in the direction α. The function f is called a *spring of further expansion* of the stain. Finally, ψ is called an *intensity function* of ξ.

Actually, the stain (1.2) is of the form

$$(1.3)\qquad \xi(t,\alpha)=\int_0^t \sum_{\{i:\tau_i<\tau\}} \psi(\tau_i)\, f(\alpha-\alpha_i,\ \tau-\tau_i)\, d\tau.$$

In the sequel we distinguish two cases:

1° the function f has a compact support, say $G\times[0, b]$;

2° the function f does not depend on $t\in[0, \infty)$.

The case 1° corresponds to the situation when the random factor of the velocity

$$(1.4)\qquad \psi(\tau_{i_0})\, f(\alpha-\alpha_{\pi(\tau_{i_0})},\ t-\tau_{i_0})$$

in (1.1) is temporary, that is, it disappears for $t>\tau_{i_0}+b$.

The case 2° corresponds to the situation when all the factors (1.4) act constantly.

One of the natural questions concerning the asymptotic behaviour of a stain is a comparison of the behaviour with the circle. Main results of our work are just devoted to the description of a situation which will be called *rounding out* of the stain.

Precisely, putting

$$R(t)=\sup_{\alpha\in G}\xi(t,\alpha),\qquad r(t)=\inf_{\alpha\in G}\xi(t,\alpha),$$

we say that the stain ξ *rounds out* when

$$(1.5)\qquad \frac{R(t)}{r(t)}\to 1 \quad \text{as } t\to\infty.$$

If the limit (1.5) exists with probability one (in probability, respectively), we shall say that the stain ξ *rounds out strongly* (*weakly*, respectively).

We say that a stain ξ *petrifies* if there exists a random function $(\eta(\alpha))_{\alpha\in G}$ such that

$$(1.6)\qquad \lim_{t\to\infty}\sup_{\alpha\in G}|\xi(t,\alpha)-\eta(\alpha)|=0$$

with probability one.

2. Main results. Let $\xi=\xi(t,\alpha)$ be a stain of the form (1.3). Obviously, we can write ξ as

$$(2.1)\qquad \xi(t,\alpha)=\sum_{i=1}^{\infty}\psi(\tau_i)\int_0^t f(\alpha-\alpha_i,\ \tau-\tau_i)\, d\tau.$$

All the results formulated in the sequel concern the asymptotic behaviour of a stain ξ of the form (2.1) as $t\to\infty$. They depend heavily on the rate of increase (decrease) of the intensity function $\psi(t)$ as $t\to\infty$. That is why we introduce two classes of ψ's.

DEFINITION 1. We say that $\psi \in \mathscr{E}$ if ψ is nondecreasing and

$$(2.2) \qquad \sup_{t\in(0,\infty)} \frac{t\psi(t)}{\int_0^t \psi(s)\,ds} < \infty .$$

DEFINITION 2. We say that $\psi \in \mathscr{R}$ if ψ is nondecreasing and

$$(2.3) \qquad \sup_{t\in(0,\infty)} \frac{t^2\psi(t)}{\int_0^t (t-s)\psi(s)\,ds} < \infty .$$

For example, $\psi(t) = t^r$ with $r > 0$ belongs to both classes $\mathscr{E}$ and $\mathscr{R}$ whereas $\psi(t) = e^t$ belongs neither to $\mathscr{E}$ nor to $\mathscr{R}$.

More generally, one can easily check that all the increasing functions varying regularly with exponent $r > 0$ belong to $\mathscr{E} \cap \mathscr{R}$.

THEOREM 1. *Let $\xi = \xi(t, \alpha)$ be of the form* (2.1) *with a function f having a compact support. Assume additionally that G-valued random variables α_i $(i = 1, 2, \ldots)$ are uniformly distributed (with respect to the Haar measure).*

If ψ is a nondecreasing function belonging to the class $\mathscr{E}$ or a nonincreasing one with $\int_0^\infty \psi(s)\,ds = \infty$, then the stain $\xi(t, \alpha)$ rounds out strongly.

If ψ is a nonincreasing function with $\int_0^\infty \psi(s)\,ds < \infty$, then the stain ξ petrifies.

THEOREM 2. *Let $\xi = \xi(t, \alpha)$ be a stain of the form* (1.3) *with a function f not depending on $\tau \in [0, \infty)$, that is*

$$(2.4) \qquad \xi = \xi(t, \alpha) = \int_0^t \sum_{\{i:\tau_i<\tau\}} \psi(\tau_i) f(\alpha-\alpha_i)\,d\tau, \qquad \alpha \in G,\ t \in [0, \infty).$$

Assume additionally that G-valued random variables α_i $(i = 1, 2, \ldots)$ are uniformly distributed on G.

If ψ is a nondecreasing function belonging to the class $\mathscr{R}$, then the stain $\xi(t, \alpha)$ rounds out strongly.

A stain may suffer from the lack of rounding out even weakly. An example of such a behaviour of the stain appears when the intensity ψ increases exponentially. Moreover, one can construct an example of the intensity function ψ leading to a stain rounding out weakly but not rounding out strongly. The results just mentioned will be discussed in Section 6.

3. Auxiliary results. The proof of Theorem 2 is based on the strong rounding out theorem of the stain $\bar{\xi}$ with discrete time which is defined as follows.

Let us put for $n = 0, 1, \ldots;\ \alpha \in G$

$$(3.1) \qquad \bar{V}(n, \alpha) = \sum_{k=0}^{n} c_k \sum_{\{i:k\leqslant\tau_i<k+1\}} f(\alpha-\alpha_i),$$

where $(c_n)_{n=0,1,\ldots}$ is a sequence of positive numbers (describing the intensity of expansion of the stain).

We set

$$\bar{\xi}(n,\alpha) = \sum_{k=0}^{n-1} \bar{V}(k,\alpha), \qquad n = 1, 2, \ldots \tag{3.2}$$

Actually, the stain $\bar{\xi} = (\bar{\xi}(n,\alpha),\ n = 0, 1, 2, \ldots;\ \alpha \in G)$ is of the form

$$\bar{\xi}(n,\alpha) = \sum_{k=0}^{n-1} (n-k)\, c_k X_k(\alpha), \tag{3.3}$$

where

$$X_k(\alpha) = \sum_{\{i: k \leqslant \tau_i < k+1\}} f(\alpha - \alpha_i), \qquad k = 0, 1, \ldots \tag{3.4}$$

Let us formulate two conditions for a sequence $(c_n)_{n=0,1,2,\ldots}$ (analogous to (2.2) and (2.3)).

We say that a sequence $(c_n)_{n=0,1,2,\ldots}$ of positive numbers satisfies the *condition* (E) if it is nondecreasing and

$$\frac{n c_{n-1}}{\sum_{k=0}^{n-1} c_k} = O(1) \qquad \text{as } n \to \infty \tag{3.5}$$

or satisfies the *condition* (R) if it is nondecreasing and

$$\frac{n^2 c_{n-1}}{\sum_{k=0}^{n-1} (n-k)\, c_k} = O(1) \qquad \text{as } n \to \infty. \tag{3.6}$$

Clearly, for $c_n = \psi(n)$, $n = 0, 1, \ldots$, the condition (2.2) implies (3.5) and (2.3) does (3.6) as well.

In the sequel we shall need the following three lemmas.

LEMMA 1. *Let $(c_k)_{k=0,1,\ldots}$ be a nondecreasing sequence of positive numbers satisfying the condition* (E) (see (3.5)). *Then, for every sequence $(a_n)_{n=0,1,2,\ldots}$ of real numbers, the condition $n^{-1}\sum_{k=0}^{n-1} a_k \to 0$ as $n \to \infty$ implies*

$$\frac{\sum_{k=0}^{n-1} c_k a_k}{\sum_{k=0}^{n-1} c_k} \to 0 \qquad \textit{as } n \to \infty.$$

LEMMA 2. *Let $(c_k)_{k=0,1,\ldots}$ be a nondecreasing sequence of positive numbers satisfying the condition* (R) (see (3.6)). *Then, for every sequence $(a_n)_{n=0,1,2,\ldots}$ of real numbers, the condition $n^{-1}\sum_{k=0}^{n-1} a_k \to 0$ as $n \to \infty$ implies*

$$\frac{\sum_{k=0}^{n-1} (n-k)\, c_k a_k}{\sum_{k=0}^{n-1} (n-k)\, c_k} \to 0 \qquad \textit{as } n \to \infty.$$

LEMMA 3. *Let* $(c_k)_{k=0,1,\ldots}$ *be a nonincreasing sequence of positive numbers. Then, for every sequence* $(a_n)_{n=0,1,2,\ldots}$, *the condition* $n^{-1}\sum_{k=0}^{n-1} a_k \to 0$ *as* $n\to\infty$ *implies*

$$\frac{\sum_{k=0}^{n-1} c_k a_k}{\sum_{k=0}^{n-1} c_k} \to 0 \quad \text{as } n\to\infty$$

and

$$\frac{\sum_{k=0}^{n-1} c_{k+1} a_k}{\sum_{k=0}^{n-1} (n-k) c_k} \to 0 \quad \text{as } n\to\infty .$$

Lemmas 1, 2 and 3 can be obtained by applying the Toeplitz theorem on summability of sequences by matrix methods (cf. [3], p. 238).

THEOREM 3 (Strong rounding out of the stain with discrete time). *Let* $\bar{\xi} = (\bar{\xi}(n,\alpha),\, n = 0, 1, \ldots;\, \alpha\in G)$ *be a stain with discrete time defined by* (3.3) *and* (3.4) *with* $(c_n)_{n=0,1,\ldots}$ *satisfying the condition* (R) (see (3.5)). *Assume that* (α_i), $i = 1, 2, \ldots$, *are independent identically distributed random variables uniformly distributed on* G. *Then*

$$\frac{\sup_{\alpha\in G}\bar{\xi}(n,\alpha)}{\inf_{\alpha\in G}\bar{\xi}(n,\alpha)} \to 1 \quad \text{as } n\to\infty \text{ with probability one.} \tag{3.7}$$

Proof. We have, according to (3.3),

$$\bar{\xi}(n,\alpha) = \sum_{k=0}^{n-1} (n-k) c_k X_k(\alpha),$$

where for any fixed α the random variables $X_k(\alpha)$ given by (3.4) are independent identically distributed random variables with common distribution not depending on α.

Put $\tilde{X}_k(\alpha) = X_k(\alpha) - EX_k$. In the sequel, if possible, α will be omitted.

First, we show that for each $\alpha\in G$

$$\eta(n,\alpha) = \frac{\bar{\xi}(n,\alpha) - E\bar{\xi}(n,\alpha)}{E\bar{\xi}(n,\alpha)} \to 0 \quad \text{as } n\to\infty . \tag{3.8}$$

Indeed, we have

$$E\bar{\xi}(n,\alpha) = \sum_{k=0}^{n-1} (n-k) c_k EX_0 .$$

Thus

$$\eta(n,\alpha) = \frac{\sum_{k=0}^{n-1} (n-k) c_k \tilde{X}_k(\alpha)}{\sum_{k=0}^{n-1} (n-k) c_k EX_0} .$$

By the classical SLLN, we get

$$\frac{1}{n}\sum_{k=0}^{n-1} \tilde{X}_k(\alpha) \to 0 \quad \text{as } n \to \infty.$$

Thus, applying Lemma 2 with $a_k = \tilde{X}_k(\alpha)$, we obtain (3.8).

Let us put

$$\bar{R}(n) = \sup_{\alpha\in G} \bar{\xi}(n, \alpha), \quad \bar{r}(n) = \inf_{\alpha\in G} \bar{\xi}(n, \alpha).$$

Now, we prove that

(3.9) $$\frac{\bar{r}(n)}{E\bar{\xi}(n)} \to 1 \quad \text{and} \quad \frac{\bar{R}(n)}{E\bar{\xi}(n)} \to 1 \quad \text{as } n \to \infty \text{ with probability one.}$$

To do this we consider a process $\Delta(n)$, $n = 1, 2, \ldots$, being the stain $\bar{\xi}(n, \alpha)$ when we take $f \equiv 1$ in (3.4). By (3.8) we get immediately

$$\frac{\Delta(n)}{E\Delta(n)} \to 1 \text{ a.s.} \quad \text{as } n \to \infty.$$

Thus,

(3.10) $$\frac{\Delta(n)}{E\bar{\xi}(n)} \to \frac{1}{EX_0} \text{ a.s.} \quad \text{as } n \to \infty.$$

Take $\varepsilon > 0$. By the uniform continuity of f on G there exists a finite ε-net $\Sigma = \{\bar{\alpha}, \ldots\} \subset G$ such that for every $\alpha \in G$ there is $\bar{\alpha} \in \Sigma$ such that $|f(\alpha) - f(\bar{\alpha})| < \varepsilon$. Fix $\alpha \in G$ and take $\bar{\alpha} \in \Sigma$. Then

$$\bar{\xi}(n, \alpha) = \bar{\xi}(n, \bar{\alpha}) + \bar{\xi}(n, \alpha) - \bar{\xi}(n, \bar{\alpha}) \geqslant \min_{\bar{\alpha}\in\Sigma} \bar{\xi}(n, \bar{\alpha}) - \varepsilon\Delta(n).$$

Hence

$$r(n) = \inf_{\alpha\in G} \bar{\xi}(n, \alpha) \geqslant \min_{\bar{\alpha}\in\Sigma} \bar{\xi}(n, \bar{\alpha}) - \varepsilon\Delta(n)$$

and, consequently,

$$\frac{\min_{\bar{\alpha}\in\Sigma} \bar{\xi}(n, \bar{\alpha}) - \varepsilon\Delta(n)}{E\xi(n)} < \frac{r(n)}{E\xi(n)} < \frac{\min_{\bar{\alpha}\in\Sigma} \bar{\xi}(n, \bar{\alpha})}{E\xi(n)}.$$

The right-hand side of the above inequality, by (3.8), is convergent to 1 as $n \to \infty$ with probability one. The left-hand side tends to $1 - \varepsilon(EX_0)^{-1}$ by (3.10). Thus, we obtain the first convergence in (3.9). Similarly, we can prove the second one. (3.9) gives immediately (3.7). ■

4. Proof of Theorem 1. Assume that $\psi \in \mathscr{E}$ is nondecreasing. By (2.1) we have

$$\xi(t, \alpha) = \sum_{\substack{i=1 \\ \tau_i < t}} \psi(\tau_i) \int_0^t f(\alpha - \alpha_i, \tau - \tau_i)\, d\tau.$$

Let $\operatorname{supp}(f) \subset G \times [0, b]$. Obviously, we can assume that b is a positive integer. Putting

$$X(i, \alpha) = \int_0^\infty f(\alpha - \alpha_i, \tau)\, d\tau,$$

we obtain the estimation

$$\sum_{\tau_i < t-b} \psi(\tau_i) X(i, \alpha) \leqslant \xi(t, \alpha) \leqslant \sum_{\tau_i \leqslant t} \psi(\tau_i) X(i, \alpha). \tag{4.1}$$

Let us set

$$Y(k, \alpha) = \sum_{\{i: k \leqslant \tau_i < k+1\}} X(i, \alpha), \qquad k = 0, 1, \ldots \tag{4.2}$$

By the assumptions for a fixed $\alpha \in G$, $(Y(k, \alpha)\colon k = 0, 1, \ldots)$ is a sequence of independent identically distributed random variables. Moreover, the mean value

$$\boldsymbol{E} Y(0, \alpha) = \boldsymbol{E} X(1, \alpha) \boldsymbol{E}^* \{i\colon 0 \leqslant \tau_i < 1\} = \boldsymbol{E} \int_0^b f(\alpha - \alpha_i, \tau)\, d\tau$$

does not depend on α since α_1 has the uniform distribution on G. Let us put $\beta = \boldsymbol{E} Y(0, \alpha)$ and $\tilde{Y}(k, \alpha) = Y(k, \alpha) - \beta$.

First, we shall prove that for every $\alpha \in G$

$$\frac{\xi(t, \alpha)}{\sum_{k=0}^{[t]+2} \psi(k)} \to \beta \quad \text{as } t \to \infty \text{ with probability one.} \tag{4.3}$$

In fact, by (4.1) and (4.2), we obtain

$$\sum_{k=0}^{[t-b]-1} \psi(k)\, Y(k, \alpha) \leqslant \xi(t, \alpha) \leqslant \sum_{k=0}^{[t]+1} \psi(k+1)\, Y(k, \alpha). \tag{4.4}$$

Thus

$$\frac{\sum_{k=0}^{[t-b]-1} \psi(k)\, \tilde{Y}(k, \alpha) - \beta \sum_{k=[t]-b}^{[t]+2} \psi(k)}{\beta \sum_{k=0}^{[t]+2} \psi(k)} \leqslant \frac{\xi(t, \alpha)}{\beta \sum_{k=0}^{[t]+2} \psi(k)} - 1 \tag{4.5}$$

$$\leqslant \frac{\sum_{k=0}^{[t]+1} \psi(k+1)\, \tilde{Y}(k, \alpha) - \beta \psi(0)}{\beta \sum_{k=0}^{[t]+2} \psi(k)}.$$

On the other hand, by the SLLN, we have

$$\frac{1}{n} \sum_{k=0}^{n-1} \tilde{Y}(k, \alpha) \to 0 \text{ a.s.} \quad \text{as } n \to \infty. \tag{4.6}$$

Now, in Lemma 1 we put $a_0 = 0$, $a_k = \tilde{Y}(k-1, \alpha)$, $k = 1, 2, \ldots$, and we obtain by (4.6), where $1/n$ is replaced by $1/(n+1)$,

$$\frac{\sum_{k=0}^{n-1} \psi(k+1)\, \tilde{Y}(k, \alpha)}{\sum_{k=0}^{n} \psi(k)} \to 0 \text{ a.s.} \quad \text{as } n \to \infty.$$

The assumption $\psi \in \mathscr{E}$ implies immediately $\sum_{k=0}^{\infty} \psi(k) = \infty$. Thus we have proved that the right-hand side of (4.5) converges to 0 as $t \to \infty$ with probability one.

Similarly, by (4.6) and Lemma 1, we get

$$\frac{\sum_{k=0}^{[t]-b-1} \psi(k)\, \tilde{Y}(k, \alpha)}{\sum_{k=0}^{[t]-b-1} \psi(k)} \to 0 \quad \text{as } t \to \infty,$$

so

$$\frac{\sum_{k=0}^{[t]-b-1} \psi(k)\, \tilde{Y}(k, \alpha)}{\sum_{k=0}^{[t]+2} \psi(k)} \to 0 \text{ a.s.} \quad \text{as } t \to \infty.$$

Moreover, we have, by $\psi \in \mathscr{E}$,

$$\frac{\sum_{k=[t]-b}^{[t]+2} \psi(k)}{\sum_{k=0}^{[t]+2} \psi(k)} \leqslant \frac{(b+3)\,\psi([t]+3)}{\sum_{k=0}^{[t]+2} \psi(k)} = O(1)\frac{1}{[t]+3} \to 0 \quad \text{as } t \to \infty.$$

The left-hand side of (4.5) converges to 0 and (4.3) is thus proved.

Now, we prove that

$$\frac{\sup_{\alpha \in G} \xi(t, \alpha)}{\sum_{k=0}^{[t]+2} \psi(k)} \to \beta \quad \text{as } t \to \infty. \tag{4.7}$$

To do this we consider

$$\Delta(t) = \sum_{\tau_i < t} \psi(\tau_i).$$

Remark that

$$\frac{\Delta(t)}{\sum_{k=0}^{[t]+2} \psi_k(t)} \to 1 \text{ a.s.} \quad \text{as } t \to \infty. \tag{4.8}$$

Indeed, we have

$$\sum_{\tau_i < [t]} \psi(\tau_i) \leqslant \Delta(t) \leqslant \sum_{\tau_i < [t]+1} \psi(\tau_i).$$

Setting

$$Y_k = {}^{\#}\{i\colon\ k \leqslant \tau_i < k+1\}, \quad k = 0, 1, \ldots,$$

we obtain

$$\sum_{k=0}^{[t]-1} \psi(k)\, Y_k \leqslant \Delta(t) \leqslant \sum_{k=0}^{[t]} \psi(k+1)\, Y_k.$$

Obviously, $(Y_k\colon k = 0, 1, \ldots)$ is a sequence of independent identically distributed random variables with $EY_k = 1$. Moreover, we have

$$\sum_{k=0}^{[t]-1} \psi(k)\, Y_k \leqslant \Delta(t) \leqslant \sum_{k=0}^{[t]} \psi(k+1)\, Y_k \leqslant \sum_{k=0}^{[t]+1} \psi(k+1)\, Y_k.$$

Thus, the argument similar to that one used in the proof of (4.3) gives us (4.8).

Now, take $\varepsilon > 0$. By the uniform continuity of f on $G \times [0, b]$ there exists a finite net $\Sigma = \{\tilde{\alpha}, \ldots\} \subset G$ such that for every $\alpha \in G$ there is $\tilde{\alpha} \in \Sigma$ such that

$$\int_0^b |f(\alpha, \tau) - f(\tilde{\alpha}, \tau)|\, d\tau < \varepsilon.$$

Then for $\alpha \in G$ one can find a $\tilde{\alpha} \in \Sigma$ such that

$$|\xi(t, \alpha) - \xi(t, \tilde{\alpha})| \leqslant \sum_{\tau_i < t} \psi(\tau_i) \int_0^t |f(\alpha - \alpha_i, \tau - \tau_i) - f(\tilde{\alpha} - \alpha_i, \tau - \tau_i)|\, d\tau \leqslant \varepsilon \Delta(t).$$

Consequently,

$$\xi(t, \alpha) \leqslant \xi(t, \tilde{\alpha}) + \varepsilon \Delta(t).$$

Thus

$$\max_{\tilde{\alpha} \in \Sigma} \frac{\xi(t, \tilde{\alpha})}{\beta \sum_{k=0}^{[t]+2} \psi(k)} \leqslant \frac{\max_{\alpha \in \Sigma} \xi(t, \alpha)}{\beta \sum_{k=0}^{[t]+2} \psi(k)} \leqslant \max_{\tilde{\alpha} \in \Sigma} \frac{\xi(t, \tilde{\alpha})}{\beta \sum_{k=0}^{[t]+2} \psi(k)} + \varepsilon \frac{\Delta(t)}{\beta \sum_{k=0}^{[t]+2} \psi(k)}.$$

Now, by (4.3) and (4.8), taking $\varepsilon \to 0$, we get (4.7).

Similarly,

$$\frac{\inf_{\alpha \in G} \xi(t, \alpha)}{\sum_{k=0}^{[t]+2} \psi(k)} \to \beta \quad \text{as } t \to \infty. \tag{4.9}$$

Finally, by (4.7) and (4.9) we obtain

$$\frac{R(t)}{r(t)} \to 1 \quad \text{as } t \to \infty \text{ with probability one.}$$

Now assume that ψ is nonincreasing with $\int_0^\infty \psi(s)\, ds = \infty$. In this case the proof is similar to the previous part. It is enough to apply Lemma 3 instead of Lemma 1.

We pass to the case where ψ is nonincreasing with $\int_0^\infty \psi(s)\, ds < \infty$. Observe that the function $t \to \xi(t, \alpha)$ is nondecreasing for every $\alpha \in G$, so the limit

$\lim_{t\to\infty} \xi(t, \alpha) = \eta(\alpha)$ always exists with $\eta(\alpha)$ finite with probability one. Indeed,

$$E\xi(t, \alpha) \leqslant \text{Const} \sum_{\tau_i < t} E\psi(\tau_i) \leqslant \text{Const} \sum_{n=0}^{\infty} \psi(n) < \infty,$$

since the process π is homogeneous.

We shall show that $\sup_{\alpha\in G} |\xi(t, \alpha) - \eta(\alpha)| \to 0$ as $t \to \infty$ with probability one.

Obviously, the limit $\gamma = \lim_{t\to\infty} \sum_{\{i:\tau_i<t\}} \psi(\tau_i)$ exists with probability one. A priori it may be infinite. But

$$E\gamma = \lim_{t\to\infty} \sum_{\{i:\tau_i<t\}} \psi(\tau_i) \leqslant \sum_{n=1}^{\infty} \psi(n) < \infty$$

(since π is homogeneous with $\lambda = 1$).

For $\varepsilon > 0$ we can find a K satisfying $\Pr(\gamma \geqslant K) < \varepsilon$.

By the compactness of G and the continuity of f on $G \times [0, b]$ we find a finite system $B \subset G$ such that for $\alpha \in G$ there exists a $\beta = \beta(\alpha) \in B$ satisfying

$$\sup_{i,\omega} \left| \int_0^b f(\alpha - \alpha_i(\omega), t)\, dt - \int_0^b f(\beta(\alpha) - \alpha_i(\omega), t)\, dt \right| < \frac{\varepsilon}{4K}.$$

Thus under the condition $\{\omega\colon \gamma(\omega) < K\}$ for an arbitrarily fixed $\alpha \in G$ we have

$$|\xi(t, \alpha) - \eta(\alpha)| \leqslant |\xi(t, \alpha) - \xi(t, \beta)| + |\xi(t, \beta) - \eta(\beta)| + |\eta(\beta) - \eta(\alpha)|$$

$$\leqslant 2\frac{\varepsilon}{4K}\gamma + \frac{\varepsilon}{2} = \varepsilon$$

for t large enough, since

$$\max_{\beta\in B} |\xi(t, \beta) - \eta(\beta)| \to 0 \quad \text{as } t \to \infty.$$

Thus $\sup_{\alpha\in G} |\xi(t, \alpha) - \eta(\alpha)| \to 0$ almost uniformly with respect to probability measure. By the Egorov theorem the stain petrifies with probability one. ∎

5. Proof of Theorem 2. Let $\xi = (\xi(t, \alpha), t \geqslant 0, \alpha \in G)$ be a stain of the form (2.4). We shall show that, for a fixed $\alpha \in G$,

$$\frac{\xi(t, \alpha)}{E\xi(t, \alpha)} \to 1 \quad \text{as } t \to \infty \text{ with probability one.} \tag{5.1}$$

Then the rest of the proof is essentially the same as for the stain with discrete time (Theorem 3) or for the stain with the compact support of the function f (Theorem 1) and will be omitted.

The proof of (5.1) will be reduced to the rounding out theorem for a stain with discrete time.

We have

$$\xi(t,\alpha)=\int_0^t V(\tau,\alpha)\,d\tau, \qquad \text{where } V(\tau,\alpha)=\sum_{\tau_i<\tau}\psi(\tau_i)\,f(\alpha-\alpha_i).$$

Putting $n=[t]$ we can write

$$\begin{aligned}\xi(t,\alpha) &= \int_{[0,n)} V\,d\tau+\int_{[n,t)} V\,d\tau=\int_{[0,n)}\sum_{\tau_i<\tau}\psi(\tau_i)\,f(\alpha-\alpha_i)\,d\tau+\int_{[0,n)} V\,d\tau\\ &= \int_{[0,n)}\Big(\sum_{\tau_i<[\tau]+1}\psi(\tau_i)\,f(\alpha-\alpha_i)-\sum_{\tau\leqslant\tau_i<[\tau]+1}\psi(\tau_i)\,(\alpha-\alpha_i)\Big)\,d\tau+\int_{[n,t)} V\,d\tau\\ &= \sum_{k=0}^{n-1}\int_{[k,k+1)}\Big(\sum_{\nu=0}^{[\tau]}\sum_{\nu\leqslant\tau_i<\nu+1}\psi(\tau_i)\,f(\alpha-\alpha_i)\Big)\,d\tau\\ &\quad -\int_{[0,n)}\Big(\sum_{\tau\leqslant\tau_i<[\tau]+1}\psi(\tau_i)\,f(\alpha-\alpha_i)\Big)\,d\tau+\int_{[n,t)} V\,d\tau\\ &= \sum_{k=0}^{n-1}\sum_{\nu=0}^{k}\sum_{\nu\leqslant\tau_i<\nu+1}\psi(\tau_i)\,f(\alpha-\alpha_i)\\ &\quad -\int_{[0,n)}\sum_{\tau\leqslant\tau_i<[\tau]+1}\psi(\tau_i)\,f(\alpha-\alpha_i)\,d\tau+\int_{[n,t)} V\,d\tau.\end{aligned}$$

Now we set

$$\bar{\xi}(n,\alpha)=\sum_{k=0}^{n-1}\sum_{\nu=0}^{k}\psi(\nu)\,X_\nu, \qquad n=1,2,\ldots, \tag{5.2}$$

where

$$X_\nu=\sum_{\nu\leqslant\tau_i<\nu+1} f(\alpha-\alpha_i), \qquad \nu=0,1,\ldots \tag{5.3}$$

X_ν are independent identically distributed random variables. In the sequel we assume that $EX_0=1$.

Obviously, $\bar{\xi}(n,\alpha)$ is a stain with discrete time. Moreover, the sequence $c_k=\psi(k)$, $k=0,1,\ldots$, satisfies the condition (R). Then we can write

$$\begin{aligned}\xi(t,\alpha) &= \bar{\xi}(n,\alpha)+\sum_{k=0}^{n-1}\sum_{\nu=0}^{k}\sum_{\nu\leqslant\tau_i<\nu+1}\big(\psi(\tau_i)-\psi(\nu)\big)\,f(\alpha-\alpha_i)\\ &\quad -\int_{[0,n)}\sum_{\tau\leqslant\tau_i<[\tau]+1}\psi(\tau_i)\,f(\alpha-\alpha_i)\,d\tau+\int_{[n,t)} V\,d\tau\\ &\equiv \bar{\xi}(n,\alpha)+\zeta^{(1)}_{(n,\alpha)}-\zeta^{(2)}_{(n,\alpha)}+\zeta^{(3)}_{(t,\alpha)}, \qquad \text{where } n=[t].\end{aligned} \tag{5.4}$$

Let us put $M_n = E\bar{\xi}(n, \alpha)$, $n = 1, 2, \ldots$ To prove (5.1) it is enough to show that

$$\frac{\xi(t,\alpha)}{M_{[t]}} \to 1 \quad \text{as } t\to\infty \text{ with probability one} \tag{5.5}$$

and

$$\frac{E\xi(t,\alpha)}{M_{[t]}} \to 1 \quad \text{as } t\to\infty. \tag{5.6}$$

By (3.8) in the proof of Theorem 3 we obtain

$$\frac{\bar{\xi}(n,\alpha)}{M_n} \to 1 \quad \text{as } n\to\infty \text{ with probability one.} \tag{5.7}$$

We have the following estimation:

$$\begin{aligned}
\zeta_{(n,\alpha)}^{(1)} &\leqslant \sum_{k=0}^{n-1}(n-k)\psi(k+1)\tilde{X}_k - \sum_{k=0}^{n-1}(n-k)\psi(k)\tilde{X}_k + \sum_{k=0}^{n-1}\psi(k+1) - n\psi(0),\\
\zeta_{(n,\alpha)}^{(2)} &\leqslant \sum_{k=0}^{n-1}\psi(k+1)\tilde{X}_k + \sum_{k=0}^{n-1}\psi(k+1),\\
\zeta_{(n,\alpha)}^{(3)} &\leqslant \sum_{k=0}^{n}\psi(k+1)\tilde{X}_k + \sum_{k=0}^{n}\psi(k+1).
\end{aligned} \tag{5.8}$$

For a fixed $\alpha\in G$, it is easy to check that

$$\beta_n^{(j)} = \frac{\zeta_{(n,\alpha)}^{(j)}}{M_n} \to 0 \quad \text{as } n\to\infty \text{ with probability one}$$

for $j = 1, 2, 3$, by Lemma 2 and the SLLN. We shall indicate some details only for $j = 1$.

Clearly,

$$\frac{\sum_{k=0}^{n-1}(n-k)\psi(k)\tilde{X}_k}{M_n} \to 0 \text{ a.e.}$$

Moreover, we have (with $\tilde{X}_{-1} = 0$)

$$\begin{aligned}
\sum_{k=0}^{n-1}(n-k)\psi(k+1)\tilde{X}_k &= \sum_{k=1}^{n}(n-k+1)\psi(k)\tilde{X}_{k-1}\\
&= \sum_{k=0}^{n}(n-k)\psi(k)\tilde{X}_{k-1} + \sum_{k=0}^{n}\psi(k)\tilde{X}_{k-1} = \beta_n^{(1)} + \beta_n^{(2)}.
\end{aligned}$$

Dividing by $M_n = \sum_{k=0}^{n-1}\psi(k)$ and applying Lemma 2 with $a_0 = 0$, $a_k = \tilde{X}_{k-1}$ and $c_k = \psi(k)$, we get, by the SLLN, $\beta_n^{(1)} \to 0$ a.e.

Finally, by the condition (R), i.e. the formula (3.6) for $c_k = \psi(k)$, obviously,

$$\frac{\sum_{k=0}^{n-1} \psi(k+1) - n\psi(0)}{M_n} \to 0.$$

The proof of (5.6) is similar to that of (5.5), even easier. ∎

6. Examples. In this section we give two examples. In the first one we indicate a stain (with discrete time) which does not round out weakly. In the second one we construct a stain rounding out strongly but not rounding out weakly.

EXAMPLE 1. Keeping the notation of previous sections, let us consider the stain of the form

$$\bar{\xi}(n, \alpha) = \sum_{k=0}^{n-1} (n-k) e^k X_k(\alpha)$$

with $X_k(\alpha) = \sum_{k \leqslant \tau_i < k+1} f(\alpha - \alpha_i)$, where f is continuous but not constant. We shall show that $\bar{\xi}$ does round out weakly.

Indeed, the random variables $X_k(\alpha)$ have the same distribution for all $\alpha \in G$ and $k \geqslant 0$. By the additional assumption, a continuous function f is not constant. All these imply the existence of $\alpha', \alpha'' \in G$ and $\delta > 0$ such that

$$P\big(X_k(\alpha') - X_k(\alpha'') > \delta\big) > \delta \tag{6.1}$$

and

$$P\big(X_k(\alpha') - X_k(\alpha'') < -\delta\big) > \delta$$

for all $t \geqslant 0$.

Assume for simplicity that $EX_k(\alpha) = 1$ and put

$$M(n) = E\bar{\xi}(n, \alpha) = \sum_{k=0}^{n-1} (n-k) e^k.$$

Let us note that

$$e^{-n} M(n) \to e \quad \text{as } n \to \infty. \tag{6.2}$$

For δ as above we fix a $B > 1$ satisfying the inequality

$$P\big(\bar{\xi}(n, \alpha) > BM(n)\big) < \delta/3 \quad \text{for all } n \geqslant 0.$$

This is possible because

$$P\big(\bar{\xi}(n, \alpha) - M(n) > (B-1) M(n)\big) \leqslant \frac{D^2 \bar{\xi}(n, \alpha)}{(B-1)^2 M(n)^2},$$

and $D^2 \bar{\xi}(n, \alpha) \leqslant M(n)^2$.

For $R(n) = \max_{\alpha\in G} \bar{\xi}(n, \alpha)$ and $r(n) = \min_{\alpha\in G} \bar{\xi}(n, \alpha)$, obviously, we have

$$\frac{R(n)}{r(n)} - 1 \geqslant \frac{|\bar{\xi}(n, \alpha') - \bar{\xi}(n, \alpha'')|}{r(n)} \geqslant \frac{|\bar{\xi}(n, \alpha') - \bar{\xi}(n, \alpha'')|}{\bar{\xi}(n, \alpha')}.$$

Thus, putting

$$U(n) = \sum_{k=0}^{n-1} (n-k) e^k \frac{X_k(\alpha') - X_k(\alpha'')}{\xi(n, \alpha')},$$

$$V(n) = \sum_{k=0}^{n-2} (n-k) e^k \frac{X_k(\alpha') - X_k(\alpha'')}{\xi(n, \alpha')},$$

$$W(n) = e^k \big(X(n-1, \alpha') - X(n-1, \alpha'')\big),$$

for $\delta' = \delta/(2e^2)$ we have

$$P\left(\frac{R(n)}{r(n)} - 1 \geqslant \frac{\delta'}{B}\right) \geqslant P\left(|U(n)| \geqslant \frac{\delta'}{B}\right)$$

$$\geqslant \max\left[P\left(|U(n)| \geqslant \frac{\delta'}{B}\right), P\left(U(n) \leqslant -\frac{\delta'}{B}\right)\right]$$

$$\geqslant \max\left[P\left(V(n) \geqslant 0, \frac{W(n)}{\bar{\xi}(n, \alpha')} \geqslant \frac{\delta'}{B}\right), P\left(V(n) \leqslant 0, \frac{W(n)}{\bar{\xi}(n, \alpha')} \leqslant -\frac{\delta'}{B}\right)\right]$$

$$\geqslant \max\left[P\left(V(n) \geqslant 0, \frac{W(n)}{M(n)} \geqslant \delta', \frac{M(n)}{\bar{\xi}(n, \alpha')} \geqslant \frac{1}{B}\right),\right.$$

$$\left. P\left(V(n) \leqslant 0, \frac{W(n)}{M(n)} \leqslant -\delta', \frac{M(n)}{\bar{\xi}(n, \alpha')} \geqslant \frac{1}{B}\right)\right].$$

In the sequel we can assume that $P(V(n) \geqslant 0) \geqslant \frac{1}{2}$ (if not, we take the event $\{V(n) \leqslant 0\}$ and continue the argument in a similar way).

By (6.1), (6.2) and the independence of X_k's, we get

$$P\left(V(n) \geqslant 0, \frac{W(n)}{M(n)} \geqslant \delta'\right) \geqslant \frac{1}{2}\delta$$

for n large enough.

Thus

$$P\left(\frac{R(n)}{r(n)} - 1 \geqslant \frac{\delta'}{B}\right) \geqslant \frac{1}{2}\delta - \frac{\delta}{3} = \frac{\delta}{6}$$

(since $p(AB) \geqslant p(A) - p(B^c)$), so $R(n)/r(n)$ does not converge to one in probability as $n \to \infty$. ∎

Before formulating Example 2, we begin with some general remark.

Remark 1. For any system of events $A_1, B_1, A_2, B_2, \ldots$ such that A_{s+1} is independent of $\sigma(A_1, B_1, \ldots, A_s, B_s, B_{s+1})$ and $P(B_s) \geqslant \eta > 0$ the condition $\sum_i P(A_i) = +\infty$ implies the inequality

$$P\Big(\bigcup_s (B_s \cap A_s)\Big) \geqslant \eta.$$

In fact, on the contrary, suppose that $P\big(\bigcup_{q=1}^{s} A_q \cap B_q\big) < \varrho < \eta$ for every s. Then

$$P\Big(A_{s+1} \setminus \bigcup_{q=1}^{s} A_q \cap B_q\Big) > (1-\varrho) P(A_{s+1}),$$

so

$$\sum_s P\Big(A_{s+1} \cap B_{s+1} \setminus \bigcup_{q=1}^{s} A_q \cap B_q\Big) > \sum_s (\eta - \varrho) P(A_{s+1}) = \infty,$$

a contradiction.

As in the previous sections we put

$$X_k(\alpha) = \sum_{\{i: k \leqslant \tau_i < k+1\}} f(\alpha - \alpha_i).$$

Example 2. We shall show that there exists a sequence $c_0, c_1, \ldots$ of positive numbers such that for any nonconstant continuous function f a stain

$$\xi(N, \alpha) = \sum_{k=0}^{N-1} (N-k) c_k X_k(\alpha)$$

rounds out weakly but it does not round out strongly.

Proof. We split the proof into several steps.

Step 1 (Construction of the sequence (c_k)). Let $n(s)$ be a sequence of all positive even integers increasing 'slowly' to infinity in the following sense:

$$\sum_{s=1}^{\infty} \delta^{n(s)} = \infty \quad \text{for each } \delta > 0, \tag{6.3}$$

$$\lim_{s \to \infty} \frac{m(s)}{n(s)^2} = \infty, \tag{6.4}$$

where

$$m(s) = n(1) + \ldots + n(s), \quad s = 1, 2, \ldots \tag{6.5}$$

Both the above conditions will be satisfied if we take for example a sequence

$$\big(n(s)\big) = (\underbrace{2, \ldots, 2}_{k(1)}, \underbrace{4, \ldots, 4}_{k(2)}, \ldots, \underbrace{2j, \ldots, 2j}_{k(j)}, \ldots)$$

with $k(j) \geqslant j^2$, $j = 1, 2, \ldots$

The sequence (c_k) will be defined by induction with respect to s. Let us put

$$c(0) = c(1) = c\big(m(1)-1\big) = 1 \tag{6.6}$$

and

$$c\big(m(s)\big) = c\big(m(s)+1\big) = \ldots = c\big(m(s+1)-1\big) \tag{6.7}$$

$$= \frac{2}{n(s+1)\big(n(s+1)+1\big)} \sum_{k=0}^{m(s)-1} \big(m(s+1)-k\big)c(k), \qquad s = 1, 2, \ldots$$

By (6.4) we have

$$c\big(m(s)\big) \geqslant \frac{2\big(m(s+1)\big)c(0)}{n(s+1)\big(n(s)+1\big)} \to \infty,$$

so

$$c(k) \to \infty \quad \text{as } k \to \infty. \tag{6.8}$$

Thus Step 1 is finished.

Step 2. Now, we prove that for each $\alpha \in G$

$$\frac{\xi(N, \alpha)}{E\xi(N, \alpha)} \to 1 \quad \text{in probability as } N \to \infty.$$

Fix $\varepsilon > 0$ and take s_0 satisfying

$$n(s_0) > \varepsilon^{-3}. \tag{6.9}$$

By (6.8), there exists a $k_0 > m(s_0)$ such that

$$c(k) > \frac{c(r)}{\varepsilon^{3/2}} \quad \text{for } r < m(s_0),\ k \geqslant k_0. \tag{6.10}$$

By (6.4) we can choose s_1 satisfying

$$m(s_1) > 2k_0 \quad \text{and} \quad m(s+1) \leqslant 2m(s) \text{ for } s > s_1. \tag{6.11}$$

Then, for $m(s) \leqslant N < m(s+1)$, $s > s_1$, we have

$$P\left(\left|\frac{\xi(N, \bar{\alpha}) - E\xi(N, \bar{\alpha})}{E\xi(N, \bar{\alpha})}\right| \geqslant \varepsilon\right) \leqslant \frac{D^2\xi(N, \bar{\alpha})}{\big(E\xi(N, \bar{\alpha})\big)^2 \varepsilon^2}$$

$$= \frac{\sigma^2}{\varepsilon^2}\left[\frac{\sum_{k=0}^{m(s_0)-1}(N-k)^2 c(k)^2}{\big(\sum_{k=0}^{N-1}(N-k)c(k)\big)^2} + \frac{\sum_{k=m(s_0)}^{m(s)-1}(N-k)^2 c(k)^2}{\big(\sum_{k=0}^{N-1}(N-k)c(k)\big)^2} + \frac{\sum_{k=m(s)}^{N-1}(N-k)^2 c(k)^2}{\big(\sum_{k=0}^{N-1}(N-k)c(k)\big)^2}\right].$$

The stochastic convergence of $\xi(N, \alpha)/E\xi(N, \alpha)$ to one as $N \to \infty$ can be obtained from (6.10) and (6.11) by a standard analysis of polynomial degree.

Step 3. The proof of the weak rounding out of the stain $\xi(N, \alpha)$ is now rather standard. The detailed calculations can be found in [2].

Step 4. Let us fix a positive nonconstant continuous function f on G. Without loss of generality we can assume that $Ef(\alpha-\alpha_i) = 1$. Let us write $\sigma^2 = D^2 f(\alpha-\alpha_i) > 0$. There exist a $\delta > 0$, $\alpha', \alpha'' \in G$ and $B \geqslant 1$ satisfying the inequalities

$$\begin{aligned} &P\big(X_k(\alpha')-X_k(\alpha'') > \delta,\ X_k(\alpha'') \leqslant B\big) > \delta, \\ &P\big(X_k(\alpha')-X_k(\alpha'') < -\delta,\ X_k(\alpha') \leqslant B\big) > \delta. \end{aligned} \tag{6.12}$$

We shall show that the stain $\xi(N, \alpha)$ does not round out strongly, i.e.

$$P\left(\frac{R(N)}{r(N)} \nrightarrow 1\right) > 0. \tag{6.13}$$

We can assume that

$$\sum_{s\in Z} \delta^{n(s)} = \infty \tag{6.14}$$

for

$$Z = \left\{s \geqslant 1\colon\ P\Big(\sum_{k=0}^{m(s)-1} \big(m(s+1)-k\big)c(k)\big(X(k, \alpha')-X(k, \alpha'')\big) \geqslant \delta\Big) \geqslant \frac{1}{2}\right\}, \tag{6.15}$$

where δ, α', α'' are taken as in (6.12). We set

$$A_s = \bigcup_{k=m(s)}^{m(s+1)-1} \{X(k, \alpha')-X(k, \alpha'') > \delta,\ X(k, \alpha'') \leqslant B\}.$$

We have

$$\begin{aligned} P\Big(\sum_{k=0}^{m(s)-1} \big(m(s+1)-k\big)c(k)X(k, \alpha'') \geqslant (1+2\sigma) \sum_{k=0}^{m(s)-1} \big(m(s+1)-k\big)c(k)\Big) \\ \leqslant \frac{\sum_{k=0}^{m(s)-1} \big(\big(m(s+1)-k\big)c(k)\big)^2 \sigma^2}{4\sigma^2 \big(\sum_{k=0}^{m(s)-1} \big(m(s+1)-k\big)c(k)\big)^2} \leqslant \frac{1}{4}. \end{aligned}$$

The definition (6.15) implies $P(B_s) \geqslant \frac{1}{4}$ for

$$\begin{aligned} B_s = \Big\{ &\sum_{k=0}^{m(s)-1} \big(m(s+1)-k\big)c(k)\big(X(k, \alpha')-X(k, \alpha'')\big) \geqslant 0 \quad \text{and} \\ &\sum_{k=0}^{m(s)-1} \big(m(s+1)-k\big)c(k)X(k, \alpha'') < (1+2\sigma) \sum_{k=0}^{m(s)-1} \big(m(s+1)-k\big)c(k)\Big\} \end{aligned}$$

with $s \in Z$. Moreover, $\sum_{s\in Z} P(A_s) = \infty$ by (6.12) and (6.14).

By Remark 1, for each $N \geqslant 1$,

$$P\Big(\bigcup_{\substack{s\in Z\\ s>N}} A_s \cap B_s\Big) \geqslant \frac{1}{4}.$$

This implies (6.13) in rather a standard way. ■

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