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ON THE SUPREMUM FROM GAUSSIAN PROCESSES OVER INFINITE HORIZON

BY

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Abstract. In the paper we study the asymptotic of the tail of distribution function $P(A(X, c) > x)$ for $x \to \infty$, where $A(X, c)$ is the supremum of $X(t) - ct$ over $[0, \infty)$. In particular, $X(t)$ is the fractional Brownian motion, a nonlinearly scaled Brownian motion or some integrated stationary Gaussian processes. For the fractional Brownian motion we give a stronger result than a recent one of Duffield and O'Connell [5].

Introduction. In this paper we study the asymptotic behaviour of the tail of distribution function of

$$A(X, c) = \sup\{X(t) - ct\colon\ t \geqslant 0\},$$

where $X(t)$ is a mean zero Gaussian process. Thus X is:

FBM: a fractional Brownian motion with parameter H $(0 < H \leqslant 1)$, which will be denoted by $B_H(t)$;

SBM: a scaled Brownian motion $S_H(t) = B(t^{2H})$ with parameter $H > 0$;

FiBM: a filtered Brownian motion

$$B^{\circ}(t) = \sqrt{2H} \int_0^t (t-s)^{H-1/2}\, dB(s)$$

with parameter $H > 0$;

DG: a degenerated Gaussian

$$N_H(t) = t^H \mathcal{N}$$

with parameter $H > 0$, where $\mathcal{N}$ is the standard normal random variable and $H > 0$;

GI: the integral $\int_0^t Z(s)\, ds$, where Z is a stationary short range dependent stationary process.

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The common feature of all considered above processes, except for the **GI** case, is that the variance $D^2 X(t) = t^{2H}$. For the **GI** case the variance is an asymptotically linear function of t.

For $H = 1/2$ the **FBM** is the Brownian motion $B_{1/2}(t) = B(t)$ and it is well known that

$$P\big(A(B_{1/2}, c) > u\big) = \exp(-2cu).$$

Dębicki and Rolski [4] studied some **GI** processes of the form $X(t) = \int_0^t Z(s)\,ds$, where $Z(t)$ is a short range dependent stationary Gaussian process with mean 0, for which they proved that

$$P\big(A(X, c) > u\big) \leqslant \text{const} \exp(-\gamma u) + o\big(\exp(-\gamma u)\big).$$

The short range dependence means that $0 < \int_0^\infty R(t)\,dt < \infty$, and then $\gamma = c/\int_0^\infty R(t)\,dt$, however for their result Dębicki and Rolski needed more assumptions. In Section 5, under the condition that $R(t) \geqslant 0$, we show some improvements of bounds and their proofs from [4]. However, recently, models with $X(t)$ having higher irregularity than Brownian motion and also dependent increments have been more often required. In the class of Gaussian processes there is a lot of interest in studying the **FBM**. Note that if $H \geqslant 1/2$, then the process has positively correlated increments. We would like to point out that **FiBM** and **DG** are not processes with stationary increments.

The tool to study $\psi(u) = P\big(A(B_H, c) > u\big)$ is the inequality

$$P\big(A(B_H, c) > u\big) \leqslant P\big(A(S_H, c) > u\big),$$

which follows from Slepian's theorem (recalled in Section 4). The **FiBM** appeared in Mandelbrot and Van Ness [17], where the authors referred for this process to Levy [16].

Since

$$P\big(\sup_{t\geqslant 0}\{B(t^{2H}) - ct\} > u\big) = P\big(\sup_{t\geqslant 0}\{B(t) - ct^{1/(2H)}\} > u\big),$$

the problem for the **SBM** can be reduced to studying a Brownian motion with the nonlinear drift $B(t) - ct^{1/(2H)}$. The supremum of the Brownian motion with nonlinear drift $B(t) - t^{1/2}$ was studied by Klüppelberg and Mikosch [11]. Other papers with nonlinear drift are Ferebee [7], [8] and Jennen [10].

There is a vast literature dealing with the distribution of the supremum of centered Gaussian processes on a compact interval, with the most celebrated Borell inequality (see e.g. Adler [1]). Berman [2] obtained bounds for the supremum of Gaussian processes with stationary increments over finite intervals by the use of Slepian's inequality. Michna [18] has recently studied bounds for the supremum of the **FBM** with drift over a finite interval. In contrast to such results, the tail of the distribution function of the supremum over unbounded sets (in our case $\boldsymbol{R}_+$) of a Gaussian process with negative drift has usually a different asymptotic. For $X(t)$ being the **FBM** the first result in this direction seems to be of Norros [19] who showed a lower bound for $\Psi(b)$ and we found his bounds out while considering **DG** case. Duffield and O'Connell [5] studied the asymptotic

$$\log\big(P(A(B_H, c) > u)\big) \sim \alpha u^{2-2H} \quad \text{for } u \to \infty. \tag{1.1}$$

It turns out that their result is typical for a larger class of Gaussian processes and that the variance function is responsible for the asymptotics of (1.1). In the papers [5] and [19], the buffer content in a stationary fluid model was studied, where for the net-input process $X(t) = B_H(t) - ct$, the buffer content process was given by $Y(t) = \sup_{s \leq t}(X(t) - X(s))$. This study is motivated by problems of queue theory, risk theory and other applications. For example, recent measurements and statistical analyses of traffic data are best explained by traffic models having long-range dependence, in particular by the **FBM**; see for instance Leland et al. [15] and Willinger et al. [23].

The study of the case **SBM** is pithier. In Section 4 we use Slepian's theorem to compare different models.

2. An application of local times for scaled Brownian motion. An important role plays the function

$$l_{H,c}(u) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{t^H} \exp\left(-\frac{(ct+u)^2}{2t^{2H}}\right) dt,$$

where $0 < H < 1$. In the special case we get $l_{1/2,c}(u) = c^{-1} \exp(-2xu)$; see the table of Laplace transforms in [14]. The following result will be useful:

LEMMA 2.1. *As* $u \to \infty$

$$l_{H,c}(u) \sim \frac{1}{c}\left(\frac{H}{1-H}\right)^{1/2} \exp\left(-\frac{1}{2}\left(\frac{c}{H}\right)^{2H} \left(\frac{1}{1-H}\right)^{2-2H} u^{2-2H}\right).$$

Proof. For the proof, Theorem 2.3 and Corollary 2.1 from Fedoryuk [6] should be used. The details how to check assumptions of this theorem are given in the Appendix. ∎

The following considerations are valid for a stochastic process $X(t)$, where $X(t) = B_H(t) - ct$ or $X(t) = S_H(t) - ct$. Let

$$L(u; X) = L(u) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^\infty \mathbb{1}[X(t) \in (u-\varepsilon, u+\varepsilon)]\, dt$$

be the local time of the process $X(t)$ at level u, where the limit exists almost surely. However, for this paper we take another approach to define the local time following Geman and Horowitz [9], and Berman [3]. Let $X(t)$ be a stochastic process fulfilling:

- $X(t)$ has continuous trajectories;

• for all $t \geqslant 0$ the distribution of $X(t)$ is absolutely continuous with respect to the Lebesgue measure m with density $g_{X(t)}$;

• $\int_0^\infty g_{X(t)}(x)\,dt < \infty$ for all $x \in \boldsymbol{R}$.

Define a random measure M_ω by

$$M_\omega(A) = m(\{t \geqslant 0\colon\ X(t) \in A\}) = \int_0^\infty \mathbf{1}(X_\omega(t) \in A)\,dt, \qquad A \in \mathscr{B}(\boldsymbol{R}_+).$$

We have

$$EM_\omega(A) = \int_A \Big(\int_0^\infty g_{X(t)}(x)\,dt\Big)\,dx.$$

Since the measure EM is absolutely continuous with respect to m, for almost all ω the measure M_ω is absolutely continuous with respect to m. Therefore there exists $L_\omega(u)$ such that

$$\int_A L_\omega(u)\,du = M_\omega(A), \qquad A \in \mathscr{B}(\boldsymbol{R}_+).$$

Thus

$$\int_A EL(u)\,du = EM(A) = \int_A \Big(\int_0^\infty g_{X(t)}(u)\,dt\Big)\,du$$

for all $A \in \mathscr{B}(\boldsymbol{R}_+)$, and hence

$$EL(u) = l_{H,c}(u) = \int_0^\infty g_{X(t)}(u)\,dt, \qquad m\text{-a.e.}$$

LEMMA 2.2. *If we choose a continuous version of* $EL(u)$, *then*

$$EL(u;\ \{B_H(s)-cs\}) = EL(u;\ \{S_H(s)-cs\}) = l_{H,c}(u).$$

Let

$$\tau(u) = \inf\{S_H(t)-ct > u\colon\ t \geqslant 0\}$$

be the crossing time of the level u by $S_H(t)-ct$, and $\mu_u(\cdot)$ its distribution. Note that μ_u is defective for $H < 1$ (use the law of iterated logarithm). By $\mu_u^\circ(\cdot)$ we denote the conditional distribution of $\tau(u)$ under the condition that $\tau(u) < \infty$. Consider jointly $(L(u), \tau(u))$ under the condition that $\tau(u) < \infty$. Let $E(L(u) \mid \tau(u) = t)$ denote a version of the conditional probability $E(L(u) \mid \tau(u))$. The idea is to consider

$$\frac{EL(u)}{P(\sup_{t\geqslant 0}\{S_H(t)-ct\} > u)} = \frac{\int_0^\infty E(L(u) \mid \tau(u) = t)\,\mu_u(dt)}{P(\tau(u) < \infty)} \tag{2.1}$$

$$= \int_0^\infty E(L(u) \mid \tau(u) = t)\,\mu_u^\circ(dt).$$

Now using the property that the process $S_H(t) - ct$ has independent increments (but not stationary), we obtain

$$E\big(L(u) \mid \tau(u) = t\big) = E\Big(L\big(0; \{B\big((t+s)^{2H} - t^{2H}\big) - cs\}\big)\Big) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{(t+x)^{2H} - t^{2H}}} \exp\left(-\frac{c^2 x^2}{2[(x+t)^{2H} - t^{2H}]}\right) dx. \tag{2.2}$$

LEMMA 2.3. (i) *The function* $y(z) = z/\sqrt{(z+1)^{2H} - 1}$ *is strictly increasing from* 0 *to* ∞ *and similarly its inverse* $z(t)$.

(ii) *For* $0 < H < 1/2$ *the function*

$$f^*(z) = \frac{(z+1)^{2H} - 1}{(z+1)^{2H} - 1 - Hz(z+1)^{2H-1}} \tag{2.3}$$

is decreasing.

(iii) *For* $1/2 < H < 1$ *the function*

$$f^*(z) = \frac{(z+1)^{2H} - 1}{(z+1)^{2H} - 1 - Hz(z+1)^{2H-1}} \tag{2.4}$$

is increasing.

(iv) *For* $0 < H < 1$,

$$\lim_{z \to 0} f^*(z) = \lim_{y \to 0} f(y) = 2, \tag{2.5}$$

$$\lim_{z \to \infty} f^*(z) = \lim_{y \to \infty} f(y) = \frac{1}{1-H}. \tag{2.6}$$

Proof. The proof goes by standard calculations. We only show that $f^*(z)$ is increasing. Since for all $z > 0$

$$\frac{d}{dz} f^*(z) = \frac{H(z+1)^{2H-2}[(z+1)^{2H} - 1 - 2Hz]}{[(z+1)^{2H} - 1 - Hz(z+1)^{2H-1}]^2} \geqslant 0,$$

the proof is completed. ∎

The key idea is given in the following lemma:

LEMMA 2.4. *If* $1/2 \leqslant H < 1$, *then for all* $u \geqslant 0$

$$\frac{1}{2-2H} \frac{1}{c} \geqslant E\big(L(u) \mid \tau(u) = t\big) \geqslant \frac{1}{c}. \tag{2.7}$$

Proof. Putting $z = x/t$ in (2.2) we receive

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{1}{\sqrt{(x+t)^{2H}-t^{2H}}}\exp\left(-\frac{c^2x^2}{2[(x+t)^{2H}-t^{2H}]}\right)dx = \frac{1}{\sqrt{2\pi}}t^{1-H}\int_0^\infty \frac{1}{\sqrt{(z+1)^{2H}-1}}\exp\left(-\frac{c^2t^{2-2H}z^2}{2[(z+1)^{2H}-1]}\right)dz.$$

We now make the next substitution $y = z/\sqrt{(z+1)^{2H}-1}$. Note that, by Lemma 2.3, $y(z) = z/\sqrt{(z+1)^{2H}-1}$ is strictly increasing from 0 to ∞, and therefore we infer that the inverse $z(y)$ exists. Thus

$$\frac{1}{\sqrt{2\pi}}t^{1-H}\int_0^\infty \frac{1}{\sqrt{(z+1)^{2H}-1}}\exp\left(-\frac{c^2t^{2-2H}z^2}{2[(z+1)^{2H}-1]}\right)dz$$
$$= \frac{1}{\sqrt{2\pi}}t^{1-H}\int_0^\infty \frac{(z+1)^{2H}-1}{(z+1)^{2H}-1-Hz(z+1)^{2H-1}}\exp\left(-\frac{c^2t^{2-2H}y^2}{2}\right)dy$$
$$= \frac{1}{\sqrt{2\pi}}t^{1-H}\int_0^\infty f^*(z(y))\exp\left(-\frac{c^2t^{2-2H}y^2}{2}\right)dy,$$

where $f^*(z)$ was defined in (2.3). Note that by Lemma 2.3 the function $f(y) = f^*(z(y))$ is increasing. Since $f(y)$ is increasing and (2.5) holds, we obtain

$$E(L(u) \mid \tau(u) = t) = \frac{t^{1-H}}{\sqrt{2\pi}}\int_0^\infty f(y)\exp\left(-\frac{c^2t^{2-2H}y^2}{2}\right)dy \tag{2.8}$$
$$\geqslant \frac{2t^{1-H}}{\sqrt{2\pi}}\int_0^\infty \exp\left(-\frac{c^2t^{2-2H}y^2}{2}\right)dy = \frac{1}{c}.$$

Analogously, from (2.6) we get

$$E(L(u) \mid \tau(u) = t) = \frac{t^{1-H}}{\sqrt{2\pi}}\int_0^\infty f(y)\exp\left(-\frac{c^2t^{2-2H}y^2}{2}\right)dy \tag{2.9}$$
$$\leqslant \frac{t^{1-H}}{\sqrt{2\pi}(1-H)}\int_0^\infty \exp\left(-\frac{c^2t^{2-2H}y^2}{2}\right)dy = \frac{1}{2(1-H)c}.$$

The proof is completed. ■

Remark 2.1. Note that if $0 < H < 1/2$, then after analogous calculations we can obtain

$$\frac{1}{2-2H}\frac{1}{c} \leqslant E(L(u) \mid \tau(u) = t) \leqslant \frac{1}{c}. \tag{2.10}$$

The only difference is that for $0 < H < 1/2$ from Lemma 2.3 the function $f^*(x)$ given by (2.3) is decreasing.

3. Suprema from scaled Brownian motion with drift. In this section we study the asymptotic for $P(A(S_H, c) > u)$. We put

$$\alpha = \frac{1}{2}\left(\frac{c}{H}\right)^{2H}\left(\frac{1}{1-H}\right)^{2-2H}. \tag{3.1}$$

THEOREM 3.1. *Let* $1/2 \leqslant H < 1$.

(i) *For all* $u \geqslant 0$,

$$2(1-H)\, c l_{H,c}(u) \leqslant P(A(S_H, c) > u) \leqslant c l_{H,c}(u). \tag{3.2}$$

(ii) *For all* $u \geqslant 0$,

$$P(A(S_H, c) > u) \geqslant \exp(-\alpha u^{2-2H}). \tag{3.3}$$

(iii) *For* $u \to \infty$,

$$\begin{aligned} 2\sqrt{H(1-H)}\exp(-\alpha u^{2-2H}) + o(\exp(-\alpha u^{2-2H})) &\leqslant P(A(S_H, c) > u) \\ &\leqslant \sqrt{\frac{H}{1-H}}\exp(-\alpha u^{2-2H}) + o(\exp(-\alpha u^{2-2H})). \end{aligned} \tag{3.4}$$

Proof. We get (3.2) inserting the inequality (2.7) into (2.1). Now (3.4) follows from the result of Lemma 2.1.

The proof of (3.3) is based on the observation that

$$k_x(t) = \frac{c}{2H} x^{(1-2H)/(2H)}\, t + c\frac{1-2H}{2H} x^{1/(2H)}$$

is the tangent function to the function $m(t) = ct^{1/(2H)}$ at the point $x\,(x > 0)$ and

$$\begin{aligned} P(A(S_H, c) > u) &= P\big(\sup_{t\geqslant 0}(B(t) - ct^{1/(2H)}) > u\big) \\ &\geqslant P\big(\sup_{t\geqslant 0}(B(t) - k_x(t)) > u\big) \qquad (3.5) \\ &= P\left(\sup_{t\geqslant 0}\left(B(t) - \frac{c}{2H}x^{(1-2H)/(2H)}\, t - c\frac{2H-1}{2H}x^{1/(2H)}\right) > u\right) \\ &= P\left(\sup_{t\geqslant 0}\left(B(t) - \frac{c}{2H}x^{(1-2H)/(2H)}\, t\right) > c\frac{2H-1}{2H}x^{1/(2H)} + u\right) \\ &= \exp\left(-\frac{c}{H}x^{(1-2H)/(2H)}\left(c\frac{2H-1}{2H}x^{1/(2H)}\right) + u\right) \\ &\geqslant \exp\left(-\frac{c}{H}\left(\frac{uH}{c(1-H)}\right)^{1-2H}\left(\frac{2H-1}{2(1-H)}u + u\right)\right) \qquad (3.6) \\ &= \exp(-\alpha u^{2-2H}) \end{aligned}$$

for all $x > 0$. Inequality (3.5) is a consequence of the concavity of $m(t)$. Inequality (3.6) follows from the fact that the function

$$-\frac{c}{H} x^{(1-2H)/(2H)}\left(c\frac{2H-1}{2H}x^{1/(2H)}+u\right)$$

takes its maximum at the point $x = ((uH)/c(1-H))^{2H}$.

The following result gives the asymptotic of $P(A(S_H, c) > u)$.

THEOREM 3.2. *For* $0 < H < 1$ *and* $u \to \infty$

$$P(A(S_H, c) > u) \sim \sqrt{\frac{H}{1-H}}\exp(-\alpha u^{2-2H}).$$

Proof. From the proof of Lemma 2.4 we have

$$\text{(3.7)}\quad E\big(L(u; \{S_H(t)-ct\}) \mid \tau(u) = t\big)$$
$$= \frac{1}{\sqrt{2\pi}}\frac{1}{c}\int_0^\infty f^*(z(y))\frac{1}{\sqrt{c^{-2}t^{2H-2}}}\exp\left(-\frac{y^2}{2c^{-2}t^{2H-2}}\right)dy,$$

where $f^*(z)$ was defined in (2.3) and the function $z(y)$ is the inverse to

$$y = z/\sqrt{(z+1)^{2H}-1}.$$

Now (3.7) can be written as $Ef(z(N_t))/(2c)$, where N_t has the distribution as the conditional $c^{-1}t^{H-1}N$ under the condition that $N \geqslant 0$ and N is the standard normal random variable. By simple calculations we see that, for all $x \geqslant 0$, $P(N_t \geqslant x) \to 0$ as $t \to \infty$, which means that $N_t \to 0$ in distribution. Since $f^*(z(y))$ is bounded and continuous, we have

$$E\big(L(u; \{S_H(t)-ct\}) \mid \tau(u) = t\big) \to 1/c.$$

Now the result follows from (2.1) and Lemma 2.1. ∎

Remark 3.1. Note that, by the law of iterated logarithm, for $H \geqslant 1$ and $u \geqslant 0$ we have

$$P(A(S_H, c) > u) = P\big(\sup_{t\geqslant 0}\{B(t)-ct^{1/(2H)}\} > u\big) = 1.$$

Remark 3.2. The right-hand side of (3.4) can be derived by using a result for the first passage density for the Brownian motion to a barrier. Thus, since $sB(1/s)$ is a Brownian motion, we get

$$P(A(S_H, c) > u) = P\big(\inf_{s>0}(u+cs^{1/(2H)}-B(s)) < 0\big)$$
$$= P\big(\inf_{s>0}(u+cs^{-1/(2H)}-B(1/s)) < 0\big)$$
$$= P\big(\inf_{s>0}(us+cs^{1-1/(2H)}-B(s)) < 0\big).$$

To calculate the last probability we need the first passage density for the Brownian motion to the barrier $f(s) = us + cs^{1-1/(2H)}$. From the integral equation for the first passage density for Brownian motion to the barrier f (see e.g. Ferebee [7]) it follows that

$$P\big(\inf_{s>0}(us + cs^{1-1/(2H)} - B(s)) < 0\big)$$
$$\leqslant \frac{c}{H\sqrt{8\pi}} \int_0^\infty t^{-1/2-1/(2H)} \exp\{-\tfrac{1}{2}(ut^{1/2} + ct^{1/2-1/(2H)})^2\}\, dt.$$

Putting

$$s = \frac{2H}{1-H} t^{1/2-1/(2H)}$$

we obtain

$$P\big(\inf_{s>0}(us + cs^{1-1/(2H)} - B(s)) < 0\big)$$
$$\leqslant \frac{c}{H\sqrt{8\pi}} \int_0^\infty \exp\left\{-\frac{1}{2}\left(u\left(\frac{2H}{1-H}\right)^{H/(1-H)} s^{-H/(1-H)} + c\frac{1-H}{2H}s\right)^2\right\} ds.$$

We have to treat the above integral for large u. In order to conclude the upper bound in (3.4) it is enough to make an appropriate use of Theorem 2.3 and Corollary 2.1 from Fedoryuk [6], which shows the behavior of such integrals. The details how to check the assumptions of this theorem are given in the Appendix.

4. Application of Slepian's theorem. We use the following result of Slepian [22] in the form presented in Piterbarg [20], p. 6.

THEOREM 4.1. *Let Gaussian processes* $\{X_1(t);\ t \geqslant 0\}$ *and* $\{X_2(t);\ t \geqslant 0\}$ *with covariance functions* $r_{X_1}(s, t)$ *and* $r_{X_2}(s, t)$, *respectively, be separable (in particular, continuous). If for all* $s, t \geqslant 0$

$$r_{X_1}(t, t) \equiv r_{X_2}(t, t), \tag{4.1}$$

$$r_{X_1}(s, t) \geqslant r_{X_2}(s, t), \tag{4.2}$$

$$EX_1(t) = EX_2(t), \tag{4.3}$$

then for all $u \geqslant 0$

$$P\big(\sup_{t\geqslant 0}\{X_1(t)\} \geqslant u\big) \leqslant P\big(\sup_{t\geqslant 0}\{X_2(t)\} \geqslant u\big).$$

We use Theorem 4.1 to compare $P(A(X, c) > x)$ for different Gaussian processes X. Recall that the covariance function of the FBM with parameter H is $r_{B_H}(s, t) = (1/2)(t^{2H} + s^{2H} - |t-s|^{2H})$ and in the SBM $S_H(t)$ the covariance function is $r_{S_H}(s, t) = \min(s^{2H}, t^{2H})$. Michna [18] showed that $r_{S_H}(s, t) \leqslant$

$r_{B_H}(s,t)$ for $1/2 \leqslant H \leqslant 1$ and in the following lemma we find a further relationship between useful covariance functions.

LEMMA 4.1. *The covariance function of the* **FiBM** B_H° *is*

$$r_{B_H^\circ}(s,t) = 2H \int_0^{s\wedge t} \big((s-x)(t-x)\big)^{H-1/2}\,dx$$

and of the process **DG** N_H *is* $r_{N_H}(s,t) = (st)^H$. *For all* $t \geqslant 0$

$$r_{S_H}(t,t) = r_{B_H^\circ}(t,t) = r_{B_H}(t,t) = r_{N_H}(t,t) = t^{2H}, \quad t \geqslant 0.$$

Moreover, for $1/2 \leqslant H \leqslant 1$

$$r_{S_H}(s,t) \leqslant r_{B_H^\circ}(s,t) \leqslant r_{B_H}(s,t) \leqslant r_{N_H}(s,t), \quad s,t \geqslant 0,$$

and for $0 < H \leqslant 1/2$

$$r_{B_H^\circ}(s,t) \leqslant r_{B_H}(s,t) \leqslant r_{S_H}(s,t) \leqslant r_{N_H}(s,t), \quad s,t \geqslant 0.$$

Proof. In the proof we assume that $s \leqslant t$. We have

$$r_{B_H^\circ}(s,t) = E2H \int_0^t (s-x)_+^{H-1/2}\,dB(x) \int_0^t (t-x)^{H-1/2}\,dB(x)$$

$$= 2H \int_0^t (s-x)_+^{H-1/2}(t-x)^{H-1/2}\,dx \tag{4.4}$$

$$= 2H \int_0^s v^{H-1/2}(t-s+v)^{H-1/2}\,dv \tag{4.5}$$

and

$$r_{N_H}(s,t) = Es^H\mathcal{N}t^H\mathcal{N} = (st)^H.$$

If $1/2 \leqslant H \leqslant 1$, then

$$r_{S_H}(s,t) = s^{2H} = 2H \int_0^s (s-x)^{2H-1}\,dx \tag{4.6}$$

$$\leqslant 2H \int_0^s (s-x)^{H-1/2}(t-x)^{H-1/2}\,dx = r_{B_H^\circ}(s,t).$$

We now obtain

$$r_{B_H}(s,t) = \tfrac{1}{2}\big(t^{2H}+s^{2H}-(t-s)^{2H}\big) = H\int_0^s v^{2H-1}\,dv + H\int_{t-s}^t v^{2H-1}\,dv$$

$$= H\int_0^s \Big((x^{H-1/2})^2 + \big((t-s+x)^{H-1/2}\big)^2\Big)\,dx$$

$$\geqslant 2H \int_0^s x^{H-1/2}(t-s+x)^{H-1/2}\,dx = r_{B_H^\circ}(s,t).$$

Since

$$t^H \leqslant (t-s)^H + s^H,$$

we have

$$(t^H - s^H)^2 \leqslant (t-s)^{2H},$$

and hence

$$(t^H)^2 + (s^H)^2 - \left((t-s)^H\right)^2 \leqslant 2s^H t^H,$$

which proves $r_{B_N}(s, t) \leqslant r_{N_H}(s, t)$. If $0 < H \leqslant 1/2$, then the inequality in (4.6) is reversed. ∎

From Lemma 4.1 and Slepian's theorem we get the following chain of inequalities:

$$P\big(A(N_H, c) > u\big) \leqslant P\big(A(B_H, c) > u\big) \leqslant P\big(A(B_H^\circ, c) > u\big)$$
$$\leqslant P\big(A(S_H, c) > u\big), \quad u \geqslant 0,$$

provided that $1/2 \leqslant H < 1$; otherwise,

$$P\big(A(N_H, c) > u\big) \leqslant P\big(A(S_H, c) > u\big) \leqslant P\big(A(B_H, c) > u\big)$$
$$\leqslant P\big(A(B_H^\circ, c) > u\big), \quad u \geqslant 0.$$

The following result was given by Norros [19] but his proof is different.

Lemma 4.2. *We have*

$$P\big(A(N_H, c) > u\big) \geqslant \frac{1}{\sqrt{2\pi}} \frac{2\alpha u^{2-2H} - 1}{(2\alpha)^{3/2} u^{3-3H}} \exp(\alpha u^{2-2H})$$
$$\sim \frac{1}{2\pi} \frac{1}{2\alpha u^{1-H}} \exp(\alpha u^{2-2H}), \quad u \to \infty.$$

Proof. The function $h(x,t) = \sup\{xt^H - ct\}$ achieves its maximum with respect to t for

$$t_x = (xH/c)^{1/(1-H)},$$

and then

$$h(x) = h(x, t_x) = cx^{1/(1-H)} \left(\frac{H}{c}\right)^{1/(1-H)} \left(\frac{1-H}{H}\right).$$

Thus

$$P\big(h(\mathcal{N}) > u\big) = P\left(\mathcal{N} > \left(\frac{H}{1-H}\frac{1}{c}\right)^{1-H} \frac{c}{H} u^{1-H}\right)$$
$$\geqslant \frac{1}{\sqrt{2\pi}} \frac{2\alpha u^{2-2H} - 1}{(2\alpha)^{3/2} u^{3-3H}} \exp(\alpha u^{2-2H})$$
$$\sim \frac{1}{2\pi} \frac{1}{2\alpha u^{1-H}} \exp(\alpha u^{2-2H}), \quad u \to \infty. \ ∎$$

In the following corollary the first equation is due to Duffield and O'Connell [5].

COROLLARY 4.1. *For* $1/2 \leqslant H \leqslant 1$

$$\lim_{t\to\infty} \frac{1}{t^{2-2H}} \log P\big(A(B_H, c) > u\big) = -\alpha$$

and

$$\lim_{t\to\infty} \frac{1}{t^{2-2H}} \log P\big(A(B_H^\circ, c) > u\big) = -\alpha.$$

5. Exponential bounds. Using Slepian's inequality we can obtain improvements of results from Dębicki and Rolski [4]. Consider a stationary, centered Gaussian process $\{Z(t);\, t \geqslant 0\}$ with covariance function $R(t) = EZ(0)Z(t)$ such that

(i) R is continuous;

(ii) $0 < R(t) < \infty$ for all $t \geqslant 0$;

(iii) $\int_0^\infty R(s)\,ds < \infty$.

Define

$$X(t) = \int_0^t Z(s)\,ds.$$

Let

$$\gamma = \frac{c}{\int_0^\infty R(s)\,ds}.$$

We have

THEOREM 5.1. *Under the assumptions* (i)–(iii) *and* $c > 0$, *for each* $u \geqslant 0$

$$P\big(A(X, c) > u\big) \leqslant e^{-\gamma u}.$$

Proof. Let $s < t$. We have

$$\operatorname{Cov}\big(X(t), X(s)\big) = D^2\big(X(s)\big) + \int_0^s dw \int_{s-w}^{t-w} R(v)\,dv \tag{5.1}$$

and

$$D^2\big(X(t)\big) = 2\int_0^t ds \int_0^s R(v)\,dv \leqslant \frac{2c}{\gamma} t. \tag{5.2}$$

From Theorem 4.1 and (5.1) we obtain

$$P\big(A(X, c) > u\big) \leqslant P\Big(\sup_{t\geqslant 0}\big(B\big(D^2 X(t)\big) - ct\big) > u\Big).$$

Now using (5.2) we get

$$P\left(\sup_{t\geqslant 0}\left(B\left(D^2 X(t)\right)-ct\right)>u\right)\leqslant P\left(\sup_{t\geqslant 0}\left(B\left(\frac{2c}{\gamma}t\right)-ct\right)>u\right)$$

$$=P\left(\sup_{t\geqslant 0}\left(B(t)-\frac{\gamma}{2}t\right)>u\right)=e^{-\gamma t}. \blacksquare$$

Remark 5.1. Note that if the increments of the process $\{X(t);\, t\geqslant 0\}$ are positively correlated and the covariance function $R(t)$ is continuous, then $R(t)>0$ for all $t\geqslant 0$.

Remark 5.2. Kulkarni and Rolski [13] studied the process $Z(t)=\sum_{i=1}^{n}\varrho_i Z_i(t)$, where $Z_i(t)$ are independent stationary Ornstein–Uhlenbeck processes with covariance function $R_i(t)=e^{-\alpha_i t}$ $(i=1,\ldots,n)$ and ϱ_i $(i=1,\ldots,n)$ are positive constants. It is easy to check that the assumptions (i)–(iii) hold for $Z(t)$. From Theorem 5.1 we have

$$P\left(A(X,c)>u\right)\leqslant \exp\left(-\frac{cu}{\sum_{i=1}^{n}\varrho_i^2/\alpha_i}\right)$$

for all $u\geqslant 0$. Note that Kulkarni and Rolski [14] have obtained

$$P\left(A(X,c)>u\right)\leqslant \exp\left(-\frac{c^2}{2\varrho^2}\right)e^{-\gamma u},\qquad u\geqslant 0,$$

for the special case of $Z(t)$ being the pure Ornstein–Uhlenbeck process with $R(t)=\varrho^2 e^{-\alpha t}$. If we additionally assume that

(iv) $\int_0^\infty t^2 R(t)\,dt<\infty$,

then we can prove the following

THEOREM 5.2. *Under the assumptions* (i)–(iv) *and* $c>0$, *for* $u\to\infty$

$$P\left(A(X,c)>u\right)\leqslant \exp\left(-\gamma^2\beta\right)e^{-\gamma u}+o\left(e^{-\gamma u}\right),$$

where $\beta=\int_0^\infty tR(t)\,dt$.

Proof. The proof of Theorem 5.2 is analogous to the proof of Theorem 5.1. It is based on the observation that if (i)–(iv) hold, then

$$D^2\left(X(t)\right)=\frac{2c}{\gamma}t-2\beta+r(t), \tag{5.3}$$

where $r(t)=2\int_t^\infty (s-t)R(s)\,ds=o(t^{-1})$ (see [4]). $\blacksquare$

The following theory gives a lower bound for the supremum of $P\left(A(X,c)>u\right)$. The idea of the proof is analogous to the proof of the lower bound for $X(t)=B_H(t)$ given by Norros [19]. However, we think that, as in the Norros paper, the prefactor $u^{-1/2}$ is redundant.

THEOREM 5.3. *Under the assumptions* (i)–(iv), *for* $u \to \infty$

$$P(A(X, c) > u) \geqslant \frac{1}{2\sqrt{\pi\gamma}} u^{-1/2} \exp(-\gamma u) + o(u^{-1/2} \exp(-\gamma u)).$$

Proof. Let $\mathcal{N}$ be a standard Gaussian random variable. We have

$$P(A(X, c) > u) \geqslant \sup_{t \geqslant 0} P((X(t) - ct) > u) = \sup_{t \geqslant 0} P\left(\mathcal{N} > \frac{u+ct}{\sqrt{D^2(X(t))}}\right)$$

$$(5.4) \qquad \geqslant P\left(\mathcal{N} > \frac{2u}{\sqrt{D^2(X(u/c))}}\right)$$

$$\geqslant \frac{1}{\sqrt{2\pi}} \left(\frac{4u^2}{D^2(X(u/c))} - 1\right) \left(\frac{\sqrt{D^2(X(u/c))}}{2u}\right)^3 \exp\left(-\frac{2u^2}{D^2(X(u/c))}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \left[4u^2 - D^2\left(X\left(\frac{u}{c}\right)\right)\right] \frac{\sqrt{D^2(X(u/c))}}{(2u)^3} \exp\left(-\frac{2u^2}{D^2(X(u/c))}\right),$$

where in (5.4) we take $t = u/c$.

From (i)–(iv) we have the expansion (5.3) and

$$\frac{1}{\sqrt{2\pi}} \left[4u^2 - D^2\left(X\left(\frac{u}{c}\right)\right)\right] \frac{\sqrt{D^2(X(u/c))}}{(2u)^3} \exp\left(-\frac{2u^2}{D^2(X(u/c))}\right)$$

$$= \frac{1}{2\sqrt{\pi\gamma}} u^{-1/2} \exp(-\gamma u) + o(u^{-1/2} \exp(-\gamma u))$$

for $u \to \infty$ ∎

Remark 5.3. Note that from Theorems 5.2 and 5.3 we infer that if the assumptions (i)–(iv) hold, then the constant γ in the exponent is asymptotically the best:

$$\lim_{u \to \infty} \frac{\log P(A(X, c) > u)}{u} = -\gamma.$$

APPENDIX

In the Appendix we study the asymptotic of the integral

$$\int_0^\infty \exp(S(x, u))\, dx \qquad \text{for } u \to \infty$$

for particular forms of $S(x, u)$. We need the following notation. Let $x_0(u)$ denote the point at which the function $S(x, u)$ of x achieves its maximum over

$[0, \infty)$. For some suitably chosen function $q(u)$ we put

$$U(x_0(u)) = \{x\colon |x-x_0(u)| \leqslant q(u)\left|S''_{x,x}(x_0(u), u)\right|^{-1/2}\}.$$

We state now the result of Theorem 2.3 and Corollary 2.1 from Fedoryuk [6].

THEOREM A.1. *Suppose that*

(a) *there exists a function* $q(u) \to \infty$ *as* $u \to \infty$ *such that*

$$S''_{x,x}(x, u) = S''_{x,x}(x_0(u), u)[1+o(1)]$$

as $u \to \infty$ *uniformly for* $x \in U(x_0(u))$;

(b) $S''_{x,x}(x, u) < 0$ *for all* x, u;

(c) $\lim_{u\to\infty} x_0(u)\sqrt{|S''_{x,x}(x_0(u), u)|} = \infty$.

Then as $u \to \infty$

$$\int_0^\infty \exp(S(x, u))\,dx \sim \sqrt{-\frac{2\pi}{S''_{x,x}(x_0(u), u)}}\exp(S(x_0(u), u)).$$

We now apply Theorem A.1 to get the following asymptotic:

LEMMA A.1. *For* $0 < H < 1$ *and* $u \to \infty$,

$$\int_0^\infty \frac{1}{t^H}\exp\left(-\frac{(ct+u)^2}{2t^{2H}}\right)dt \sim \frac{\sqrt{2\pi}}{c}\left(\frac{H}{1-H}\right)^{1/2}\exp\left(-\left(\frac{c}{H}\right)^{2H}\left(\frac{1}{1-H}\right)^{2-2H}u^{2-2H}\right).$$

Proof. We apply Theorem A.1 to the equality

$$\int_0^\infty \frac{1}{t^H}\exp\left(-\frac{(ct+u)^2}{2t^{2H}}\right)dt = \frac{1}{1-H}\int_0^\infty \exp\left(-\frac{(ct^{1/(1-H)}+u)^2}{2t^{2H/(1-H)}}\right)dt,$$

where $0 \leqslant H < 1$. We take

$$S(x, u) = -\frac{(cx^{1/(1-H)}+u)^2}{2t^{2H/(1-H)}}.$$

By examination we obtain

- $x_0(u) = [H/(c(1-H))]^{1-H}u^{1-H}$,
- $S''_{x,x}(x_0(u), u) = -c^2/(H(1-H))$.

Hence we infer that condition (c) of Theorem A.1 holds true.

To check condition (a) we choose

$$q(u) = \frac{c}{\sqrt{H(1-H)}}u^{H(1-H)}.$$

We show now that

$$S''_{x,x}(x, u) - S''_{x,x}(x_0(\lambda), \lambda) \to 0$$

as $u \to \infty$ uniformly in

$$x \in \{y: |y - x_0(u)| \leqslant u^{H(1-H)}\}. \tag{A.1}$$

It suffices to check that (1.1) holds for $x = x_0(u) \pm u^{H(1-H)}$ because in this case the difference in (A.1) is the biggest. Computing we get

$$\begin{aligned} S''_{x,x}(x, u) - S''_{x,x}(x_0(u), u) &= -c^2 - \frac{(2H-1)H}{(1-H)^2} c \left[\left(\frac{H}{c(1-H)}\right)^{1-H} + u^{(H-1)(1-H)}\right]^{-1/(1-H)} \\ &\quad - \frac{H(H+1)}{(1-H)^2}\left[\left(\frac{H}{c(1-H)}\right)^{1-H} + u^{(H-1)(1-H)}\right]^{-2/(1-H)} \\ &\quad + \frac{c^2}{H(1-H)} \to 0 \quad \text{as } u \to \infty. \end{aligned}$$

To check condition (b) we evaluate

$$S''_{x,x}(x, u) = -c^2 - \frac{(2H-1)H}{(1-H)^2} cux^{-1/(1-H)} - \frac{(H+1)H}{(1-H)^2} u^2 x^{-2/(1-H)}.$$

Putting $z = x^{-1/(1-H)}$ we obtain the square function

$$-c^2 - \frac{(2H-1)H}{(1-H)^2} cuz - \frac{(H+1)H}{(1-H)^2} u^2 z^2$$

of variable z. By standard calculation it can be checked that for all u and z (so for all x)

$$S''_{x,x}(x, u) = -c^2 - \frac{(2H-1)H}{(1-H)^2} cuz - \frac{(H+1)H}{(1-H)^2} u^2 z^2 < 0$$

and the proof is completed. ∎

Analogously we can prove:

LEMMA A.2. *For* $1/2 \leqslant H < 1$ *and* $u \to \infty$,

$$\frac{c}{H\sqrt{8\pi}} \int_0^\infty \exp\left\{-\frac{1}{2}\left(u\left(\frac{2H}{1-H}\right)^{H/(1-H)} s^{-H/(1-H)} + c\frac{1-H}{2H} s\right)^2\right\} ds \sim \left(\frac{H}{1-H}\right)^{1/2} \exp\left(-\left(\frac{c}{H}\right)^{2H} \left(\frac{1}{1-H}\right)^{2-2H} u^{2-2H}\right).$$

Proof. We need to study the following function:

$$S(x, u) = -\frac{1}{2}\left(u\left(\frac{2H}{1-H}\right)^{H/(1-H)} x^{-H/(1-H)} + c\frac{1-H}{2H} x\right)^2. \tag{A.2}$$

The first and second derivatives of the function S have the following form:

(A.3)
$$S'_x(x, u) = -\left(u\left(\frac{2H}{1-H}\right)^{H/(1-H)} x^{-H/(1-H)} + c\frac{1-H}{2H}x\right) \times\left(-u\left(\frac{2H}{1-H}\right)^{H/(1-H)} \frac{H}{1-H}x^{-1/(1-H)} + c\frac{1-H}{2H}\right),$$

(A.4) $S''_{xx}(x, u)$
$$= -\left(-u\left(\frac{2H}{1-H}\right)^{H/(1-H)} \frac{H}{1-H}x^{-1/(1-H)} + c\frac{1-H}{2H}\right)^2 -\left(u^2\left(\frac{2H}{1-H}\right)^{2H/(1-H)} \frac{H}{(1-H)^2}x^{-2/(1-H)} + u\frac{c}{2(1-H)}\left(\frac{2H}{1-H}\right)^{H/(1-H)} x^{-1/(1-H)}\right).$$

For fixed u the function $S(x, u)$ takes its maximum at the point

(A.5)
$$x_0(u) = 2\left(\frac{u}{c}\right)^{1-H}\left(\frac{H}{1-H}\right)^{2-H},$$

and then

(A.6)
$$S''_{xx}(x_0(u), u) = -c^2\frac{1-H}{4H^3}.$$

Define the function

(A.7)
$$q(u) = u^{H(1-H)}c\sqrt{\frac{1-H}{4H^3}}.$$

Using (A.2)–(A.7) we can check the following conditions:

- the function $q(u) \to \infty$ as $u \to \infty$ and
$$S''_{xx}(x, u) - S''_{xx}(x_0(u), u) \to 0$$
uniformly on the interval $(x_0(u) - u^{H(1-H)}, x_0(u) + u^{H(1-H)})$ as $u \to \infty$;
- for all x and u, $S''_{xx}(x, u) < 0$;
- $\lim_{u\to\infty} x_0(u)\sqrt{|S''_{xx}(x_0(u), u)|} = \infty$.

Now it is sufficient to apply Theorem A.1 in order to complete the proof. ∎

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