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WEAK MARTINGALE HARDY SPACES*

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Abstract. Weak martingale Hardy spaces generated by an operator T are investigated. The concept of weak atoms is introduced and an atomic decomposition of the space wH_p^T is given if the operator T is predictable. Martingale inequalities between weak Hardy spaces generated by two different operators are considered. In particular, we obtain inequalities for the maximal function, for the q-variation, and for the conditional q-variation. The duals of the weak Hardy spaces generated by these special operators are characterized.

1. Introduction. We consider martingale operators and weak martingale Hardy spaces generated by them. The Hardy space H_p^T and the weak Hardy space wH_p^T of martingales are introduced with the L_p-norm and weak wL_p-norm of the maximal operator T^*, respectively. We define also the weak BMO spaces.

The martingale Hardy spaces H_p^T and their atomic decomposition were investigated in Weisz [19]. In this paper, besides the p-atoms a new concept of atoms, the so-called *weak atoms*, is introduced. Then the martingales from wH_p^T $(0 < p < \infty)$ are decomposed into the sum of weak atoms, and an equivalent norm of wH_p^T is also given whenever the operator T is predictable. The atomic decomposition of wH_p^T is also obtained in the case where T is adapted and the stochastic basis is regular. The atomic decomposition of wH_1 was shown by Fefferman and Soria [8] in the classical case.

In Section 5 martingale inequalities are verified. We show that if an inequality holds for a number p, then, by the weak atomic decomposition, it also holds for all parameters less than p. As special operators the maximal operator M, the q-variation S_q, and the conditional q-variation s_q are considered. The weak type Burkholder–Davis–Gundy inequality is obtained from the general results.

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The dual spaces of H_p^T were considered in Garsia [9] ($T = S_2$, $p = 1$), Herz [10], [11] ($T = S_2$, $T = s_2$, $p < \infty$), Lepingle [13] ($T = s_q$, $p = 1$) and Weisz [18], [19] ($T = S_q$, $T = s_q$, $p < \infty$). In Section 6 we extend these results and investigate the duals of the Hardy spaces $\bar{H}_p^T$, the wH_p^T closure of H_∞^T. More exactly, the duals of $\bar{H}_p^M$, $\bar{H}_p^{S_q}$ and $\bar{H}_p^{s_q}$ generated by the maximal operator, q-variation and conditional q-variation are characterized, respectively. It is proved that the dual of $\bar{H}_p^{s_q}$ is $w\mathcal{BMO}_{q'}(\alpha)$ ($0 < p < q$, $1 \leqslant q < \infty$, $\alpha = 1/p-1$, $1/q+1/q' = 1$). Besides these conditions, if the stochastic basis is regular, then the dual of $\bar{H}_p^{S_q}$ is also $w\mathcal{BMO}_{q'}(\alpha)$. The equivalence of the $wBMO_q$ spaces is obtained as well.

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2. Preliminaries and notation. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\mathcal{F} = (\mathcal{F}_n, n \in N)$ be a non-decreasing sequence of σ-algebras. The σ-algebra generated by an arbitrary set system $\mathcal{H}$ will be denoted by $\sigma(\mathcal{H})$. We suppose that $\mathcal{A} = \sigma\big(\bigcup_{n\in N} \mathcal{F}_n\big)$.

The expectation operator and the conditional expectation operators relative to $\mathcal{F}_n$ ($n \in N$) are denoted by E and E_n, respectively. We briefly write L_p instead of the $L_p(\Omega, \mathcal{A}, P)$ space while the norm (or quasinorm) of this space is defined by $\|f\|_p := (E|f|^p)^{1/p}$ ($0 < p \leqslant \infty$). For simplicity, we assume that for a function $f \in L_1$ and for a martingale $f = (f_n, n \in N)$ we have $E_0 f = 0$ and $f_0 = 0$, respectively.

The stochastic basis $\mathcal{F}$ is said to be *regular* if there exists a number $R > 0$ such that for every non-negative and integrable function f

$$E_n f \leqslant R E_{n-1} f \qquad (n \in N).$$

We define $E_{-1} := E_0$. The simplest example for a regular stochastic basis is the sequence of dyadic σ-algebras where $\Omega = [0, 1)$, $\mathcal{A}$ is the σ-algebra of Borel measurable sets, P is the Lebesgue measure and

$$\mathcal{F}_n = \sigma\{[k2^{-n}, (k+1)2^{-n}) \colon 0 \leqslant k < 2^n\}.$$

In this paper the constants C_p are depending only on p and may denote different constants in different contexts.

We define the *martingale differences* as follows:

$$d_0 f := 0, \qquad d_n f := f_n - f_{n-1} \ (n \geqslant 1).$$

The concept of a *stopped martingale* is well known in the martingale theory: if ν is a stopping time (briefly, $\nu \in \mathcal{T}$) and f is a martingale, then the stopped martingale $f^\nu = (f_n^\nu, n \in N)$ is defined by

$$f_n^\nu := \sum_{k=0}^{n} \chi(\nu \geqslant k)\, d_k f,$$

where $\chi(A)$ is the characteristic function of a set A. We know that f_n^ν has the property: $f_n^\nu = f_m$ on the set $\{\nu = m\}$ whenever $n \geqslant m$. Especially, in the case $\nu = n$ $(n \in N)$ we have

$$f^n = (f_0, f_1, \ldots, f_n, f_n, \ldots).$$

We shall consider the following special martingale operators. The *maximal function* of a martingale $f = (f_n, n \in N)$ is denoted by

$$f_n^* := \sup_{k \leqslant n} |f_k|, \qquad f^* := \sup_{k \in N} |f_k|.$$

The *q-variation* $S_q(f)$ and the *conditional q-variation* $s_q(f)$ $(1 \leqslant q < \infty)$ of a martingale f are defined as follows:

$$S_{q,n}(f) := \Big(\sum_{k=0}^{n} |d_k f|^q\Big)^{1/q}, \qquad S_q(f) := \Big(\sum_{k=0}^{\infty} |d_k f|^q\Big)^{1/q},$$

and

$$s_{q,n}(f) := \Big(\sum_{k=0}^{n} E_{k-1} |d_k f|^q\Big)^{1/q}, \qquad s_q(f) := \Big(\sum_{k=0}^{\infty} E_{k-1} |d_k f|^q\Big)^{1/q}$$

while for $q = \infty$ let

$$S_{\infty,n}(f) := s_{\infty,n}(f) := \sup_{k \leqslant n} |d_k f|, \qquad S_\infty(f) := s_\infty(f) := \sup_{k \in N} |d_k f|.$$

Usually the 2-variations are delt with, however in Lepingle [12], [13], Pisier and Xu [15] and Weisz [19] the q-variations are also considered.

Following Burkholder and Gundy [4] we investigate more general martingale operators T that map the set of the martingales stopped by n for any $n \in N$ into the set of non-negative $\mathscr{A}$ measurable functions. Throughout the paper we will assume the following conditions:

(B1) T is subadditive, i.e. if $f = \sum_{k=0}^{\infty} f_k$ in the sense of $f_m = \sum_{k=0}^{\infty} f_{k;m}$ a.e. for all $m \in N$, then

$$T(f^n) \leqslant \sum_{k=0}^{\infty} T(f_k^n) \qquad (n \in N),$$

where f_k $(k \in N)$ are martingales.

(B2) T is homogeneous, i.e. $T(cf) = |c|\, T(f)$.

(B3) T is local, i.e. $T(f) = 0$ on the set $\{s_2(f) = 0\}$.

(B4) T is symmetric, i.e. $T(f) = T(-f)$.

We define, for an arbitrary martingale f,

$$T_n(f) := T(f^n) \ (n \in N), \qquad T^*(f) := \sup_{n \in N} T_n(f).$$

Under these conditions the operator T has some natural properties. For example, $T_0(f) = 0$, $T(f-g) \leqslant T(f) + T(g)$ and $T(f^\mu - f^\nu) = 0$ on the set $\{\mu = \nu\}$.

Moreover, if we set $T_\nu(f) = T_n(f)$ on $\{\nu = n\}$, where $\nu \in \mathscr{T}$ is a finite stopping time, then we have $T_\nu(f) = T(f^\nu)$. It is easy to see that the operator T^* also satisfies all the above conditions. For more details and examples we refer to Burkholder and Gundy [4].

An operator T is said to be *adapted* (respectively, *predictable*) if $T_n(f)$ is $\mathscr{F}_n$ (respectively, $\mathscr{F}_{n-1}$) measurable for all martingales f and for all $n \in N$. If $M(f^n) := |f_n|$, then $M_n^*(f) = f_n^*$ and $M^*(f) = f^*$ $(n \in N)$. One can easily check that the operators M, S_q and s_q $(1 \leqslant q < \infty)$ satisfy the condition (B), moreover, that M and S_q are adapted, and s_q is predictable.

The *predictable operator* of an operator T satisfying (B) is to be introduced. We consider all the non-decreasing, non-negative and predictable sequences $\lambda = (\lambda_n, n \in N)$ of functions for which

$$T_n(f) \leqslant \lambda_n \quad (n \in N).$$

Set

$$T_n^-(f) := \inf_\lambda \lambda_n \ (n \in N), \qquad T^-(f) := \sup_{n \in N} T_n^-(f).$$

One can simply prove that T^- satisfies (B) and is predictable, moreover, that $T_n^-(f)$ is non-decreasing in n. We remark that $T^-(f)$ is not necessarily finite a.e. whenever $T^*(f)$ is finite a.e. Note that T^- was introduced and investigated for the maximal operator by Garsia [9] while for S_2 by Weisz [18].

3. Weak martingale Hardy and *BMO* spaces. The *weak L_p space wL_p* $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{wL_p} := \sup_{y>0} yP(\{|f| > y\})^{1/p} < \infty$$

while we set $wL_\infty = L_\infty$.

The *martingale Hardy space H_p^T* and the *weak martingale Hardy space wH_p^T* $(0 < p \leqslant \infty)$ generated by T denote the space of martingales for which

$$\|f\|_{H_p^T} := \|T^*(f)\|_p < \infty \quad \text{and} \quad \|f\|_{wH_p^T} := \|T^*(f)\|_{wL_p} < \infty,$$

respectively. It is known that

$$H_p^T \subset wH_p^T \ (0 < p \leqslant \infty) \quad \text{and} \quad wH_p^T \subset H_q^T \ (0 < q < p \leqslant \infty).$$

It is interesting to remark that $L_1 \subset wH_1^M, wH_1^{S_2}$ because of the inequalities

$$\|f\|_{wH_1^M} = \sup_{y>0} yP(f^* > y) \leqslant \|f\|_1, \qquad \|f\|_{wH_1^{S_2}} = \sup_{y>0} yP(S_2(f) > y) \leqslant 3\,\|f\|_1$$

(cf. Neveu [14], Burkholder [3]). Moreover, $H_p^M \sim H_p^{S_2}$ for $1 \leqslant p < \infty$ and $H_p^M \sim H_p^{S_2} \sim L_p$ for $1 < p < \infty$, where $\sim$ denotes the equivalence of the spaces and norms (see Neveu [14], Burkholder [3], Davis [7]). Using the interpolation results of Weisz [17] and [19] we can prove that $wH_p^M \sim wH_p^{S_2} \sim wL_p$ $(1 < p < \infty)$.

It is known that the dual of the Hardy space $H_p^{s_2}$ is $BMO_2(\alpha)$ (Herz [10], Weisz [18]) and the dual of $H_p^{s_q}$ is $\mathcal{BMO}_{q'}(\alpha)$ (Lepingle [13], $p = 1$; Weisz [19]) $(0 < p \leqslant 1,\ 1 \leqslant q < \infty,\ 0 \leqslant \alpha = 1/p-1$ and $1/q+1/q' = 1)$, where the BMO spaces are defined with the norms

$$\|f\|_{BMO_q(\alpha)} := \sup_{\nu\in\mathcal{T}} P(\nu \neq \infty)^{-1/q-\alpha} \|f-f^\nu\|_q,$$

$$\|f\|_{\mathcal{BMO}_q(\alpha)} := \sup_{\nu\in\mathcal{T}} P(\nu \neq \infty)^{-1/q-\alpha} \Big[E\Big(\sum_{k=1}^{\infty} |d_k f|^q \chi(\nu < k)\Big)\Big]^{1/q}$$

and

$$\|f\|_{\mathcal{BMO}_\infty(\alpha)} = \sup_{\nu\in\mathcal{T}} P(\nu \neq \infty)^{-\alpha} \sup_{k\in N} \|d_k f\chi(\nu < k)\|_\infty .$$

It is easy to see that

$$\|f\|_{BMO_2(\alpha)} = \|f\|_{\mathcal{BMO}_2(\alpha)}.$$

Let us introduce the weak BMO spaces. Set

$$t_\alpha^q(f, x) := t_\alpha^q(x) := x^{-1/q-\alpha} \sup_{\nu\in\mathcal{T}:P(\nu\neq\infty)\leqslant x} \|f-f^\nu\|_q$$

and

$$u_\alpha^q(f, x) := u_\alpha^q(x) := x^{-1/q-\alpha} \sup_{\nu\in\mathcal{T}:P(\nu\neq\infty)\leqslant x} \Big[E\Big(\sum_{k=1}^{\infty} |d_k f|^q \chi(\nu < k)\Big)\Big]^{1/q},$$

where $1 \leqslant q < \infty$ and $-1/q < \alpha$. For $q = \infty$ and $\alpha > 0$ we define

$$u_\alpha^\infty(f, x) := u_\alpha^\infty(x) := x^{-\alpha} \sup_{\nu\in\mathcal{T}:P(\nu\neq\infty)\leqslant x} \|d_k f\chi(\nu < k)\|_\infty .$$

We say that $f \in wBMO_q(\alpha)$ and $f \in w\mathcal{BMO}_q(\alpha)$ $(-1/q < \alpha)$ if

$$\|f\|_{wBMO_q(\alpha)} := \int_0^\infty \frac{t_\alpha^q(x)}{x}\,dx < \infty \qquad (1 \leqslant q < \infty)$$

and

$$\|f\|_{w\mathcal{BMO}_q(\alpha)} := \int_0^\infty \frac{u_\alpha^q(x)}{x}\,dx < \infty \qquad (1 \leqslant q \leqslant \infty),$$

respectively. Set

$$wBMO_q := wBMO_q(0) \quad \text{and} \quad w\mathcal{BMO}_q := w\mathcal{BMO}_q(0)\ (1 \leqslant q < \infty).$$

Obviously,

(1) $\quad t_\alpha^2(x) = u_\alpha^2(x)$ and $\|f\|_{wBMO_2(\alpha)} = \|f\|_{w\mathcal{BMO}_2(\alpha)}$ $\quad (\alpha > -1/2)$.

It is easy to see that, for $x \geqslant 1$,

$$t_\alpha^q(x) \sim x^{-1/q-\alpha} \|f\|_q \quad \text{and} \quad u_\alpha^q(x) = x^{-1/q-\alpha} \|S_q(f)\|_q.$$

Thus, if $f \neq 0$ and $\alpha \leqslant -1/q$, then $\|f\|_{wBMO_q(\alpha)} = \|f\|_{w\mathscr{BMO}_q(\alpha)} = \infty$ $(1 \leqslant q < \infty)$. However, if $\alpha > -1/q$, then we can write

$$\|f\|_{wBMO_q(\alpha)} \sim \int_0^1 \frac{t_\alpha^q(x)}{x} dx + \frac{\|f\|_q}{1/q+\alpha}$$

and

$$\|f\|_{w\mathscr{BMO}_q(\alpha)} = \int_0^1 \frac{u_\alpha^q(x)}{x} dx + \frac{\|S_q(f)\|_q}{1/q+\alpha}.$$

PROPOSITION 1. *If* $1 \leqslant q_1 < q_2 < \infty$ *and* $\alpha > -1/q_2$, *then*

$$\|f\|_{wBMO_{q_1}(\alpha)} \leqslant \|f\|_{wBMO_{q_2}(\alpha)}.$$

Proof. By Hölder's inequality,

$$t_\alpha^{q_1}(x) = x^{-1/q_1-\alpha} \sup_{\nu \in \mathscr{T}: P(\nu \neq \infty) \leqslant x} \left[E\left(|f-f^\nu|^{q_1} \chi(\nu \neq \infty)\right)\right]^{1/q_1}$$
$$\leqslant x^{-1/q_1-\alpha} \sup_{\nu \in \mathscr{T}: P(\nu \neq \infty) \leqslant x} \left(E|f-f^\nu|^{q_2}\right)^{1/q_2} P(\nu \neq \infty)^{(1-q_1/q_2)(1/q_1)}$$
$$\leqslant x^{-1/q_2-\alpha} \sup_{\nu \in \mathscr{T}: P(\nu \neq \infty) \leqslant x} \|f-f^\nu\|_{q_2} = t_\alpha^{q_2}(x),$$

which shows the proposition. ■

4. Weak atomic decomposition. The atomic decomposition is a useful characterization of Hardy spaces used in proving some duality theorems and martingale inequalities (see e.g. Coifman and Weiss [6] and Weisz [17]). Let us introduce first the concepts of atoms. A martingale a is a *p-atom relative to an operator* T if there exists a stopping time ν such that

(i) $a_n = 0$ if $\nu \geqslant n$;

(ii) $\|T^*(a)\|_\infty \leqslant P(\nu \neq \infty)^{-1/p}$.

It is proved by the author [19] that the martingales from H_p^T $(0 < p \leqslant 1)$ can be decomposed into the sum of p-atoms whenever T is predictable. Special cases of this atomic decomposition can be found in Bernard and Maisonneuve [2], Chevalier [5] and Herz [10]. To give the atomic decomposition of the weak Hardy spaces let us define the concept of weak atoms. A martingale a is a *weak atom relative to an operator* T if there exists a stopping time ν such that (i) is satisfied and

(ii′) $\|T^*(a)\|_\infty < \infty$

holds. The atomic decomposition of wH_p^T is stated as follows:

THEOREM 1. *Assume that* T *is a predictable operator. A martingale* $f = (f_n, n \in N)$ *is in* wH_p^T $(0 < p < \infty)$ *if and only if there exists a sequence*

$(a_k,\ k \in \mathbf{Z})$ of weak atoms relative to T with the corresponding stopping times ν_k such that

(i) $$\sum_{k=-\infty}^{\infty} a_{k;n} = f_n \quad \text{for all } n \in \mathbf{N},$$

(ii) $$\sup_{k \in \mathbf{Z}} 2^{kp} P(\nu_k \neq \infty) < \infty,$$

(iii) $$T^*(a_k) \leqslant A 2^k, \quad \text{where } A \text{ is an absolute constant.}$$

Moreover, the following equivalence of norms holds:

(2) $$\|f\|_{wH_p^T} \sim \inf \sup_{k \in \mathbf{Z}} 2^k P(\nu_k \neq \infty)^{1/p},$$

where the infimum is taken over all preceding decompositions of f.

Proof. The first half of the proof will be sketched only. Assume that $f \in wH_p$ $(0 < p < \infty)$. Let us define the stopping times

$$\nu_k := \inf\{n \in \mathbf{N}\colon\ T_{n+1}^*(f) > 2^k\}$$

and martingales

(3) $$a_k := f^{\nu_{k+1}} - f^{\nu_k} \qquad (k \in \mathbf{Z}).$$

Since T is local, we have $T^*(a^k) \leqslant 3 \cdot 2^k$; thus a^k (with the stopping time ν_k) is a weak atom for each $k \in \mathbf{Z}$. It is easy to see that (i) holds (cf. also Weisz [19], p. 44). There are no convergence problems in (i), because only finitely many terms (depending on ω) are non-zero. As $\{\nu_k \neq \infty\} = \{T^*(f) > 2^k\}$, by the definition we have

$$2^{kp} P(\nu_k \neq \infty) = 2^{kp} P\big(T^*(f) > 2^k\big) \leqslant \|f\|_{wH_p^T}^p,$$

which proves (ii) and one side of (2).

Conversely, suppose that (i), (ii) and (iii) are satisfied and let

$$D := \sup_{k \in \mathbf{Z}} 2^{kp} P(\nu_k \neq \infty).$$

For a fixed $y > 0$ choose $j \in \mathbf{Z}$ such that $2^j \leqslant y < 2^{j+1}$. Then

$$f_n = \sum_{k=-\infty}^{j-1} a_{k;n} + \sum_{k=j}^{\infty} a_{k;n} =: g_n + h_n \qquad (n \in \mathbf{N})$$

implies that $T^*(f) \leqslant T^*(g) + T^*(h)$ and

$$P\big(T^*(f) > 2Ay\big) \leqslant P\big(T^*(g) > Ay\big) + P\big(T^*(h) > Ay\big).$$

From the inequality

$$T^*(g) \leqslant \sum_{k=-\infty}^{j-1} T^*(a_k) \leqslant A2^j$$

we get

$$P\big(T^*(g) > Ay\big) \leqslant P\big(T^*(g) > A2^j\big) = 0.$$

It follows from the definitions that

$$T^*(a_k) = T^*(a_k - a_k^{\nu_k}) = 0 \quad \text{on } \{\nu_k = \infty\}. \tag{4}$$

This and the inequality $T^*(h) \leqslant \sum_{k=j}^{\infty} T^*(a_k)$ imply that

$$\{T^*(h) \neq 0\} \subset \bigcup_{k=j}^{\infty} \{\nu_k \neq \infty\}.$$

Consequently,

$$P\big(T^*(f) > 2Ay\big) \leqslant P\big(T^*(h) > Ay\big) \leqslant P\big(T^*(h) > 0\big) \leqslant \sum_{k=j}^{\infty} P(\nu_k \neq \infty)$$

$$\leqslant \sum_{k=j}^{\infty} D2^{-kp} \leqslant C_p D2^{-jp} \leqslant C_p D y^{-p},$$

which shows that $\|f\|_{wH_p^T}^p \leqslant C_p D$. The proof of the theorem is complete. ∎

Note that the definitions of a_k and ν_k $(k \in Z)$ in the first part of the proof of Theorem 1 are independent of p.

With the usual method we can extend Theorem 1 to adapted operators in the case where $\mathscr{F}$ is regular.

THEOREM 2. *If T is adapted and $\mathscr{F}$ is regular, then Theorem* 1 *holds as well.*

Proof. Let

$$\tau_k := \inf\{n \in N:\ T_n^*(f) > 2^k\}$$

and

$$F_n^k := \{E_{n-1}(\chi(\tau_k = n)) \geqslant 1/R\},$$

where R is the regularity constant. It is clear that $F_n^k \in \mathscr{F}_{n-1}$ and, by the regularity of $\mathscr{F}$, $F_n^k \supset \{\tau_k = n\}$. Define

$$\nu_k(\omega) := \inf\{n \in N:\ \omega \in F_{n+1}^k\}.$$

Then $\{\tau_k(\omega) = n\}$ implies $\omega \in F_n^k$, which yields $\{\nu_k(\omega) \leqslant n-1\}$. In other words, $\nu_k < \tau_k$ on the set $\{\tau_k \neq \infty\}$. This implies that if a_k is defined again by (3), then $T^*(a_k) \leqslant 3 \cdot 2^k$. By Chebyshev's inequality we obtain

$$P(F_n^k) \leqslant RE\big[E_{n-1}(\chi(\tau_k = n))\big] = RP(\tau_k = n).$$

Hence

$$P(\nu_k \neq \infty) \leqslant \sum_{n=1}^{\infty} P(F_n^k) \leqslant \sum_{n=1}^{\infty} P(\tau_k = n) = RP(\tau_k \neq \infty) = RP\big(T^*(f) > 2^k\big).$$

The proof can be completed in the same way as in Theorem 1. ∎

5. Martingale inequalities. In this section the connection of the weak martingale Hardy spaces is investigated. The idea of the method is the following. If a strong inequality holds for a number p, then by the atomic decomposition we can verify its weak version for all parameters less than p. We single out the results for some special operators. As a consequence the weak version of the well-known Burkholder–Davis–Gundy inequality is obtained.

THEOREM 3. *Assume that T is predictable and U is adapted, moreover, that there exists $0 < p_1 \leqslant \infty$ such that for all martingales f*

$$\|U^*(f)\|_{p_1} \leqslant C\,\|T^*(f)\|_{p_1}. \tag{5}$$

Then

$$\|f\|_{wH_p^U} \leqslant C_p\,\|f\|_{wH_p^T} \qquad (0 < p < p_1).$$

Proof. Taking the atomic decomposition and the martingales g and h given in the proof of Theorem 1 we get $U^*(f) \leqslant U^*(g) + U^*(h)$ and

$$P\big(U^*(f) > 2y\big) \leqslant P\big(U^*(g) > y\big) + P\big(U^*(h) > y\big).$$

By (iii) of Theorem 1, (2) and (5),

$$\|U^*(g)\|_{p_1} \leqslant \sum_{k=-\infty}^{j-1} \|U^*(a_k)\|_{p_1} \leqslant C \sum_{k=-\infty}^{j-1} \|T^*(a_k)\|_{p_1} \leqslant 3C \sum_{k=-\infty}^{j-1} 2^k P(\nu_k \neq \infty)^{1/p_1}$$

$$\leqslant C \sum_{k=-\infty}^{j-1} 2^{k(1-p/p_1)}\,\|f\|_{wH_p^T}^{p/p_1} \leqslant C_p\, y^{1-p/p_1}\,\|f\|_{wH_p^T}^{p/p_1},$$

where $2^j \leqslant y < 2^{j+1}$. Hence

$$P\big(U^*(g) > y\big) \leqslant y^{-p_1} E\,|U^*(g)|^{p_1} \leqslant C_p\, y^{-p}\,\|f\|_{wH_p^T}^{p}. \tag{6}$$

On the other hand, Theorem 1 and (4) for the operator U^* imply

$$P\big(U^*(h) > y\big) \leqslant P\big(U^*(h) > 0\big) \leqslant \sum_{k=j}^{\infty} P\big(U^*(a_k) > 0\big)$$

$$\leqslant \sum_{k=j}^{\infty} P(\nu_k \neq \infty) \leqslant C_p\, y^{-p}\,\|f\|_{wH_p^T}^{p}.$$

This and (6) show the theorem. ∎

The equivalence $H_\infty^T \sim H_\infty^{T^-}$ and the inequality

$$\|f\|_{(w)H_p^T} \leqslant \|f\|_{(w)H_p^{T^-}} \quad (0<p<\infty) \tag{7}$$

are clear from the definition. The next result is a consequence of Theorem 3 and (7).

COROLLARY 1. *Assume that U and T are adapted operators, moreover, that there exists $0<p_1\leqslant\infty$ such that (5) holds. Then*

$$\|f\|_{wH_p^U} \leqslant C_p\|f\|_{wH_p^{T^-}} \quad (0<p<p_1).$$

By Theorem 2, for a regular stochastic basis we can omit the predictability of T in Theorem 3.

COROLLARY 2. *Let $\mathcal{F}$ be regular. Assume that U and T are adapted and for all martingales f*

$$\|U^*(f)\|_{p_1} \sim \|T^*(f)\|_{p_1}, \quad \text{where } 0<p_1\leqslant\infty.$$

Then

$$\|f\|_{wH_p^U} \sim \|f\|_{wH_p^T} \quad (0<p<p_1).$$

Consequently, we obtain

COROLLARY 3. *If $\mathcal{F}$ is regular and T is adapted, then $wH_p^T \sim wH_p^{T^-}$ $(0<p\leqslant\infty)$.*

Now we consider the quadratic variations and the maximal operator.

PROPOSITION 2. *If $1\leqslant q<\infty$, then*

$$\|f\|_{wH_p^{S_q}} \leqslant C_p\|f\|_{wH_p^{s_q}} \quad (0<p<q),$$

$$\|f\|_{wH_p^{s_q}} \leqslant C_p\|f\|_{wH_p^{S_q}} \quad (q<p<\infty),$$

$$\|f\|_{wH_p^M} \leqslant C_p\|f\|_{wH_p^{s_q}} \quad (0<p<q\leqslant 2).$$

Proof. The first and third inequalities follow from

$$\|S_q(f)\|_q = \|s_q(f)\|_q \quad (1\leqslant q<\infty), \tag{8}$$

$$\|f^*\|_q \leqslant C_q\,\|s_q(f)\|_q \quad (1\leqslant q\leqslant 2) \tag{9}$$

and from Theorem 3. Note that (9) is due to Lepingle [13]. The second inequality of Proposition 2 comes from the concavity lemma (cf. Garsia [9]). ∎

COROLLARY 4. *If $\mathcal{F}$ is regular and $1\leqslant q<\infty$, then $wH_p^{S_q} \sim wH_p^{s_q}$ $(0<p<\infty,\ p\neq q)$ and $wH_p^M \sim wH_p^{S_2}$ $(0<p<\infty)$.*

Proof. The equivalence $wH_p^{S_q} \sim wH_p^{s_q}$ for $q<p<\infty$ comes easily from the regularity and from the second inequality of Proposition 2. If $0<p<q$,

then the equivalence follows from (8) and Corollary 2. The Burkholder–Davis–Gundy inequality $H_p^M \sim H_p^{S_2}$ and Corollary 2 imply $wH_p^M \sim wH_p^{S_2}$ $(0 < p < \infty)$. ∎

6. Duality results. The dual spaces of the weak Hardy spaces generated by the operators S_q and s_q are going to be characterized. The spaces L_p or L_∞ are not dense in wL_p $(0 < p < \infty)$. A characterization of the wL_p closure of L_∞ can be found on p. 47 of Bergh and Löfström [1]. Then the space H_p^T is not necessarily dense in wH_p^T (cf. also Fefferman and Soria [8]), so we take its closure. More exactly, let $\bar{H}_p^T$ be the wH_p^T closure of H_∞^T.

THEOREM 4. *The dual of* $\bar{H}_p^{s_q}$ *is* $w\mathcal{BMO}_{q'}(\alpha)$, *where* $0 < p < q$, $1 \leqslant q < \infty$, $\alpha = 1/p - 1$ *and* $1/q + 1/q' = 1$.

Proof. Since $H_\infty^{s_q}$ is dense in $H_p^{s_q}$ and $H_q^{S_q}$ is dense in $H_p^{s_q}$ $(p < q)$ (see Weisz [19]), we infer that $H_p^{s_q}$ and $H_p^{S_q}$ are also dense in $\bar{H}_p^{s_q}$. Let $g \in w\mathcal{BMO}_{q'}(\alpha)$; then $g \in H_{q'}^{S_{q'}}$. Define the linear functional l_g by

$$l_g(f) = E\Big(\sum_{k=1}^{\infty} d_k f\, d_k g\Big) \qquad (f \in H_q^{S_q}). \tag{10}$$

It is clear that

$$d_k f = \sum_{l=-\infty}^{\infty} d_k a_l \text{ a.e.}$$

for all $k \in N$, where the weak atoms a_l are the same as in Theorem 1. Moreover, it was proved in Weisz [19] that the last series converges to $d_k f$ also in $H_q^{s_q}$-norm. Hence

$$l_g(f) = \sum_{k=1}^{\infty} \sum_{l=-\infty}^{\infty} E(d_k a_l\, d_k g).$$

Applying the identity $a_{l;n} = a_{l;n}\chi(\nu_l < n)$ and Hölder's inequality we get

$$|l_g(f)| \leqslant \sum_{l=-\infty}^{\infty} E\Big(\sum_{k=1}^{\infty} |d_k a_l|\, \chi(\nu_l < k)\, |d_k g|\Big)$$
$$\leqslant \sum_{l=-\infty}^{\infty} \Big(E \sum_{k=1}^{\infty} |d_k a_l|^q\Big)^{1/q} \Big(E \sum_{k=1}^{\infty} |d_k g|^{q'} \chi(\nu_l < k)\Big)^{1/q'}.$$

By (iii) of Theorem 1, $s_q(a_l) \leqslant 3 \cdot 2^l$ $(l \in \boldsymbol{Z})$. Thus

$$\Big(E \sum_{k=1}^{\infty} |d_k a_l|^q\Big)^{1/q} = \big[E\big(s_q^q(a_l)\big)\big]^{1/q} \leqslant 3 \cdot 2^l P(\nu_l \neq \infty)^{1/q}.$$

Using (2) we have

$$P(\nu_l \neq \infty) \leqslant 2^{-lp} \|f\|_{wH_p^{s_q}}^p.$$

Therefore

$$|l_g(f)| \leqslant 3 \sum_{l=-\infty}^{\infty} (2^{-lp} \|f\|^p_{wH_p^{sq}})^{-1/q'+1-1/p} \|f\|_{wH_p^{sq}} \Big(E \sum_{k=1}^{\infty} |d_k g|^{q'} \chi(\nu_l < k)\Big)^{1/q'}$$

$$\leqslant 3 \|f\|_{wH_p^{sq}} \sum_{l=-\infty}^{\infty} u_\alpha^{q'}(B2^{-lp}),$$

where $B = \|f\|^p_{wH_p^{sq}}$. Since

$$x_1^{1/q'+\alpha} u_\alpha^{q'}(x_1) \leqslant x_2^{1/q'+\alpha} u_\alpha^{q'}(x_2) \qquad (x_1 < x_2),$$

we can show that

$$\text{(1)} \qquad \sum_{l=-\infty}^{\infty} u_\alpha^{q'}(B2^{-lp}) \sim \int_0^{\infty} \frac{u_\alpha^{q'}(x)}{x} dx \qquad (B > 0).$$

This implies

$$|l_g(f)| \leqslant C_{p,q} \|f\|_{wH_p^{sq}} \|g\|_{w\mathcal{BMO}_{q'}} \qquad (f \in H_q^{S_q}).$$

Conversely, if l is in the dual of $\bar{H}_p^{s_q}$, then it is also in the dual of $H_q^{S_q}$. Henceforth, there exists $g \in H_{q'}^{S_{q'}}$ such that l has the form (10) for all $f \in H_q^{S_q}$. Let ν_l be stopping times satisfying

$$\text{(12)} \qquad P(\nu_l \neq \infty) \leqslant 2^{-lp} \qquad (l \in \boldsymbol{Z})$$

and let

$$a_l := \frac{\sum_{k=1}^{\infty} \big[|d_k g|^{q'-1} \operatorname{sign}(d_k g) - E_{k-1}\big(|d_k g|^{q'-1} \operatorname{sign}(d_k g)\big)\big] \chi(\nu_l < k)}{2^{-lp(1/p-1/q)} \big(E \sum_{k=1}^{\infty} |d_k g|^{q'} \chi(\nu_l < k)\big)^{1/q}}.$$

If the denominator is 0, then let $a_l = 0$. a_l is not necessarily a weak atom, however (i) holds, namely, $a_{l;n} = 0$ on the set $\{\nu_l \geqslant n\}$. Let the martingales f_N, g_N and h_N $(N \in \boldsymbol{N})$ be defined by

$$\text{(13)} \qquad f_{N;n} := \sum_{l=-N}^{N} a_{l;n} = \sum_{l=-N}^{j-1} a_{l;n} + \sum_{l=j}^{N} a_{l;n} =: g_{N;n} + h_{N;n} \qquad (n \in \boldsymbol{N}),$$

where $2^j \leqslant y < 2^{j+1}$ for a fixed $y > 0$. We will show that

$$\text{(14)} \qquad \|f_N\|_{wH_p^{sq}} \leqslant C_p \qquad (N \in \boldsymbol{N}),$$

where C_p is independent of N. As in Theorem 3,

$$\|s_q(g_N)\|_q \leqslant \sum_{l=-N}^{j-1} \|S_q(a_l)\|_q \leqslant 2 \sum_{l=-N}^{j-1} 2^{-lp(1/q-1/p)} \leqslant C_p y^{1-p/q}$$

and

$$P\big(s_q(g_N) > y\big) \leqslant y^{-q} E\big(s_q^q(g_N)\big) \leqslant C_p y^{-p}.$$

The inequality

(15) $$P(s_q(h_N) > y) \leqslant C_p y^{-p}$$

can be verified in the same way as in Theorem 3. Thus we have proved (14).

Hence

$$C_p \|l\| \geqslant |l(f_N)| = \frac{\sum_{l=-N}^{N} \sum_{k=1}^{\infty} E(|d_k g|^{q'} \chi(v_l < k))}{2^{-lp(1/p-1/q)} (E \sum_{k=1}^{\infty} |d_k g|^{q'} \chi(v_l < k))^{1/q}}$$
$$= \sum_{l=-N}^{N} (E \sum_{k=1}^{\infty} |d_k g|^{q'} \chi(v_l < k))^{1/q'} 2^{-lp(1/q-1/p)}.$$

Taking the supremum over all stopping times satisfying (12) and over all $N \in N$ and using (11) we obtain

$$\|l\| \geqslant C_p \sum_{l=-\infty}^{\infty} u_\alpha^{q'}(2^{-lp}) \geqslant C_p \|g\|_{w\mathscr{B}\mathscr{M}\mathscr{O}_{q'}(\alpha)}.$$

With a slight modification the theorem can also be shown in the case $q = 1$. ∎

Taking into account Corollary 4 and (1) we can point out the next results.

Corollary 5. *If $\mathscr{F}$ is regular, then the dual of $\bar{H}_p^{S_q}$ is $w\mathscr{B}\mathscr{M}\mathscr{O}_{q'}(\alpha)$, where $0 < p < q$, $1 \leqslant q < \infty$, $\alpha = 1/p-1$, and $1/q+1/q' = 1$.*

Corollary 6. *The dual of $\bar{H}_p^{s_2}$ is $wBMO_2(\alpha)$, where $0 < p < 2$ and $\alpha = 1/p-1$.*

Now we consider the weak Hardy spaces generated by the maximal operator.

Theorem 5. *$wBMO_1(\alpha)$ is equivalent to a subspace of the dual of $\bar{H}_p^{M^-}$, where $0 < p < \infty$ and $\alpha = 1/p-1$.*

Proof. Let $g \in wBMO_1(\alpha)$ and define l_g for $f \in L_\infty$ again by (10). Then

$$|l_g(f)| = |E(fg)| = \Big|\sum_{l=-\infty}^{\infty} E(a_l g)\Big| = \Big|\sum_{l=-\infty}^{\infty} E(a_l(g-g^{v_l}))\Big|$$
$$\leqslant 3\|f\|_{wH_p^{M^-}} \sum_{l=-\infty}^{\infty} (2^{-lp} \|f\|_{wH_p^{M^-}}^p)^{-1/p} \|g-g^{v_l}\|_1$$
$$\leqslant 3\|f\|_{wH_p^{M^-}} \sum_{l=-\infty}^{\infty} t_\alpha^1(B2^{-lp}) \leqslant C_p \|f\|_{wH_p^{M^-}} \|g\|_{wBMO_1(\alpha)},$$

where we used (11) for t_α instead of u_α.

Conversely, suppose that there exists $g \in L_1$ such that the bounded linear functional l is of the form (10) for all $f \in L_\infty$. Let

$$a_l := 2^l(h_l - h_l^{v_l}) \qquad (l \in N),$$

where the stopping times ν_l satisfy (12) and $h_l := \operatorname{sign}(g-g^{\nu_l})$. It is easy to see that each a_l ($l \in N$) is a weak atom with respect to M^-. Thus, by Theorem 1, if f_N is again defined by (13), then $\|f_N\|_{wH_p^{M^-}} \leqslant C_p$ ($N \in N$). Therefore

$$C_p \|l\| \geqslant |l(f_N)| = |E(f_N g)| = \Big|\sum_{l=-N}^{N} E(a_l g)\Big| = \Big|\sum_{l=-N}^{N} 2^l E\big((h_l - h_l^{\nu_l}) g\big)\Big|$$

$$= \Big|\sum_{l=-N}^{N} 2^l E\big(h_l (g-g^{\nu_l})\big)\Big| = \sum_{l=-N}^{N} 2^{-lp(-1/p)} \|g-g^{\nu_l}\|_1.$$

As above we derive

$$\|l\| \geqslant C_p \sum_{l=-\infty}^{\infty} t_\alpha^1 (2^{-lp}) \geqslant C_p \|g\|_{wBMO_1(\alpha)},$$

which shows the theorem. ∎

By the duality between L_q and $L_{q'}$ ($1 < q < \infty$) we can infer that for every linear functional l on $\bar{H}_p^M$ there exists $g \in L_1$ such that l has the form (10). Corollaries 3 and 4 imply

Corollary 7. *If $\mathscr{F}$ is regular, then the dual of $\bar{H}_p^M$ and $\bar{H}_p^{S_2}$ is $wBMO_1(\alpha)$, where $0 < p < \infty$ and $\alpha = 1/p - 1$.*

Remark. It follows from the equivalence $wL_p \sim wH_p^M$ ($1 < p < \infty$) that if $\mathscr{F}$ is regular, then the dual of the wL_p closure of L_∞ is $wBMO_1(\alpha)$ ($1 < p < \infty$, $\alpha = 1/p - 1$).

Now we extend Corollary 7.

Theorem 6. *Suppose that $\mathscr{F}$ is regular and $1 \leqslant q < \infty$, $0 < p < q/(q-1)$ and $\alpha = 1/p - 1$. Then the dual of $\bar{H}_p^M$ and $\bar{H}_p^{S_2}$ is $wBMO_q(\alpha)$.*

Proof. By Proposition 1 and Theorem 5 we have

$$|l_g(f)| = |E(fg)| \leqslant C_p \|f\|_{wH_p^M} \|g\|_{wBMO_1(\alpha)} \leqslant C_p \|f\|_{wH_p^M} \|g\|_{wBMO_q(\alpha)},$$

where $g \in wBMO_q(\alpha)$ and $f \in L_{q'}$ ($1/q + 1/q' = 1$).

Conversely, since $p < q'$, there exists $g \in L_q$ such that the bounded linear functional l equals l_g on $L_{q'}$. Let

$$a_l := 2^{-lp(-1/q+1-1/p)} (h_l - h_l^{\nu_l}) \qquad (l \in N),$$

where the stopping times ν_l satisfy (12) and

$$h_l := \frac{|g-g^{\nu_l}|^{q-1} \operatorname{sign}(g-g^{\nu_l})}{\|g-g^{\nu_l}\|_q^{q-1}}.$$

Of course, $\|h_l\|_{q'} = 1$. Define f_N, g_N and h_N ($N \in N$) again by (13). Then

$$\|g_N^*\|_{q'} \leqslant \|g_N\|_{q'} \leqslant \sum_{l=-N}^{j-1} \|a_l\|_{q'} \leqslant 2 \sum_{l=-N}^{j-1} 2^{l(1-p+p/q)} \leqslant C_p y^{1-p+p/q}$$

and

$$P(g_N^* > y) \leqslant y^{-q'} E(g_N^{*q'}) \leqslant C_p y^{-p}.$$

Applying the analogue of (15) we can conclude that $\|f_N\|_{wH_p^M} \leqslant C_p$ $(N \in N)$. Consequently,

$$C_p \|l\| \geqslant |l(f_N)| = \Big|\sum_{l=-N}^{N} E(a_l g)\Big| = \Big|\sum_{l=-N}^{N} 2^{-lp(-1/q+1-1/p)} E\big(h_l(g-g^{\nu_l})\big)\Big|$$

$$= \sum_{l=-N}^{N} 2^{-lp(-1/q+1-1/p)} \|g-g^{\nu_l}\|_q,$$

and hence

$$\|l\| \geqslant C_p \sum_{l=-\infty}^{\infty} t_\alpha^q(2^{-lp}) \geqslant C_p \|g\|_{wBMO_q(\alpha)}.$$

The proof of the theorem is complete. ∎

Now we formulate the weak version of the John–Nirenberg theorem.

COROLLARY 8. *Suppose that $\mathscr{F}$ is regular and $1 \leqslant q < \infty$. If $-1 < \alpha q$ for a fixed α, then the $wBMO_q(\alpha)$ spaces are all equivalent. In particular, the $wBMO_q$ spaces are all equivalent if $1 \leqslant q < \infty$.*

Finally, some martingale inequalities are formulated.

COROLLARY 9. *We have*

$$c\|g\|_{\mathscr{BMO}_{q'}(\alpha)} \leqslant \|g\|_{w\mathscr{BMO}_{q'}(\alpha)} \leqslant C\|g\|_{\mathscr{BMO}_{q'}(\beta)} \qquad (1 < q' \leqslant \infty,\ 0 \leqslant \alpha < \beta)$$

and

$$c\|g\|_{BMO_1(\alpha)} \leqslant \|g\|_{wBMO_1(\alpha)} \leqslant C\|g\|_{BMO_1(\beta)} \qquad (0 \leqslant \alpha < \beta).$$

Proof. It is proved in Weisz [19], [18] that the dual of $H_p^{s_q}$ is $\mathscr{BMO}_{q'}(\alpha)$ and that $BMO_1(\alpha)$ is equivalent to a subspace of the dual of $H_p^{M^-}$, where $0 < p \leqslant 1$, $1 \leqslant q < \infty$, $\alpha = 1/p - 1$ and $1/q + 1/q' = 1$. The corollary follows from Theorems 4 and 5 and from the inequalities

$$\|f\|_{H_r^{s_q}} \leqslant \|f\|_{wH_p^{s_q}} \leqslant \|f\|_{H_p^{s_q}} \qquad (0 < r < p),$$

$$\|f\|_{H_r^{M^-}} \leqslant \|f\|_{wH_p^{M^-}} \leqslant \|f\|_{H_p^{M^-}} \qquad (0 < r < p). \ ∎$$

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