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TIME DEPENDENT MALLIAVIN CALCULUS ON MANIFOLDS AND APPLICATION TO NONLINEAR FILTERING

BY

JANG SCHILTZ (METZ)

Abstract. In this paper, we prove, using Malliavin calculus, that under a global Hörmander condition the law of a Riemannian manifold valued stochastic process, a solution of a stochastic differential equation with time dependent coefficients, admits a $\mathscr{C}^\infty$-density with respect to the Riemannian volume element. This result is applied to a nonlinear filtering problem with time dependent coefficients on manifolds.

1. Introduction. The purpose of this paper is to investigate the regularity of the probability law of the image of a time dependent diffusion process on manifolds through a $\mathscr{C}^\infty$-mapping. We suppose that the local coordinates of the diffusion coefficients are Hölder continuous in the time variable and smooth in the space variable. We prove that under a global Hörmander condition the solution of such an equation admits a $\mathscr{C}^\infty$-density with respect to the Riemannian volume element. This is an improvement of the results of Tanigushi [17] in which the coefficients of the stochastic differential equation are not supposed to be time dependent.

The results are used to prove that the filter associated with some nonlinear filtering problem with manifold-valued time dependent system and time dependent observation process admits a smooth density with respect to the Riemannian volume element.

The development of the stochastic analysis in order to give a stochastic proof of Hörmander's theorem has been initiated by Malliavin [11], then continued and precised by Stroock [16], Bismut [2], Norris [13], Nualart [14] and Zakai [19]. Let us notice that Chaleyat-Maurel and Michel [3] have proved a similar result for continuous coefficients under a global Hörmander condition, by means of partial differential equations techniques.

The application of the Malliavin calculus to stochastic differential equations with time depending coefficients has been used by Kusuoka and Stroock [9] for an elliptic system with bounded coefficients, then Florchinger [6] showed the existence of a smooth density for a diffusion with time depending coefficients under a local Hörmander condition.

Nonlinear filtering problems where the observation process evolves on a Riemannian manifold have been studied by Duncan [4] and Pontier and Szpirglas [15]. On the other hand, Ng and Caines [12] gave a general formulation of the nonlinear filtering problem when the system process and the observation process are both with values on a Riemannian manifold. A Bayes formula for the conditional expectation of smooth functions of the system process is proved. Furthermore, they proved that the density of the filter, provided it exists, verifies a Zakai equation (cf. [18]).

In [7], Florchinger has proved, by means of Malliavin calculus, that the filter associated with a nonlinear filtering problem on Riemannian manifolds admits a smooth density.

This paper is divided in four sections organized as follows. In the first section we recall some results of the stochastic calculus of variations, that we will need later on. The aim of the second section is to prove that some time depending differential equations on manifolds admit a unique solution under our working hypotheses. In the third section we prove that under these conditions our manifold-valued stochastic diffusion process is infinitely differentiable and we compute its Malliavin derivative. Furthermore, we prove that under a global Hörmander condition its law admits a smooth density with respect to the Riemannian volume element. In the fourth section, we apply the previous results to prove the existence of a $\mathscr{C}^\infty$-density of the filter of a nonlinear filtering problem with time depending coefficients on Riemannian manifolds.

1. Some stochastic calculus of variations in Euclidean spaces. In this section we describe some results of Malliavin calculus in $\boldsymbol{R}^n$ that we will need in the sequel. We use the notation of Nualart's book on Malliavin calculus [14]. More bibliographical references on this subject may be found therein.

Let $(W, \mathscr{F}, P)$ be a d-dimensional standard Wiener space, i.e. W is the Banach space $\mathscr{C}([0, T], \boldsymbol{R}^d)$ such that $w(0) = 0$ for any w in W, equipped with the norm $\|w\| = \max_{t\in[0,T]} |w(t)|$, P is the standard Wiener measure, and $\mathscr{F}$ the completion of the Borel σ-algebra on W with respect to the measure P.

Let H be the subspace of W consisting of all functions h such that each component $h^\alpha(t)$ of $h(t)$ is absolutely continuous and admits a square integrable derivative $\dot{h}^\alpha(t)$. H is then a Hilbert space with the inner product

$$\langle h, g\rangle_H = \sum_{\alpha=1}^{d} \int_0^T \dot{h}^\alpha(t)\dot{g}^\alpha(t)\,dt, \qquad h, g \in H. \tag{1.1}$$

We will call a *smooth functional* on the Wiener space $(W, \mathscr{F}, P)$ any random variable $F\colon W \to \boldsymbol{R}$ of the form

$$F(w) = f\big(w(t_1), \ldots, w(t_p)\big), \tag{1.2}$$

where f is a function in $\mathscr{C}_b^\infty(\boldsymbol{R}^{d\times p}, \boldsymbol{R})$ (the space of all bounded $\mathscr{C}^\infty$-functions $f\colon \boldsymbol{R}^{d\times p} \to \boldsymbol{R}$ with bounded derivatives of all orders) and $t_1, \ldots, t_p$ are in $[0, T]$. We denote the space of all smooth functionals on the Wiener space by $\mathscr{S}$.

The *stochastic gradient* of a smooth functional F on the Wiener space is the random function DF with values in the Hilbert space $L^2([0, T]; \boldsymbol{R}^d)$ defined for any t in $[0, T]$ and any j in $\{1, \ldots, d\}$ by

$$(D_{j,t}F)(w) = \sum_{k=1}^{p} \frac{\partial f}{\partial x^{jk}} \big(w(t_1), \ldots, w(t_p)\big) \mathbf{1}_{[0,t_k]}(t). \tag{1.3}$$

Iterating formula (1.3), we define the stochastic gradient of order N of a smooth functional F as the random function $D^N f$ with values in the Hilbert space $L^2([0, T]^N; \boldsymbol{R}^d)$ expressed for all $s_1, \ldots, s_N$ in $[0, T]$ as

$$D^N_{s_1,\ldots,s_N} F = D_{s_1} \ldots D_{s_N} F. \tag{1.4}$$

We introduce the generalized Sobolev spaces of smooth functionals in the following manner.

For any integer $N \geqslant 1$ and any real number $p > 1$, denote by $\|\cdot\|_{N,p}$ the semi-norm on the space $\mathscr{S}$ defined by

$$\|F\|^p_{N,p} = \|F\|^p_p + \big\| \|D^N F\|_{HS} \big\|^p_p, \tag{1.5}$$

where $\|D^N F\|_{HS}$ is the Hilbert–Schmidt norm of $D^N F$, i.e.

$$\|D^N F\|^2_{HS} = \sum_{j_1,\ldots,j_N=1}^{d} \int_{[0,T]^N} (D^N_{(j_1,s_1),\ldots,(j_N,s_N)} F)^2 \, ds_1 \ldots ds_N. \tag{1.6}$$

Then for any integer $N \geqslant 1$ and any real number $p > 1$, denote by $\boldsymbol{D}_{N,p}$ the Banach space which is the completion of $\mathscr{S}$ with respect to the norm $\|\cdot\|_{N,p}$.

From the definition of the stochastic derivative operator we deduce that D is a closed unbounded linear operator from $\boldsymbol{D}_{1,2}$ into $L^2([0, T] \times W; \boldsymbol{R}^d)$.

The *space of smooth Wiener functionals* in the sense of stochastic calculus of variations $\boldsymbol{D}_\infty$ is then defined by

$$\boldsymbol{D}_\infty = \bigcap_{1<p} \bigcap_{k \in \boldsymbol{N}^*} \boldsymbol{D}_{k,p}. \tag{1.7}$$

Moreover, we have the chain rule:

PROPOSITION 1.1. *For any $\mathscr{C}^1$-function $\varphi\colon \boldsymbol{R}^m \to \boldsymbol{R}$ with bounded partial derivatives of all orders and all families of functionals $F_1, \ldots, F_m$ in $\boldsymbol{D}_{1,2}$, it follows that $\varphi(F_1, \ldots, F_m) \in \boldsymbol{D}_{1,2}$ and*

$$D\varphi(F_1, \ldots, F_m) = \sum_{i=1}^{m} \frac{\partial \varphi}{\partial x^i} (F_1, \ldots, F_m) DF_i. \tag{1.8}$$

2. Existence and uniqueness of the solution of a stochastic differential equation on manifolds. Now, let us expand these tools to manifold-valued Wiener functionals. Let M and N be σ-compact connected manifolds of class $\mathscr{C}^\infty$, of respective dimensions m and n, equipped with the Riemannian metrics g_M and g_N, respectively. Set

(2.1) $$D_\infty(M) = \{G\colon W \to M;\ F(G) \in D_\infty \text{ for all } F \in \mathscr{C}_0^\infty(M)\},$$

where $\mathscr{C}_0^\infty(M)$ denotes the space of M-valued $\mathscr{C}^\infty$-functions with compact support. Then $D_\infty(M)$ is the space of all M-valued infinitely differentiable functionals.

Consider the time depending $\mathscr{C}^\infty$-vector fields $A_0, \ldots, A_d$ on M and a $\mathscr{C}^\infty$-mapping Π from M into N such that the following two conditions hold:

(C.1) M is equipped with an atlas $\{(U_i, \phi_i), i \in I\}$ of relatively compact charts such that, for any i in I and any α in $\{0, \ldots, d\}$, if

$$A_\alpha(t, x) = \sigma_\alpha^j(t, x) \frac{\partial}{\partial \varphi_i^j}$$

denotes the representation of the vector fields A_α in the local coordinates $(\phi_i^1, \ldots, \phi_i^m)$, we can extend the functions $\sigma_\alpha^j(t, x)$ to functions on $[0, T] \times \boldsymbol{R}^m$ such that for all $\alpha \in \{0, \ldots, d\}$, the functions $\sigma_\alpha^j(t, x)$, as well as their derivatives in x are Hölder-continuous in t uniformly in $[0, T] \times \mathscr{K}$ for any compact subset $\mathscr{K}$ in $\boldsymbol{R}^m$, that they are $\mathscr{C}^\infty$-bounded in x when t is a fixed element in $[0, T]$, and that all their derivatives in x are uniformly bounded.

(C.2) Π is a proper mapping, i.e., for each compact subset K of N, the inverse image $\Pi^{-1}(K)$ is a compact subset of M.

We then have the following result:

THEOREM 2.1. *Suppose that condition* (C.1) *holds. Let x_0 be an $\mathscr{F}_0$-measurable random variable with values in M such that in every chart (U, ϕ) the moments of all orders of the $\boldsymbol{R}^m$-valued random variable $\phi(x_0)$ are square integrable. Then the stochastic differential equation*

(2.2) $$x_t = x_0 + \int_0^t A_0(s, x_s)\, ds + \int_0^t A_\alpha(s, x_s) \circ dw_s^\alpha,$$

where $w_t = (w_t^1, \ldots, w_t^d)$ denotes the standard Brownian motion on W, has a unique M-valued solution $\big(X(t, x_0, w)\big)_{t \in [0, \theta(w) \wedge T]}$, where $\theta(w)$ denotes the explosion time of the solution.

Remarks. (i) From now on, in a chart (U, ϕ), we will identify $x \in U$ with its local coordinates $\phi(x)$ in $\phi(U)$.

(ii) To avoid explosion problems we could have worked on compact manifolds. To find sensitive assumptions to ensure the non-explosion of the solution on σ-compact manifolds is rather delicate. For instance, even if M is a complete Riemannian manifold and if the generator $A_0 + \frac{1}{2} \sum_{\alpha=1}^d A_\alpha^2$ is the

Laplacian, we need conditions on the decrease at infinity of the curvature to avoid that it tends too quickly to minus infinity when the process tends to the one-point compactification of the manifold. One possibility would be to suppose that the image of the charts contains a ball of fixed radius and that there are uniform bounds on the derivatives of the coefficients of the vector fields in these local coordinates. This would give what Elworthy [5] called a *uniform cover*. There is a discussion about that in an article of Li [10].

Other possibilities can be found in the paper of Bakry [1].

Proof. For each chart (U, ϕ) of the atlas satisfying condition (C.1), consider the expression of the vector fields A_α in the local coordinates $(\phi^1, \ldots, \phi^m)$,

$$A_\alpha(t, x) = \sigma^i_\alpha(t, x)\frac{\partial}{\partial\phi^i}, \quad \alpha = 0, \ldots, d. \tag{2.3}$$

Let us extend the functions $\sigma^i_\alpha(t, x)$ to $\mathscr{C}^\infty$-functions on $\boldsymbol{R}^n$ satisfying the hypotheses of (C.1) and consider the stochastic differential equation

$$\begin{cases} dx^i_t = \sigma^i_0(t, x_t)\,dt + \sigma^i_\alpha(t, x_t)\circ dw^\alpha_t, \\ x^i_0 = x^i \in \boldsymbol{R}^m, \end{cases} \quad i = 1, \ldots, m. \tag{2.4}$$

We then know (cf. [8]) that it has a unique solution $(X(t, x, w))_{t\in[0,T]}$ which does not explode. Let us fix $x = (x^1, \ldots, x^m)$ in U and set

$$v_U(w) = \inf\{t;\ X(t, x, w) \notin U\}.$$

Define $(X_U(t, x, w))_{t\in[0,T]}$ by

$$X_U(t, x, w) = X(t\wedge v_U(w), x, w). \tag{2.5}$$

Like that we can construct a local solution X_U for each x in M and every neighbourhood U of x.

Furthermore, if (U, ϕ) and $(\tilde{U}, \tilde{\phi})$ are two coordinate neighbourhoods with a non-empty intersection and if $x \in U \cap \tilde{U}$, then $X_U(t, x, w) = X_{\tilde{U}}(t, x, w)$ for any $t \leqslant v_U(w) \wedge v_{\tilde{U}}(w)$. Indeed, if

$$A_\alpha(t, x) = \tilde{\sigma}^i_\alpha(t, \tilde{x})\frac{\partial}{\partial\tilde{\phi}^i}$$

under the local coordinate $(\tilde{\phi}^1, \ldots, \tilde{\phi}^m)$ in $\tilde{U}$, then we have

$$\tilde{\sigma}^i_\alpha(t, x) = \sigma^k_\alpha(t, x)\frac{\partial\tilde{\phi}^i}{\partial\phi^k}$$

and $X_{\tilde{U}}$ is the solution of the equation

$$d\tilde{x}^i_t = \tilde{\sigma}^i_0(t, \tilde{x}_t)\,dt + \tilde{\sigma}^i_\alpha(t, \tilde{x}_t)\circ dw^\alpha_t. \tag{2.6}$$

On the other hand, Proposition 1.1 implies that if we express the process

X_U through the local coordinates $\tilde{\phi}$ in $\tilde{U}$, i.e. if we write $\tilde{X}_t^i = \tilde{\phi}(X_U(t, x, w))$, then

$$d\tilde{X}_t^i = \frac{\partial\tilde{\phi}^i}{\partial\phi^k}(x_t)\circ dx_t^k = \frac{\partial\tilde{\phi}^i}{\partial\phi^k}(x_t)\,\sigma_0^k(t, x_t)\,dt + \frac{\partial\tilde{\phi}^i}{\partial\phi^k}(x_t)\,\sigma_\alpha^k(t, x_t)\circ dw_t^\alpha$$
$$= \tilde{\sigma}_0^i(t, \tilde{X}_t)\,dt + \tilde{\sigma}_\alpha^i(t, \tilde{X}_t)\circ dw_t^\alpha.$$

Therefore $\tilde{X}_t = \tilde{\phi}(X_U(t, x, w))$ is a solution of equation (2.6) as well as $\tilde{x}_t = X_{\tilde{U}}(t, x, w)$, and so, by the uniqueness of solutions of stochastic differential equations in Euclidean spaces, we conclude that $X_U(t, x, w) = X_{\tilde{U}}(t, x, w)$ for all $t \leqslant \nu_U(w)\wedge\nu_{\tilde{U}}(w)$.

We will now patch together the local solutions into a global solution.

Consider for each w in W the totality of charts $(U_1, \phi_1), \ldots, (U_l, \phi_l)$ such that $x_0(w)$ belongs to U_i for all i, $i = 1, \ldots, l$. Then the process $\hat{X}(t, x_0, w) = X_{U_j}(t, x_0, w)$ is well defined for $t\in[0, \hat{\nu}_{x_0}\wedge T]$, where $\hat{\nu}_{x_0}(w) = \inf_{1\leqslant i\leqslant l}\{\nu_{U_i}(w)\}$, and $j\in\{1, \ldots, l\}$ is such that $\hat{\nu}_{x_0}(w) = \nu_{U_j}(w)$.

Set $\nu_1(w) = \hat{\nu}_{x_0}(w)\wedge T$ and $x_t = \hat{x}_t$ for $t\in[0, \nu_1]$.

Inductively, if $\nu_n(w)$ and $x_t = X(t, x_0, w)$ for all $t\in[0, \nu_n(w)]$ are defined, then on the set $\{w; \nu_n(w) < T\}$, $x_n = x_{\nu_n}$, and we define $w_n = \theta_{\nu_n} w$, where $(\theta_t w)(s) = w_{t+s} - w_t$ and $\nu_{n+1} = \hat{\nu}_{x_n}(w_n)\wedge T$.

Then we set $x_t = \hat{X}(t-\nu_n, x_n, w_n)$ for t in $[\nu_n, \nu_{n+1}]$.

So, we have constructed a global solution of equation (2.1). The uniqueness follows easily since condition (C.1) implies that the local solutions $X_U(t, x, w)$ are unique for every x in M and every coordinate neighbourhood U of x. ∎

From now on, we suppose that the image of the charts contains a ball of fixed radius and that there are uniform bounds on the derivatives of the coefficients of the vector fields in these local coordinates, so the process x_t will be well defined on all $[0, T]$.

Furthermore, the family of morphisms $x\mapsto X(t, x, w)$ is a flow of diffeomorphisms of M into itself (cf. [8]) and we denote it by $(X(t, w))_{t\in[0,T]}$. Let us fix now the M-valued random variable x_0 and set

$$x_t(w) = X(t, x_0, w) \tag{2.7}$$

and

$$y_t(w) = \Pi(x_t(w)). \tag{2.8}$$

3. Stochastic calculus of variations on manifolds. Under the above hypotheses y_t is an infinitely differentiable N-valued Wiener functional for all t in $[0, T]$. Indeed, we have the following result:

THEOREM 3.1. *For each t in $[0, T]$, $y_t(w)$ is an element of $\mathbf{D}_\infty(N)$. Moreover, for any function f in $\mathscr{C}_0^\infty(N)$ and any h in H, the following equality holds for any t in $[0, T]$:*

$$\langle D(f(y_t))(w), h\rangle_H = (\Pi_*)_{x_t(w)}\int_0^t \dot{h}^\alpha(s)\{(X(t, w)\circ X(s, w)^{-1})_* A_\alpha\}_{t,x_t(w)} f\,ds. \tag{3.1}$$

Proof. Consider $f \in \mathscr{C}_0^\infty(N)$. Then condition (C.2) implies that $f \circ \Pi \in \mathscr{C}_0^\infty(M)$. So we can define, for each chart (U_i, ϕ_i) from the atlas satisfying condition (C.1), a function $\tilde{f}_i$ in $\mathscr{C}_0^\infty(R^m)$ by $\tilde{f}_i = f \circ \Pi \circ \phi_i^{-1}$ (since the charts are relatively compact). Consequently, $f \circ \Pi = \tilde{f}_i \circ \phi_i$ in the domain of the chart (U_i, ϕ_i).

Let us suppose at first that for any t in the interval $[v_n, v_{n+1}]$ the process x_t is in the domain of the chart (U_k, ϕ_k), where the v_j are the stopping times introduced above. Then, for any t in $[v_n, v_{n+1}]$,

$$\begin{aligned}\langle D(f(y_t))(w), h\rangle_H &= \langle D(\tilde{f}_k \circ \phi_k)(x_t)(w), h\rangle_H \\ &= \langle D(\tilde{f}_k(\hat{X}(t-v_n, x_n, w_n)))(w), h\rangle_H \\ &= \int_0^T \dot{h}^\alpha(s)\, \overline{D(\tilde{f}_k(\hat{X}(t-v_n, x_n, w_n)))}^\alpha (w)(s)\, ds.\end{aligned}$$

But, by Proposition 1.1 and equation V-10.3 from [8],

$$D(\tilde{f}_k(\hat{X}(t-v_n, x_n, w_n)))(w) = \sum_{j=1}^m \int_0^t \frac{\partial \tilde{f}_k}{\partial x^j}(\hat{X}(t-v_n, x_n, w_n))(\tilde{Z}_t \tilde{Z}_s^{-1})\, ds,$$

where

$$\tilde{Z}_j^i(t) = \frac{\partial}{\partial x^j} X_{U_k}^i(t, x, w).$$

Hence

$$\langle D(f(y_t))(w), h\rangle_H = \int_0^t \dot{h}^\alpha(s)\{(X_{U_k}(t, w) \circ X_{U_k}(s, w)^{-1})_* \tilde{A}_\alpha^k\}_{t, X_{U_k}(t-v_n, x_n, w_n)} \tilde{f}_k\, ds,$$

where $\tilde{A}_\alpha^k$ denotes the representation in local coordinates of the vector field A_α in the chart (U_k, ϕ_k). So,

$$\begin{aligned}\langle D(f(y_t))(w), h\rangle_H &= \int_0^t \dot{h}^\alpha(s)\{(X(t, w) \circ X(s, w)^{-1})_* A_\alpha\}_{t, x_t(w)} f\, ds \\ &= (\Pi_*)_{x_t(w)} \int_0^t \dot{h}^\alpha(s)\{(X(t, w) \circ X(s, w)^{-1})_* A_\alpha\}_{t, x_t(w)} f\, ds.\end{aligned}$$

And since the interval $[v_n(w), v_{n+1}(w)]$ forms a partition of $[0, T]$, the result follows for every t in $[0, T]$. ∎

Let us introduce now the Malliavin covariance matrix of the process y_t. We define a $\mathscr{C}^\infty$-tensor field B^0 on $[0, T] \times M$ of type (2, 0) by

$$B_{t,x}^0(u_1, u_2) = \sum_{\alpha=1}^d u_1((A_\alpha)_{t,x})\, u_2((A_\alpha)_{t,x}), \tag{3.2}$$

$$t \in [0, T],\ x \in M,\ u_1, u_2 \in T_x^* M.$$

Then the *Malliavin covariance matrix* of y_t is the non-negative definite and symmetric bilinear form $\langle\langle Dy_t, Dy_t\rangle\rangle(w)$ on $T_{y_t(w)}^*$ defined for all $u_1, u_2 \in T_{y_t(w)}^* N$

and all t in $[0, T]$ by

$$(3.3)\quad \langle\langle Dy_t, Dy_t\rangle\rangle(w)(u_1, u_2) = \int_0^t (\Pi_*)_{x_t(w)} \{(X(t, w) \circ X(s, w)^{-1})_* B^0\}(u_1, u_2)\, ds.$$

Since $T^*_{y_t(w)} N$ is equipped with the inner product assigned by the Riemannian metric g_N, we can define the determinant $\det(\langle\langle Dy_t, Dy_t\rangle\rangle(w))$ in the usual way. We put

$$(3.4)\quad g^t(w) = \begin{cases} 1/\det(\langle\langle Dy_t, Dy_t\rangle\rangle(w)) & \text{if } \det(\langle\langle Dy_t, Dy_t\rangle\rangle(w)) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let us consider the following condition:

$$(\mathrm{A})\qquad f(y_t)\, g^t \in \bigcap_{p\in[1,+\infty[} L^p(P) \quad \text{for all } f \in \mathscr{C}_0^\infty(N).$$

Under this assumption we can then prove the following two integration-by-parts formulas as in [17].

PROPOSITION 3.2. *For every differential operator ∂ on N and every function ϕ in $\mathscr{C}_0^\infty(M)$, there exist $p > 1$, $r \in N$ and a continuous linear mapping $\xi\colon \boldsymbol{D}_{p,r} \to L^1(P)$ such that, for any f in $\mathscr{C}_0^\infty(M)$ and G in $\boldsymbol{D}_{p,r}$,*

$$(3.5)\qquad E(\partial f(y_t)\, \phi(y_t)\, G) = E(f(y_t)\, \xi(G)).$$

PROPOSITION 3.3. *For each G in $\boldsymbol{D}_\infty$, the function $g_G(x) = \langle \delta_G, G\rangle$ is $\mathscr{C}^\infty$, and for any f in $\mathscr{C}_0^\infty(M)$*

$$(3.6)\qquad E(f(y_t)\, G) = \int_M f(x)\, g_G(x)\, \nu(dx),$$

where δ_x is the Dirac δ-function at x in M, and ν is the Riemannian volume element on M. In particular, the function p defined by $p(x) = \langle \delta_x, 1\rangle$ is the $\mathscr{C}^\infty$-density of the law of the stochastic process $(y_t)_{t\in[0,T]}$ with respect to ν.

Hence, if condition (A) is satisfied, then the law of the process y_t has a smooth density with respect to the Riemannian volume element. We will show that this holds under a global Hörmander condition.

With this aim, for any t in $[0, T]$ and any x in M, denote by $\mathfrak{L}_{t,x}$ the subspace of $T_x M$ generated by the vector fields $(A_i)_{t,x}$, $i = 1, \ldots, d$, and $([A_{i_q}, [\ldots, [A_{i_1}, A_{i_0}]\ldots]])_{t,x}$, $1 \leqslant i_0 \leqslant d$, $0 \leqslant i_j \leqslant d$, $j = 1, \ldots, q$, $q \in N^*$. Consider the following assumption:

$$(\mathrm{H})\qquad (\Pi_*)_x\, \mathfrak{L}_{t,x} = T_y N$$

for any t in $[0, T]$ and any x in M, where $y = \Pi(x)$.

Under this assumption the main result of this section holds.

THEOREM 3.4. *Assume that Hörmander's condition* (H) *holds. Then for all* t *in* $]0, T]$ *the probability law of the process* $y_t(w)$ *has a* $\mathscr{C}^\infty$*-density with respect to the Riemannian volume element.*

Proof. We have to prove that condition (A) holds for every t in $]0, T]$. Condition (C.2) implies that $f \circ \Pi \in \mathscr{C}_0^\infty(M)$ for any function $f \in \mathscr{C}_0^\infty(N)$. As $f(y_t) = (f \circ \Pi)(x_t)$, it suffices to prove that, for all t in $]0, T]$,

$$g^t = \bigcap_{p \in [1, +\infty[} L^p(P). \tag{3.7}$$

Let $(V_{\bar{0}}, \bar{\phi})$ be a relatively compact chart on N such that $y_0 \in V_0$ and let (U_0, ϕ) be a chart from the atlas satisfying condition (C.1) such that $x_0 \in U_0$. Then there exists a compact subset $\tilde{U}_0$ of U_0 such that $\Pi(\tilde{U}_0) \subset V_0$ and $\phi^q = \bar{\phi}^q \circ \Pi$, $1 \leqslant q \leqslant n$ in $\tilde{U}_0$.

Consider the representation of the vector fields A_α through the local coordinates $(\phi^1, \ldots, \phi^m)$,

$$A_\alpha(t, x) = \sigma_\alpha^i(t, x) \frac{\partial}{\partial \phi^i}, \quad \alpha = 0, \ldots, d. \tag{3.8}$$

Extend the functions $\sigma_\alpha^i(t, x)$ to $\mathscr{C}^\infty$-functions on $\boldsymbol{R}^m$ that satisfy the hypotheses of condition (C.1) and consider the process $\tilde{x}_t(w)$, a solution of the stochastic differential equation

$$\begin{cases} d\tilde{x}_t^i = \sigma_0^i(t, \tilde{x}_t)\, dt + \sigma_\alpha^i(t, \tilde{x}_t) \circ dw_t^\alpha, \\ \tilde{x}_0^i = \phi^i(x_0) \in \boldsymbol{R}^m, \end{cases} \quad i = 1, \ldots, m. \tag{3.9}$$

Let v be the stopping time defined by

$$v(w) = \inf\{t;\ \tilde{X}(t, x_0, w) \notin \tilde{U}_0\}. \tag{3.10}$$

To prove (3.7), we construct a random process $\xi_t(w)$ such that for any t in $]0, v]$

(i) $$0 \leqslant \xi_t \leqslant \det(\langle\langle Dy_t, Dy_t \rangle\rangle) \text{ p.s.}$$

and

(ii) $$\xi_t^{-1} \in \bigcap_{p \in [1, +\infty[} L^p(P).$$

Let $\tilde{y}_t(w)$ be the solution of the stochastic differential equation

$$\tilde{y}_j^i(t) = \delta_j^i + \int_0^t \partial_k \sigma_\alpha^i(s, \tilde{x}_s)\, \tilde{y}_j^k(s) \circ dw_s^\alpha + \int_0^t \partial_k \sigma_0^i(s, \tilde{x}_s)\, \tilde{y}_j^k(s)\, ds, \tag{3.11}$$

where δ_j^i denotes the Kronecker symbol. Then, for every $0 \leqslant s < t \leqslant v$, we have

$$D_s^j \tilde{x}_t^i = \tilde{y}_l^i(t) (\tilde{y}(s)_k^s)^{-1} \sigma_j^k(s, \tilde{x}_s), \quad \text{i.e.} \quad D_s^i \tilde{x}_t = \tilde{y}_t \tilde{y}_s^{-1} \sigma_i(s, \tilde{x}_s).$$

For each $t \in [0, T]$ and $\zeta \in \boldsymbol{R}^m$, we can define the quadratic form $a_t(w)$

on R^m by

$$(3.12)\qquad a_t(w)[\zeta] = \tilde{y}_t \sum_{r=1}^{d} \int_0^v \langle \zeta, (\tilde{y}_s^{-1}\sigma_\alpha(s, \tilde{x}_s))(\tilde{y}_s^{-1}\sigma_\alpha(s, \tilde{x}_s))^\tau \zeta \rangle \, ds \, \tilde{y}_t^\tau.$$

This allows us to define for any t in $[0, T]$ the random variable ξ_t as follows:

$$(3.13)\qquad \xi_t(w) = \varepsilon_0 \{\inf_{\eta \in S^{n-1}} a_t(w)[\tilde{\eta}]\}^n,$$

where $\tilde{\eta} = (\eta, 0, \ldots, 0) \in S^{m-n}$, $\eta \in S^{n-1}$, and

$$\varepsilon_0 = \inf \left\{ \det (g_N)_{\Pi(x)} \left(\frac{\partial}{\partial \phi^q}, \frac{\partial}{\partial \phi^r} \right);\ x \in U_0 \right\}.$$

The proof of the theorem will then be complete if we show that for any t in $]0, v]$ the conditions (i) and (ii) hold. Actually, this implies that (3.7) holds for every t in $]0, v]$. But since the function $t \mapsto \det(\langle\langle Dy_t, Dy_t \rangle\rangle)(w)$ is increasing in t, we have $g^{t'} \leqslant g^v$ for any t in $[v, T]$; hence g^t is in $L^p(P)$ for all $p > 1$ and t in $[0, T]$. ∎

LEMMA 3.5. *ξ_t satisfies* (i) *for any t in $]0, v]$.*

Proof. We have for all $1 \leqslant q, r \leqslant n$, $t \in]0, v]$,

$$\begin{aligned} &\langle\langle Dy_t, Dy_t \rangle\rangle(w)(d\tilde{\phi}^q, d\tilde{\phi}^r) \\ &= \int_0^t \{[X(t, w) \circ X(s, w)^{-1}]_* B^0\}_{t, x_t(w)} (d\phi^q, d\phi^r) \, ds \\ &= B^0_{t, x_t(w)} (d\phi^q \circ X(t, w) \circ X(s, w)^{-1}, d\phi^r \circ X(t, w) \circ X(s, w)^{-1}) \\ &= \sum_{\alpha=1}^{d} \int_0^t \tilde{y}_l^q(t) (\tilde{y}_k^l(s))^{-1} \sigma_\alpha^k(s, \tilde{x}_s) \tilde{y}_i^r(t) (\tilde{y}_j^i(s))^{-1} \sigma_\alpha^j(s, \tilde{x}_s) \, ds \\ &= \tilde{y}_t \sum_{\alpha=1}^{d} \int_0^t \langle \tilde{y}_s^{-1} \sigma_\alpha(s, \tilde{x}_s), (\tilde{y}_s^{-1} \sigma_\alpha(s, \tilde{x}_s))^\tau \rangle \, ds \, \tilde{y}_t^\tau. \end{aligned}$$

Consequently,

$$\det(\langle\langle Dy_t, Dy_t \rangle\rangle)(w) \geqslant \{\inf_{\eta \in S^{n-1}} a_t(w)[\tilde{\eta}]\}^n.$$

Moreover, $a_t(w)$ is a non-negative definite quadratic form, so $\xi_t \geqslant 0$ for every t in $]0, T]$. ∎

LEMMA 3.6. *ξ_t satisfies* (ii) *for any t in $]0, T]$.*

Proof. Consider the representation of the vector fields A_α through local coordinates, introduced in (3.8), and denote by $\tilde{A}_\alpha$, $\alpha = 0, \ldots, d$, the vector fields on R^m given for each x in R^m and any t in $[0, T]$ by

$$(3.14)\qquad \tilde{A}_\alpha(t, x) = \sigma_\alpha^i(t, x) \frac{\partial}{\partial x^i}.$$

Denote by $\mathscr{L}(\tilde{A}_0, \ldots, \tilde{A}_d)$ (respectively, $\mathscr{L}(\tilde{A}_1, \ldots, \tilde{A}_d)$) the Lie algebra generated by the vector fields $\tilde{A}_0, \ldots, \tilde{A}_d$ (respectively, $\tilde{A}_1, \ldots, \tilde{A}_d$) and by $\mathscr{I}(\tilde{A}_1, \ldots, \tilde{A}_d)$ the ideal generated by the Lie algebra $\mathscr{L}(\tilde{A}_1, \ldots, \tilde{A}_d)$ in the Lie algebra $\mathscr{L}(\tilde{A}_0, \ldots, \tilde{A}_d)$. It follows from Hörmander's condition (H) that

$$\mathscr{I}(\tilde{A}_1, \ldots, \tilde{A}_d)(0, x_0) = \mathbf{R}^m. \tag{3.15}$$

On the other hand, we know from [14] that the Malliavin covariance matrix M_t associated with the stochastic process $\tilde{x}_t$ is given by

$$M_t = \tilde{y}_t \int_0^t \tilde{y}_s^{-1} \sum_{\alpha=1}^{d} \sigma_k(s, \tilde{x}_s)\sigma_k(s, \tilde{x}_s)^{\tau} (\tilde{y}_s^{-1})^{\tau} ds\, \tilde{y}_t^{\tau}. \tag{3.16}$$

Hence, since ε_0 is a constant, to conclude it suffices to prove that $(\det M_t)^{-1}$ belongs to the space $L^p(P)$ for every p in $[1, +\infty[$ and every t in $]0, v]$. But assumption (C.1) and relation (3.15) imply that the hypotheses of Theorem 1.1.3 from [6] are satisfied, and this theorem gives the result. ∎

4. Application to a nonlinear filtering problem. In this section, we will use the results of the preceding section to prove that the filter associated with some nonlinear filtering problem on Riemannian manifolds has a $\mathscr{C}^\infty$-density with respect to the Riemannian volume element. In fact, we will consider a generalisation of a nonlinear filtering model on Riemannian manifolds introduced by Ng and Caines in [12].

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space, and w and v two independent Wiener processes on this space of dimensions d and n, respectively. Let M be a σ-compact, connected and orientated Riemannian manifold of dimension m, equipped with the Riemannian metric g_M. Denote by $\mathrm{Gl}(M)$ the bundle of linear frames over M, and by p_M the projection of $\mathrm{Gl}(M)$ onto M. Let (x^i, e^i_j), $i, j = 1, \ldots, m$, be local coordinates around the element (x, e) of $\mathrm{Gl}(M)$, and $\{\Gamma^q_{il}\}$ the Christoffel symbols of the Riemannian connection on M, compatible with the metric g_M.

Consider some time depending $\mathscr{C}^\infty$-vector fields A_j, $j = 1, \ldots, d$, on $\mathrm{Gl}(M)$ whose representation through local coordinates is given by

$$A_j(t, x, e) = a^i_j(t, x, e)\left(\frac{\partial}{\partial x^i} - \Gamma^q_{il} e^l_p \frac{\partial}{\partial e^p_q}\right). \tag{4.1}$$

Furthermore, let A_0 be a time depending $\mathscr{C}^\infty$-vector field on M which is expressed in local coordinates as

$$A_0(t, x) = a^i_0(t, x) \frac{\partial}{\partial x^i}. \tag{4.2}$$

Denote its horizontal lift with respect to the connection $\{\Gamma^q_{il}\}$ by $\tilde{A}_0$. Then $\tilde{A}_0$ is expressed in local coordinates as

$$\tilde{A}_0(t, x, e) = \alpha^i_0(t, x)\left(\frac{\partial}{\partial x^i} - \Gamma^q_{il} e^l_p \frac{\partial}{\partial e^p_q}\right). \tag{4.3}$$

Now, introduce the system process $(x_t)_{t\in[0,T]}$ as the M-valued stochastic process defined for any t in $[0, T]$ by

$$x_t = p_M(r_t), \tag{4.4}$$

where $r_t = (x_t, e_t)$ is the solution of the stochastic differential equation

$$r_t = r_0 + \int_0^t \tilde{A}_0(s, r_s)\,ds + \int_0^t A_\alpha(s, r_s) \circ dw_s^\alpha \tag{4.5}$$

with $r_0 = (x_0, e_0) \in \mathrm{Gl}(M)$. In local coordinates, (4.5) can be expressed as

$$\begin{cases} dx_t^i = a_0^i(t, x_t)\,dt + a_\alpha^i(t, x_t, e_t) \circ dw_t^\alpha, \\ de_{\alpha t}^i = -\Gamma_{mk}^i(x_t)\, e_{\alpha t}^k \circ dx_t^m, \end{cases} \qquad i = 1, \ldots, m. \tag{4.6}$$

Let us notice that, unlike in Ng–Caines' model, the process r_t may leave the space $O(M)$, even if its starting point r_0 is inside this space. Theorem 2.1 insures however that the process is well defined on $[0, T]$ if the vector fields A_j satisfy condition (C.1) of the second section.

To construct the observation process y_t we follow the model given in [7]. Let N be a σ-compact, connected Riemannian manifold of dimension n, equipped with the associated Riemannian metric g_N. Denote by $O(N)$ the bundle of orthonormal frames on N and by p_N the projection of $O(N)$ onto N. Let (y^i, f_j^i), $j = 1, \ldots, n$, be the representation through local coordinates around the element (y, f) in $O(N)$, and $\{\gamma_{li}^q\}$ be the Christoffel symbols of the Riemannian connection on N compatible with respect to the metric g_N. Denote by $\{H_1, \ldots, H_n\}$ the family of canonical horizontal vector fields on $O(N)$ with respect to the Riemannian connection $\{\gamma_{li}^q\}$. Note that around (y, f) in $O(N)$, H_j $(j = 1, \ldots, n)$ is expressed as

$$H_j = f_j^i \left(\frac{\partial}{\partial y^i} - \gamma_{li}^q f_p^l \frac{\partial}{\partial f_p^q} \right). \tag{4.7}$$

Introduce a time depending $\mathscr{C}^\infty$-vector field $h(t, x_t, y)$ on N written in local coordinates as

$$h(t, x_t, y) = h^i(t, x_t, y) \frac{\partial}{\partial y^i}. \tag{4.8}$$

Let $\tilde{h}$ be its horizontal lift with respect to the connection $\{\gamma_{li}^q\}$. Then $\tilde{h}$ is expressed in local coordinates as

$$\tilde{h}(t, x_t, y) = h^i \left(\frac{\partial}{\partial y^i} - \gamma_{li}^q f_p^l \frac{\partial}{\partial f_p^q} \right). \tag{4.9}$$

We then define for any t in $[0, T]$ the observation process $(y_t)_{t\in[0,T]}$ by

$$y_t = p_N(s_t), \tag{4.10}$$

where $s_t = (y_t, f_t)$ is the solution of the stochastic differential equation

$$s_t = s_0 + \int_0^t \tilde{h}(s, x_s, s_s)\,ds + \int_0^t H_j(s_s) \circ dw_s^j \tag{4.11}$$

with $s_0 = (y_0, f_0)$ in $O(N)$. In local coordinates, (4.11) can be expressed as

$$\begin{cases} dy_t^i = \tilde{h}^i(t, x_t, y_t)\,dt + H_j^i(t, y_t, f_t) \circ dv_t^j, \\ df_{\alpha t}^i = -\gamma_{mk}^i(y_t)\,e_{\alpha t}^k \circ dy_t^m, \end{cases} \qquad i = 1, \ldots, m. \tag{4.12}$$

Let us notice that since $\{\gamma_{lt}^q\}$ is compatible with respect to the metric g_N, s_t is necessarily a process evolving in $O(N)$ (cf. [8]). Moreover, since p_N is a proper mapping, we have

$$\sigma(s_\tau;\ 0 \leqslant \tau \leqslant t) = \sigma(y_\tau;\ 0 \leqslant \tau \leqslant t), \tag{4.13}$$

which implies that it is equivalent to observe the stochastic process $(x_t)_{t\in[0,T]}$ through $\sigma(s_\tau;\ 0 \leqslant \tau \leqslant t)$ or $\sigma(y_t;\ 0 \leqslant \tau \leqslant t)$.

We then define the filter as usual by

DEFINITION 4.1. For any t in $[0, T]$ and any function ψ in $\mathscr{C}_0^\infty(M)$, denote by $\pi_t\psi$ the *filter* associated with the system-process observation pair (x_t, y_t), given by (4.4) and (4.10), defined by

$$\pi_t\psi = E[\psi(x_t)/\mathscr{Y}_t], \tag{4.14}$$

where $\mathscr{Y}_t = \sigma(y_\tau;\ 0 \leqslant \tau \leqslant t)$.

By means of a change of probability measure, we are now able to define an unnormalized filter linked with the filter π_t by an abstract Bayes formula.

Let $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ be an independent copy of the probability space $(\Omega, \mathscr{F}, P)$. Consider on the probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ the M-valued stochastic process $(\tilde{x}_t)_{t\in[0,T]}$ which has the same probability law as the stochastic process $(x_t)_{t\in[0,T]}$. This means that on the probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ the equality

$$\tilde{x}_t = p_M(\tilde{r}_t) \tag{4.15}$$

holds, where $\tilde{r}_t$ is the solution of the stochastic differential equation

$$\tilde{r}_t = \tilde{r}_0 + \int_0^t \tilde{A}_0(\tau, \tilde{r}_\tau)\,d\tau + \int_0^t A_\alpha(\tau, \tilde{r}_\tau) \circ dw_\tau^\alpha \tag{4.16}$$

with $\tilde{r}_0 = r_0$. This allows us to introduce the Girsanov exponential, associated with the stochastic processes $(\tilde{x}_t)_{t\in[0,T]}$ and $(y_t)_{t\in[0,T]}$, as usual in nonlinear filtering problems by

$$\begin{aligned} \Lambda_t(\tilde{x}_t, y_t) = \exp\Bigg(&\int_0^t \langle h(s, \tilde{x}_s, y_s), dy_s\rangle_{y_s} - \frac{1}{2}\int_0^t \sum_{j,k=1}^n \gamma_{kj}^k(y_s)\,h^j(s, \tilde{x}_s, y_s)\,ds \\ &- \frac{1}{2}\int_0^t \operatorname{tr}\left(\frac{\partial H}{\partial y}(s, \tilde{x}_s, y_s)\right) ds \\ &- \frac{1}{2}\int_0^t \langle h(s, \tilde{x}_s, y_s), h(s, \tilde{x}_s, y_s)\rangle_{y_s}\,ds\Bigg) \quad P\otimes\tilde{P}\ \text{p.s.}, \end{aligned} \tag{4.17}$$

where $H = (h^1, \ldots, h^n)$ and $\langle \cdot, \cdot \rangle_y$ denotes the inner product in $T_y N$ induced by the Riemannian metric g_N. Then the unnormalized filter associated with the system-process observation pair can be defined by

DEFINITION 4.2. For any t in $[0, T]$ and any function ψ in $\mathscr{C}_0^\infty(M)$ denote by $\varrho_t(\psi)$ the *unnormalized filter* associated with the system-process observation pair (x_t, y_t), given by (4.4) and (4.10), defined by

$$\varrho_t(\psi) = E_{\tilde{P}}[\psi(\tilde{x}_t)\Lambda(\tilde{x}_t, y_t)], \tag{4.18}$$

where $E_{\tilde{P}}$ stands for the expectation with respect to $\tilde{P}$.

We then have the abstract Bayes formula:

THEOREM 4.3 (Ng and Caines [12]). *For all t in* $[0, T]$ *and any function* ψ *in* $\mathscr{C}_0^\infty(M)$, *we have*

$$\pi_t = \frac{\varrho_t \psi}{\varrho_t 1}. \tag{4.19}$$

This allows us to prove, by means of Malliavin calculus on manifolds, that the filter π_t has a $\mathscr{C}^\infty$-density with respect to the Riemannian volume element. With that aim, for any t in $[0, T]$ and any r in $\mathrm{Gl}(M)$, denote by $\mathfrak{L}_{t,r}$ the ideal generated at the point (t, r) by the vector fields $A_1, \ldots, A_d$ in the Lie algebra generated by the vector fields $\tilde{A}_0, A_1, \ldots, A_d$. Consider the assumption:

$$(p_{M*})_r \mathfrak{L}_{t,r} = T_x M \tag{H'}$$

for any r in $\mathrm{Gl}(M)$, any t in $[0, T]$, where $x = p_M(r)$. Then we have the following result:

THEOREM 4.4. *Assume that condition* (H') *holds and that the vector fields* $\tilde{A}_0, A_1, \ldots, A_d$ *satisfy condition* (C.1) *from Section* 2. *Then, for any t in* $]0, T]$, *the probability law of the filter* π_t *has a* $\mathscr{C}^\infty$*-density with respect to the Riemannian volume element on M.*

Proof. Since the filter π_t is linked with the unnormalized filter ϱ_t by the Bayes formula (4.15), it is equivalent to show the existence of a $\mathscr{C}^\infty$-density for the filter π_t or the unnormalized filter ϱ_t.

Since the vector fields $\tilde{A}_0, A_1, \ldots, A_d$ satisfy condition (C.1) and the mapping p_M satisfies condition (C.2), we have $\tilde{x}_t \in \boldsymbol{D}_\infty$. Furthermore, since h is a bounded $\mathscr{C}^\infty$-function and p_N is a proper mapping, $y_t \in \boldsymbol{D}_\infty$. Consequently, the results on Malliavin calculus from the preceding section and the usual arguments of Malliavin calculus on Euclidean spaces imply

PROPOSITION 4.5. *For any t in* $[0, T]$, $\Lambda(\tilde{x}_t, y_t)$ *is an element of the space* $\boldsymbol{D}_\infty$.

Moreover, for any t in $[0, T]$ it follows from (4.15) that $\tilde{x}_t = p_M(\tilde{r}_t)$, where the stochastic process $(\tilde{r}_t)_{t\in[0,T]}$ is a solution of a stochastic differential equation whose coefficients satisfy conditions (C.1), (C.2) as well as Hörmander's condition (H′). Hence Proposition 4.5 applied to (4.18), together with Proposition 3.3 and Theorem 3.4, implies that for every t in $]0, T]$ there exists an M-valued $\mathscr{C}^\infty$-function p_t such that

$$\varrho_t \psi = \int_M \psi(x) p_t(x) \nu(dx). \tag{4.20}$$

So $p_t(x)$ is the $\mathscr{C}^\infty$-density of the unnormalized filter ϱ_t with respect to the Riemannian volume element on M. ∎

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U.R.A. C.N.R.S. No 399, Département de Mathématiques
Université de Metz, BP 80794
F-57012 Metz Cedex, France
schiltz@poncelet.univ-metz.fr

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