PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 19, Fasc. 1 (1999), pp. 171–180

ON CERTAIN SUBCLASSES OF THE CLASSES L_c

BY

T. RAJBA* (WROCŁAW)

Abstract. Loève in [5] introduced the classes L_c associated with number c, $c \in \boldsymbol{R}$, as the classes of probability measures satisfying the condition (1). Many authors investigated those classes ([2], [5]–[9], [20], [21]). In this paper we consider certain subclasses $L_{c_1,\ldots,c_k}$, $L_{c,(k)}$ of the classes L_c. We prove that they coincide with the classes of distributions of series of some random variables and with the classes of limit distributions of some normed sums. We give a characterization of certain classes $D_{c_1,\ldots,c_k}$ associated with $L_{c_1,\ldots,c_k}$.

Urbanik in [18] introduced the concept of the decomposability semigroup associated with probability measure P, as the set of all numbers c, such that $P \in L_c$ ([11]–[14]). The class L of selfdecomposable distributions coincides with the class of probability measures P such that $D(P) \supset [0, 1]$. The class L_m, $m \geqslant 1$, of multiply selfdecomposable distributions may be described as the class of probability measures P such that $P \in L_{c_1,\ldots,c_m}$, for every $c_1, \ldots, c_m \in [0, 1]$, or in terms of multiply decomposability semigroups it is equivalent to the inclusion $D_m(P) \supset [0, 1]^m$, where $D_m(P)$ is the multiply decomposability semigroup defined by the formula $D_m(P) = \{(c_1, \ldots, c_m);\ P \in L_{c_1,\ldots,c_m}\}$ ([3], [4], [10], [15]–[17], [19]).

Let φ be the characteristic function of a probability measure on the real line $\boldsymbol{R}$. We say ([2], [3]) that φ is *c-decomposable*, $c \in \boldsymbol{R}$, if

$$\varphi(t) = \varphi(ct)\,\varphi_c(t), \qquad t \in \boldsymbol{R}, \tag{1}$$

for some characteristic function φ_c. L_c is the family of all c-decomposable laws. L_0 and L_1 are the families of all laws. Every L_c is closed under compositions and passages to the limit.

Let X be a random variable with the characteristic function φ. The probability distribution of the random variable X will be denoted by $\mathscr{L}(X)$. Rewriting (1) in terms of random variables we obtain $\varphi \in L_c$ if and only if

$$\mathscr{L}(X) = \mathscr{L}(cX + X_c) \tag{2}$$

* Institute of Mathematics, Wrocław University.

for some random variable X_c with the characteristic function φ_c, such that X and X_c are independent.

For nondegenerate and c-decomposable laws, the inequality $|c| \leqslant 1$ is satisfied. Further, if φ is nondegenerate and c-decomposable with $0 < |c| < 1$, then φ is the characteristic function of a continuous distribution [21]. In the sequel we consider only nondegenerate laws and the numbers c such that $0 < |c| < 1$.

For nondegenerate φ, $\varphi \in L_c$, $0 < |c| < 1$, if and only if it is a characteristic function of

$$X(c) = \sum_{k=0}^{\infty} c^k Z_k, \tag{3}$$

where Z_k, $k = 0, 1, 2, \ldots$, are independent and identically distributed random variables. Then the series converges a.s. (almost surely) and $\varphi_{Z_k} = \varphi_c$, where φ_{Z_k} means the characteristic function of Z_k (see [5] and [6]).

Rewriting (3) in terms of characteristic functions, we obtain $\varphi \in L_c$ if and only if

$$\varphi(t) = \prod_{k=0}^{\infty} \varphi_c(c^k t) \tag{4}$$

for some characteristic function φ_c.

Further, $\varphi \in L_c$ if and only if it is the limit of a sequence of characteristic functions of normed sums S_n/B_n of independent random variables with $B_n/B_{n+1} \to c$:

$$\varphi_{S_n/B_n}(t) \to \varphi(t), \tag{5}$$

where $S_n = Y_1 + \ldots + Y_n$, and $Y_1, Y_2, \ldots$ are independent random variables (see [2]).

Now we define certain subclasses of the classes L_c.

Let $c_1, c_2, \ldots, c_k \in \boldsymbol{R}$, $k \geqslant 1$. We say that φ *belongs to* $L_{c_1,c_2,\ldots,c_k}$ if and only if

$$\varphi(t) = \varphi(c_1 t)\varphi_{c_1}(t),\ \varphi_{c_1}(t) = \varphi_{c_1}(c_2 t)\varphi_{c_1,c_2}(t), \ldots \\ \ldots, \varphi_{c_1,\ldots,c_{k-1}}(t) = \varphi_{c_1,\ldots,c_{k-1}}(c_k t)\varphi_{c_1,\ldots,c_k}(t) \tag{6}$$

for some characteristic functions $\varphi_{c_1}, \varphi_{c_1,c_2}, \varphi_{c_1,\ldots,c_k}$.

We note that (6) is equivalent to the following statement:

$$\varphi \in L_{c_1},\ \varphi_{c_1} \in L_{c_2}, \ldots, \varphi_{c_1,\ldots,c_{k-1}} \in L_{c_k}. \tag{7}$$

Obviously, if $\varphi \in L_{c_1,\ldots,c_k}$, then

$$\varphi(t) = \varphi(c_1 t)\varphi_{c_1}(c_2 t) \ldots \varphi_{c_1,\ldots,c_{k-1}}(c_k t)\varphi_{c_1,\ldots,c_k}(t), \tag{8}$$

$$\mathscr{L}(X) = \mathscr{L}(c_1 X + c_2 X_{c_1} + c_3 X_{c_1,c_2} + \ldots + c_k X_{c_1,\ldots,c_{k-1}} + X_{c_1,\ldots,c_k}) \tag{9}$$

for some independent random variables $X, X_{c_1}, X_{c_1,c_2}, \ldots, X_{c_1,\ldots,c_k}$ with characteristic functions $\varphi, \varphi_{c_1}, \varphi_{c_1,c_2}, \ldots, \varphi_{c_1,\ldots,c_k}$, respectively.

We say that ψ *belongs to* $D_{c_1,\ldots,c_k}$ if and only if there exist characteristic functions $\varphi, \varphi_{c_1}, \ldots, \varphi_{c_1,\ldots,c_k}$ satisfying (6) such that $\varphi_{c_1,\ldots,c_k} = \psi$. For $c_1 = c_2 = \ldots = c_k = c$, instead of $L_{c_1,\ldots,c_k}, \varphi_{c_1,\ldots,c_k}, X_{c_1,\ldots,c_k}, D_{c_1,\ldots,c_k}$ we will write $L_{c,(k)}, \varphi_{c,(k)}, X_{c,(k)}, D_{c,(k)}$, respectively.

We note that

$$L_{c_1,\ldots,c_{k-1},c_k} \subset L_{c_1,\ldots,c_{k-1}}, \; L_{c,(k)} \subset L_{c,(k-1)}. \tag{10}$$

Let Z be a random variable with the characteristic function ψ. We say that ψ *belongs to* $D_{(k)}$ if and only if

$$E\left(\ln^k(|Z|+1)\right) < \infty. \tag{11}$$

The following theorem is a generalization of the theorem proved by Zakusilo in [20] for $k = 1$.

THEOREM 1. *For each* k, $k \geqslant 1$, *the classes* $D_{c,(k)}$, $0 < |c| < 1$, *are independent of* c *and coincide with the class* $D_{(k)}$.

Proof. Given $k \geqslant 1$ and $0 < |c| < 1$. Let $\{Z_{j_1,\ldots,j_k}\}_{j_1,\ldots,j_k=0}^{\infty}$, and let Z be independent and identically distributed random variables with an arbitrary common characteristic function ψ.

Let $N = \{0, 1, 2, \ldots\}$ and $N^k = \{(j_1, \ldots, j_k), j_1, \ldots, j_k \in N\}$. Consider a one-to-one and onto mapping $x\colon N \to N^k$. For elements of N^k we put

$$x(n) = \left((x(n))_1, \ldots, (x(n))_k\right), \quad |x(n)| = \sum_{j=1}^{k} (x(n))_j, \quad n = 0, 1, 2, \ldots \tag{12}$$

We are going to investigate the convergence of the series

$$\sum_{n=0}^{\infty} c^{|x(n)|} Z_{x(n)}, \tag{13}$$

say to a random variable $X_k(c)$.

As in [20] (see also [1]), we consider two series:

$$\sum_{n=0}^{\infty} P(\{|c^{|x(n)|} Z_{x(n)}| > 1\}), \tag{i}$$

$$\sum_{n=0}^{\infty} E(|c^{|x(n)|} Z_{x(n)}|;\; |c^{|x(n)|} Z_{x(n)}| < 1). \tag{ii}$$

We observe that the convergence of the series (i) and (ii) is equivalent to the convergence of the following series (i′) and (ii′) and, consequently, (i″) and (ii″):

$$\sum_{j_1=0}^{\infty} \cdots \sum_{j_k=0}^{\infty} P(\{|c^{j_1} \ldots c^{j_k} Z_{j_1,\ldots,j_k}| > 1\}), \tag{i′}$$

$$\sum_{j_1=0}^{\infty} \cdots \sum_{j_k=0}^{\infty} E(|c^{j_1} \ldots c^{j_k} Z_{j_1,\ldots,j_k}|;\; |c^{j_1} \ldots c^{j_k} Z_{j_1,\ldots,j_k}| < 1), \tag{ii′}$$

(i″) $$\sum_{n=0}^{\infty} \dots \sum_{j_1+\dots+j_k=n} P(\{|c^{j_1}\dots c^{j_k} Z_{j_1,\dots,j_k}| > 1\}),$$

(ii″) $$\sum_{n=0}^{\infty} \dots \sum_{j_1+\dots+j_k=n} E(|c^{j_1}\dots c^{j_k} Z_{j_1,\dots,j_k}|;\ |c^{j_1}\dots c^{j_k} Z_{j_1,\dots,j_k}| < 1).$$

Taking into account the equality $\mathscr{L}(Z_{j_1,\dots,j_k}) = \mathscr{L}(Z)$ we can write the series (i″) in the form

$$\sum_{n=0}^{\infty} \binom{n+k-1}{k-1} P(\{|Z| > |c^{-1}|^n\})$$

The convergence of the above series is equivalent to the convergence of the series

$$\sum_{n=0}^{\infty} (n+1)^{k-1} P(\{\ln|Z| > n\ln|c^{-1}|\})$$
$$= \sum_{n=0}^{\infty} (n+1)^{k-1} \sum_{j=n}^{\infty} P\big(\{j\ln|c^{-1}| < \ln|Z| \leqslant (j+1)\ln|c^{-1}|\}\big)$$
$$= \sum_{n=0}^{\infty} \Big(\sum_{j=1}^{n+1} j^{k-1}\Big) P\big(\{n\ln|c^{-1}| < \ln|Z| \leqslant (n+1)\ln|c^{-1}|\}\big).$$

The above series is convergent if and only if the series

$$\sum_{n=0}^{\infty} (n+1)^{k} P\big(\{n\ln|c^{-1}| < \ln|Z| \leqslant (n+1)\ln|c^{-1}|\}\big)$$

is convergent, which is equivalent to satisfying the condition (11).

The series (ii″) equals

$$\sum_{n=0}^{\infty} \binom{n+k-1}{k-1} |c^n|\, E(|Z|;\ |Z| < |c^{-1}|^n).$$

The convergence of the above series is equivalent to the convergence of the series

(14) $$\sum_{n=0}^{\infty} (n+1)^{k-1} |c^n|\, E(|Z|;\ |Z| < |c^{-1}|^n)$$
$$= \sum_{n=0}^{\infty} (n+1)^{k-1} |c^n|\, E(|Z|;\ |Z| < 1)$$
$$+ \sum_{n=0}^{\infty} (n+1)^{k-1} |c^n| \sum_{j=0}^{n-1} E(|Z|;\ |c^{-1}|^j \leqslant |Z| < |c^{-1}|^{j+1})$$
$$\leqslant E(|Z|;\ |Z| < 1) \sum_{n=0}^{\infty} (n+1)^{k-1} |c|^n$$
$$+ \sum_{n=0}^{\infty} (n+1)^{k-1} |c|^n \sum_{j=0}^{n-1} |c^{-1}|^{j+1} P(\{|c^{-1}|^j \leqslant |Z| \leqslant |c^{-1}|^{j+1}\})$$
$$\leqslant \frac{(k-1)!}{(1-|c|)^k} + \sum_{j=0}^{\infty} P(\{|c^{-1}|^j \leqslant |Z| < |c^{-1}|^{j+1}\}) \sum_{n=j+1}^{\infty} (n+1)^{k-1} |c|^{n-j-1}.$$

Since

$$\sum_{n=j+1}^{\infty} (n+1)^{k-1}|c|^{n-j-1} = \sum_{i=0}^{\infty} (i+j+2)^{k-1}|c|^i$$

$$= \sum_{i=0}^{\infty}\left(\sum_{m=0}^{k-1}\binom{k-1}{m}(j+1)^m(i+1)^{k-1-m}\right)|c|^i$$

$$= \sum_{m=0}^{k-1}\binom{k-1}{m}(j+1)^m \sum_{m=0}^{\infty}(i+1)^{k-1-m}|c|^i$$

$$\leqslant \sum_{m=0}^{k-1}\binom{k-1}{m}(j+1)^m \frac{(k-1-m)!}{1-|c|}\left(\frac{1}{1-|c|}\right)^{k-1-m}$$

$$\leqslant \frac{(k-1)!}{1-|c|}\left(\frac{1}{1-|c|}+j+1\right)^{k-1},$$

for the series in expressions (14) we have the inequality

$$\sum_{j=0}^{\infty} P(\{|c^{-1}|^j \leqslant |Z| < |c^{-1}|^{j+1}\}) \sum_{n=j+1}^{\infty} (n+1)^{k-1}|c|^{n-j-1}$$

$$\leqslant \frac{(k-1)!}{1-|c|}\sum_{j=0}^{\infty} P(\{|c^{-1}|^j \leqslant |Z| < |c^{-1}|^{j+1}\})\left(\frac{1}{1-|c|}+j+1\right)^{k-1}.$$

Since the series in the above expression is convergent if and only if $E(\ln^{k-1}(|Z|+1)) < \infty$, this completes the proof that the convergence of two series (i) and (ii) is equivalent to satisfying the condition (11). We note that condition (11) is independent of c. Hence we conclude that the series (13) is convergent if and only if the condition (11) holds; moreover, the convergence of the series (13) is independent of the choice of the mapping x. Thus we can write $X_k(c)$ in the form

$$X_k(c) = \sum_{j_1=0}^{\infty} \cdots \sum_{j_k=0}^{\infty} c^{j_1}\cdots c^{j_k} Z_{j_1,\ldots,j_k}.$$

Putting

$$\varphi_{c,(k)}(t) = \psi(t),\ \varphi_{c,(k-1)}(t) = \prod_{n=0}^{\infty}\varphi_{c,k}(c^n t),\ \ldots$$

$$\ldots,\ \varphi_{c,(j-1)} = \prod_{n=0}^{\infty}\varphi_{c,(j)}(c^n t), \qquad 1 \leqslant j \leqslant k,$$

$$\varphi(t) = \varphi_{c,(0)}(t),$$

we complete the proof of the theorem.

Let $k \geqslant 1$ and $0 < |c_j| < 1$ for $1 \leqslant j \leqslant k$. Consider now the series

$$\sum_{j_1=0}^{\infty} \cdots \sum_{j_k=0}^{\infty} c_1^{j_1}\cdots c_k^{j_k} Z_{j_1,\ldots,j_k}$$

for $Z_{j_1,\ldots,j_k}$ as in Theorem 1.

Then taking into account the inequality

$$(\min_{1 \leqslant j \leqslant k} |c_j|)^{j_1 + \ldots + j_k} \leqslant |c_1|^{j_1} \ldots |c_k|^{j_k} \leqslant (\max_{1 \leqslant j \leqslant k} |c_j|)^{j_1 + \ldots + j_k},$$

as a corollary to Theorem 1 we obtain

THEOREM 2. *Let $k \geqslant 1$. The classes $D_{c_1, \ldots, c_k}$, $0 < |c_j| < 1$, $1 \leqslant j \leqslant k$, coincide with the class $D_{(k)}$.*

From the proof of the above two theorems we obtain immediately the following two theorems:

THEOREM 3. *Let $k \geqslant 1$ and $0 < |c_j| < 1$ for $1 \leqslant j \leqslant k$. Let φ be a characteristic function. Then the following conditions are equivalent:*

(a) $\varphi \in L_{c_1, \ldots, c_k}$.

(b) *φ is the characteristic function of*

$$X(c_1, \ldots, c_k) = \sum_{j_1=0}^{\infty} \ldots \sum_{j_k=0}^{\infty} c_1^{j_1} \ldots c_k^{j_k} Z_{j_1, \ldots, j_k}, \tag{15}$$

where $\{Z_{j_1, \ldots, j_k}\}_{j_1, \ldots, j_k=0}^{\infty}$ are independent identically distributed random variables with the same characteristic function ψ. Then the series converges a.s.

(c) *φ is the characteristic function of the form*

$$\varphi(t) = \prod_{j_1=0}^{\infty} \ldots \prod_{j_k=0}^{\infty} \psi(c_1^{j_1} \ldots c_k^{j_k}) \tag{16}$$

for some characteristic function ψ.

THEOREM 4. *Let $k \geqslant 1$ and $0 < |c| < 1$. Let φ be a characteristic function. Then the following conditions are equivalent:*

(a) $\varphi \in L_{c,(k)}$.

(b) *φ is the characteristic function of*

$$X_k(c) = \sum_{n=0}^{\infty} c^n \sum_{j=1}^{\binom{n+k-1}{k-1}} Z_{n,j}, \tag{17}$$

where $\{Z_{n,j}\}_{n,j}$, $n = 0, 1, 2, \ldots, j = 1, 2, \ldots, \binom{n+k-1}{k-1}$, are independent identically distributed random variables with the same characteristic function $\psi = \varphi_{c,(k)}$. Then the series converges a.s.

(c) *φ is the characteristic function of the form*

$$\varphi(t) = \prod_{n=0}^{\infty} [\psi(c^n t)]^{\binom{n+k-1}{k-1}} \tag{18}$$

for some characteristic function $\psi = \varphi_{c,(k)}$.

In the next theorem we show that the classes $L_{c,(k)}$ coincide with some limit distributions of normed sums.

THEOREM 5. *Let $k \geqslant 1$, $0 < |c| < 1$, and φ be a characteristic function. Then $\varphi \in L_{c,(k)}$ if and only if there exists a sequence of positive numbers*

$B_0, B_1, \ldots, B_n/B_{n+1} \to c$ and a sequence of random variables $U_0, U_1, \ldots$ such that, for independent random variables $\{X_{n,j}\}_{n,j}$, $n = 0, 1, 2, \ldots$, $j = 1, 2, \ldots, \binom{n+k-1}{k-1}$,

$$\mathscr{L}(X_{n,1}) = \mathscr{L}(U_n) \quad \text{and} \quad \mathscr{L}(X_{n,j}) = \mathscr{L}(U_i) \text{ for } k > 1,$$

$$\binom{n-i-1+k-1}{k-1} < j \leqslant \binom{n-i+k-1}{k-1}, \quad 0 \leqslant i \leqslant n-1,\ k > 1, \tag{19}$$

$$Y_{n,(k)} = \sum_{j=1}^{\binom{n+k-1}{k-1}} X_{n,j}, \tag{20}$$

$$S_{n,(k)} = Y_{1,(k)} + \ldots + Y_{n,(k)} \tag{21}$$

such that the characteristic function of $S_{n,(k)}/B_n$ is convergent to the characteristic function φ,

$$\varphi_{S_{n,(k)}/B_n}(t) \to \varphi(t). \tag{22}$$

Proof. Let $B_0, B_1, \ldots, B_n/B_{n+1} \to c$ be a sequence of positive integers and $U_0, U_1, \ldots$ be a sequence of random variables. Suppose that for independent random variables $\{X_{n,j}\}_{n,j}$, $n = 0, 1, 2, \ldots, j = 1, 2, \binom{n+k-1}{k-1}$, as in (19), and for $Y_{n,(k)}$ and $S_{n,(k)}$, $n = 1, 2, \ldots$, defined by (20) and (21), respectively, the convergence (22) holds. Since

$$\varphi_{S_{n,(k)}/B_n}(t) = \varphi_{S_{n-1,(k)}/B_{n-1}}\left(\frac{B_{n-1}}{B_n}t\right)\varphi_{Y_{n,(k)}/B_n}(t),$$

$$\text{where } \varphi_{S_{n-1,(k)}/B_{n-1}}\left(\frac{B_{n-1}}{B_n}t\right) \to \varphi(ct),$$

denoting by $\varphi_{c,(1)}$ the characteristic function which is the limit of $\varphi_{Y_{n,(k)}/B_n}(t)$ (see [2]; without loss of generality we can assume that $\varphi_{Y_{n,(k)}/B_n}(t)$ is convergent to a characteristic function, passing to a subsequence if necessary), we obtain

$$\varphi(t) = \varphi(ct)\varphi_{c,(1)}(t) \tag{23}$$

and, consequently, $\varphi \in L_c$. This completes the proof of "if" assertion for $k = 1$.

Now suppose that $k > 1$. Then

$$\varphi_{Y_{n,(k)}/B_n}(t) = [\varphi_{U_0/B_n}(t)]^{\binom{n+k-2}{k-2}} \ldots [\varphi_{U_n/B_n}(t)]^{\binom{0+k-2}{k-2}}$$
$$= \varphi_{Y_{n-1,(k)}/B_{n-1}}\left(\frac{B_{n-1}}{B_n}t\right)[\varphi_{U_0/B_n}(t)]^{\binom{n+k-3}{k-3}} \ldots [\varphi_{U_n/B_n}(t)]^{\binom{0+k-3}{k-3}},$$

where

$$\varphi_{Y_{n(k)}/B_n}(t) \to \varphi_{c,(1)}(t), \qquad \varphi_{Y_{n-1,(k)}/B_{n-1}}\left(\frac{B_{n-1}}{B_n}t\right) \to \varphi_{c,(1)}(ct).$$

Hence, as for $\varphi_{c,(1)}$, we can put

$$\varphi_{c,(2)} = \lim_{n\to\infty} [\varphi_{U_0/B_n}(t)]^{\binom{n+k-3}{k-3}} \dots [\varphi_{U_n/B_n}(t)]^{\binom{0+k-3}{k-3}}.$$

Then we obtain $\varphi_{c,(1)} \in L_c$ and, consequently, $\varphi \in L_{c,(2)}$.

It is not difficult to show (by induction) that for $1 \leqslant j \leqslant k-1$

$$\varphi_{c,(j)}(t) = \varphi_{c,(j)}(ct)\,\varphi_{c,(j+1)}(t), \tag{24}$$

where

$$\varphi_{c,(j)}(t) \doteq \lim_{n\to\infty} [\varphi_{U_0/B_n}(t)]^{\binom{n+k-(j+1)}{k-(j+1)}} \dots [\varphi_{U_n/B_n}(t)]^{\binom{0+k-(j+1)}{k-(j+1)}}, \quad 1 \leqslant j \leqslant k-1,$$

$$\varphi_{c,(k)}(t) = \lim_{n\to\infty} \varphi_{U_n/B_n}(t).$$

By (23) and (24) we obtain $\varphi \in L_{c,(k)}$. Thus, $\varphi \in L_{c,(k)}$ in both the cases, and the "if" assertion is proved.

Now suppose that $\varphi \in L_{c,(k)}$, $k \geqslant 1$. It follows from Theorem 4 (b) that there exist independent and identically distributed random variables $\{Z_{n,j}\}_{n,j}$, $n = 0, 1, 2, \dots, j = 1, 2, \dots, \binom{n+k-1}{k-1}$, with common characteristic function ψ, i.e.

$$\varphi_{Z_{n,j}} = \varphi_Z = \psi, \quad n = 0, 1, 2, \dots, j = 1, 2, \dots, \binom{n+k-1}{k-1},$$

such that φ is the characteristic function of

$$X_k(c) = \sum_{n=0}^{\infty} c^n \sum_{j=0}^{\binom{n+k-1}{k-1}} Z_{n,j}.$$

Let $B_0, B_1, \dots$ be a number sequence such that $B_n/B_{n+1} \to c$. Put $U_n = B_n Z_{n,1}$, $X_{n,1} = U_n$, $n = 0, 1, 2, \dots$ In the case $k > 1$ we put

$$X_{n,j} = B_i Z_{n,j}, \quad \binom{n-i-1+k-1}{k-1} < j \leqslant \binom{n-i+k-1}{k-1}, \; 0 \leqslant i \leqslant n-1.$$

Further, we put

$$Y_{n,(k)} = \sum_{j=1}^{\binom{n+k-1}{k-1}} X_{n,j}, \quad S_{n,(k)} = Y_{1,(k)} + \dots + Y_{n,(k)}, \quad n = 1, 2, \dots$$

In the case $k \geqslant 2$ we have

$$\varphi_{S_{n,(k)}/B_n}(t) = [\varphi_{B_n/B_n Z}(t)]^{\binom{0+k-2}{k-2}} [\varphi_{B_{n-1}/B_n Z}(t)]^{\binom{0+k-2}{k-2}+\binom{1+k-2}{k-2}} \dots$$
$$\dots [\varphi_{B_1/B_n}(t)]^{\binom{0+k-2}{k-2}+\dots+\binom{n-1+k-2}{k-2}} [\varphi_{B_0/B_n}(t)]^{\binom{0+k-2}{k-2}+\dots+\binom{n+k-2}{k-2}}.$$

Taking into account that

$$\binom{0+k-2}{k-2}+\ldots+\binom{j+k-2}{k-2}=\binom{j+k-1}{k-1}, \quad j=0, 1, 2, \ldots, n,$$

we obtain

$$(25) \qquad \varphi_{S_{n,(k)}/B_n}(t) = [\psi(t)]^{\binom{0+k-1}{k-1}}\left[\psi\left(\frac{B_{n-1}}{B_n}t\right)\right]^{\binom{1+k-1}{k-1}}\ldots$$

$$\ldots\left[\psi\left(\frac{B_0}{B_1}\frac{B_1}{B_2}\ldots\frac{B_{n-1}}{B_n}t\right)\right]^{\binom{n+k-1}{k-1}}.$$

Formula (25) in the case $k = 1$ evidently also holds. From (25) it follows that

$$\varphi_{S_{n,(k)}/B_n}(t) = \prod_{n=0}^{\infty}[\psi(c^n t)]^{\binom{n+k-1}{k-1}},$$

and this, by Theorem 4 (c), yields $\varphi_{S_{n,(k)}/B_n}(t) \to \varphi(t)$. Thus the assertion "only if" is proved.

REFERENCES

[1] W. Feller, *Wstęp do rachunku prawdopodobieństwa*, t. II (in Polish), Warszawa 1978.

[2] A. Ilinskij, *On c-decomposition of characteristic functions*, Liet. Matem. Rink. 4 (1978), pp. 45–50.

[3] Z. Jurek, *The classes $L_m(Q)$ of probability measures on Banach spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 31 (1983), pp. 51–62.

[4] — and W. Vervaat, *An integral representation for selfdecomposable Banach space valued random variables*, Z. Wahrscheinlichkeitstheorie verw. Gebiete 62 (1983), pp. 247–262.

[5] M. Loève, *Nouvelles classes de lois limites*, Bull. Soc. Math. France 73, 1–2 (1945), pp. 107–126.

[6] — *Probability Theory*, New York 1955.

[7] F. F. Misheikis, *On certain classes of limit distributions* (in Russian), Litovsk. Mat. Sb. 12,4 (1972), pp. 133–152.

[8] — *On certain classes of limit distributions* (in Russian), ibidem 12,3 (1972), pp. 101–106.

[9] — *On certain classes of probability laws* (in Russian), ibidem 15,2 (1975), pp. 60–65.

[10] Nguyen van Thu, *Multiply self-decomposable probability measures on Banach spaces*, Studia Math. 66 (1979), pp. 160–175.

[11] T. Niedbalska, *An example of the decomposability semigroup*, Colloq. Math. 39 (1978), pp. 137–139.

[12] T. Niedbalska-Rajba, *On decomposability semigroups on the real line*, ibidem 44 (1981), pp. 347–358.

[13] T. Rajba, *On decomposability semigroups for certain probability measures*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 28 (1979), pp. 415–418.

[14] — *A representation of distributions from certain classes L_S^{id}*, Probab. Math. Statist. 4 (1984), pp. 67–78.

[15] K. Sato, *Class L of multivariate distributions and its subclasses*, J. Multivariate Anal. 10 (1980), pp. 207–232.

[16] K. Urbanik, *A representation of self-decomposable distributions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16 (1968), pp. 209–214.

[17] — *Lévy's probability measures on Euclidean spaces*, Studia Math. 54 (1972), pp. 119–148.

[18] — *Operator semigroups associated with probability measures*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 23 (1975), pp. 75–76.

[19] — *Lévy's probability measures on Banach spaces*, Studia Math. 63 (1987), pp. 283–308.

[20] O. K. Zakusilo, *On classes of limit distributions there is some scheme of summing up* (in Russian), Teor. Veroyatnost. i Mat. Statist. 12 (1975), pp. 44–48.

[21] — *Some properties of classes L_c of limit distributions* (in Russian), ibidem 15 (1976), pp. 68–73.

Received on 26.6.1998;
revised version on 11.11.1998