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# AN APPLICATION OF WAVELET ANALYSIS TO PRICING AND HEDGING DERIVATIVE SECURITIES

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*Abstract.* This work provides an application of wavelet analysis to pricing and hedging path-dependent contingent claims within the framework of the Black–Scholes model.

**1. Introduction.** A European contingent claim written on an asset is a financial contract which gives its owner the right to receive a payoff at the expiration date $T$. This payoff depends on the market behaviour (the path) of the underlying asset during the time $[0, T]$ as determined in the contract. Two problems arise naturally when dealing with contingent claims. The *problem of pricing*: Calculate the fair price of the contingent claim at each time $0 \leqslant t < T$ using the behaviour of the underlying asset during the time $[0, t]$. The *problem of hedging*: Having sold the contingent claim, how can the seller insure against the upcoming random loss at the time $T$? In some cases of interest, in particular in the case of the Black–Scholes maket, we find the solution of these problems in terms of the so-called *arbitrage-free pricing*.

In the Black–Scholes model we specify three financial assets traded continuously during the time $[0, T]$. The corresponding prices are modelled by adapted stochastic processes $(\tilde{S}_t)_{t\in[0,T]}$, $(\tilde{B}_t)_{t\in[0,T]}$, and $(\tilde{Y}_t)_{t\in[0,T]}$ on the filtered probability space $(\Omega, (\mathscr{F}_t)_{t\in[0,T]}, \tilde{P})$, where $(\mathscr{F}_t)_{t\in[0,T]}$ denotes the natural filtration generated by some Brownian motion $(\tilde{W}_t)_{t\in[0,T]}$. The *stock process* $(\tilde{S}_t)_{t\in[0,T]}$ with initial value $S_0 > 0$ describes the price of a risky asset. It is given by

$$\tilde{S}_t := \exp\{\sigma\tilde{W}_t + (\mu - \tfrac{1}{2}\sigma^2)t\} S_0 \quad \text{for all } t\in[0, T].$$

The constants $\mu > 0$ and $\sigma > 0$ are known as the *appreciation rate* and the *stochastic volatility* of the stock. The *bond process* $(\tilde{B}_t)_{t\in[0,T]}$ represents a risk-free security, assumed to continuously compound in value at the fixed *interest rate* $r > 0$, meaning $\tilde{B}_t := e^{rt}$ for all $t\in[0, T]$. The terminal payoff of the contingent claim is given by an $\mathscr{F}_T$-measurable random variable $\tilde{Y}_T$. The process $(\tilde{Y}_t)_{t\in[0,T]}$, which is to be determined, corresponds to the behaviour of the market price of the contingent claim during $[0, T]$. The main idea of the arbitrage-free pricing is to introduce the *discounted* processes as $(S_t := e^{-rt}\tilde{S}_t)_{t\in[0,T]}$, and $(Y_t := e^{-rt}\tilde{Y}_t)_{t\in[0,T]}$. They describe the prices of the stock and of the claim if the risk-free security is chosen as the numeraire asset. Supposing the absence of arbitrage and following standard arguments (see, for example, [10], [5], [12]), we are led to the following statement: There exists a probability measure $P$, which is equivalent to $\tilde{P}$ such that the discounted processes $(S_t)_{t\in[0,T]}$ and $(Y_t)_{t\in[0,T]}$ are martingales under $P$. In our setting the measure $P$ is unique and the Radon–Nikodym derivative $dP/d\tilde{P}$ is explicitly obtained. Furthermore, from the theorem of Girsanov it follows that $(S_t)_{t\in[0,T]}$ satisfies

$$S_t = \exp\left\{\sigma W_t - \frac{\sigma^2}{2}t\right\} S_0 \quad \text{for all } t\in[0,T],$$

where $(W_t)_{t\in[0,T]}$ is a Brownian motion with respect to $P$ and $S_0 > 0$. In the following let us suppose that $Y_T \in L^2(\Omega, \mathscr{F}_T, P)$. In this situation the problem of pricing is easily solved: The fair (discounted) price of the contingent claim at the time $t\in[0,T]$ is found by calculating the conditional expectation of $Y_T$ under $P$: $(Y_t := E_P(Y_T \mid \mathscr{F}_t))_{t\in[0,T]}$. The problem of hedging admits the following treatment: Representing the square-integrable martingale $(Y_t)_{t\in[0,T]}$ as the stochastic integral (this is possible in our setting)

$$Y_t = Y_0 + \int_0^t y_s \, dW_s \quad \text{for all } t\in[0,T],$$

we obtain $(y_s)_{s\in[0,T]}$ and determine the process $(\eta, \theta) = ((\eta_t, \theta_t))_{t\in[0,T]}$ (the trading strategy) as $\eta_t = Y_t - y_t\sigma^{-1}$ and $\theta_t = y_t(\sigma S_t)^{-1}$ for all $t\in[0,T]$. In such a strategy, $\theta_t$ describes the number of units of risky asset held at the time $t$, and $\eta_t$ describes the amount invested in the riskless asset at the time $t$. The strategy $(\eta, \theta)$ satisfies

$$\text{(1)} \qquad \tilde{Y}_t = Y_0 + \int_0^t \eta_s \, d\tilde{B}_s + \int_0^t \theta_s \, d\tilde{S}_s \quad \text{for all } t\in[0,T],$$

$$\text{(2)} \qquad \tilde{Y}_t = \theta_t \tilde{S}_t + \eta_t \tilde{B}_t \quad \text{for all } t\in[0,T].$$

From the equation (1) it follows that, starting with initial investment $Y_0$, the trading strategy $(\eta, \theta)$ replicates the payoff $\tilde{Y}_T$ of the contingent claim. The equation (2) means that in order to attain $\tilde{Y}_T$ in this way only the investment $Y_0$ at the time $t = 0$ is needed. Such a strategy is called *self-financing*. (For additional information we refer the reader to [12], [5], [15].) The seller of contingent claim may apply this trading strategy to avoid the risk completely. However, since closed-form expressions of $Y_t$ and $y_t$ are not always available, it is important to study numerical methods (see, for example, [5], [6], [1]).

In this work we apply some results from the wavelet theory in order to obtain square-mean approximations of $(Y_t)_{t\in[0,T]}$ and of $(y_t)_{t\in[0,T]}$. Let us explain which $L^2$-approximations are meant. We choose the orthonormal basis $(\varphi^k)_{k\in N}$

of $L^2[0, T]$ by putting $\varphi_s^0 := \sqrt{1/T}$,

$$\varphi_s^{2k} := \sqrt{\frac{2}{T}} \cos\left(\frac{2\pi ks}{T}\right),$$

$$\varphi_s^{2k+1} := \sqrt{\frac{2}{T}} \sin\left(\frac{2\pi ks}{T}\right) \quad \text{for all } k \geqslant 1,\ s \in [0, T]$$

and define the family $(E^n)_{n\in N}$ of linear subspaces of $L^2[0, T]$, where $E^n$ is spanned by $(\varphi^k)_{k=0}^n$. For each $n \in N$ let us also introduce the $\sigma$-algebra $\sigma_n$, generated by $\{\int_0^T z_s\, dW_s \colon z \in E^n\}$, the random variable $Y_T^n := E_P(Y_T \mid \sigma_n)$, the martingale $(Y_t^n := E_P(Y_T^n \mid \mathscr{F}_t))_{t\in[0,T]}$, and the predictable process $(y_t^n)_{t\in[0,T]}$ defined by

$$Y_T^n = E_P(Y_T^n) + \int_0^T y_s^n\, dW_s.$$

Note that, in view of the properties of the Hermite polynomials,

$$\{X\colon X \text{ is a polynomial in } \int_0^T \varphi_s^0\, dW_s, \ldots, \int_0^T \varphi_s^n\, dW_s\}$$

is a dense subspace of $L^2(\Omega, \sigma_n, P)$. Moreover, it is well known (see [11], Chapter 4.2) that

$$\bigcup_{n=0}^{\infty} \{X\colon X \text{ is a polynomial in } \int_0^T \varphi_s^0\, dW_s, \ldots, \int_0^T \varphi_s^n\, dW_s\}$$

is a dense subspace of $L^2(\Omega, \mathscr{F}_T, P)$. Hence $\bigcup_{n=0}^{\infty} L^2(\Omega, \sigma_n, P)$ is dense in $L^2(\Omega, \mathscr{F}_T, P)$. This implies that the sequence $(Y_T^n)_{n\in N}$ converges to $Y_T$ in the square mean. Doob's maximal inequality and the isometry of stochastic integral yield:

$$\text{(3)} \qquad \lim_{n\to\infty} E_P\big(\sup_{t\in[0,T]} |Y_t^n - Y_t|^2\big) = 0, \qquad \lim_{n\to\infty} E_P\big(\int_0^T |y_t^n - y_t|^2\, dt\big) = 0.$$

Let $n \in N$ be sufficiently large. Then, in view of (3), the processes $(Y_t^n)_{t\in[0,T]}$ and $(y_t^n)_{t\in[0,T]}$ are seen to be $L^2$-approximations of $(Y_t)_{t\in[0,T]}$ and $(y_t)_{t\in[0,T]}$, respectively.

Throughout this paper let $n \in N$ be fixed. For each $t \in [0, T]$ we introduce $R^{n+1}$-valued Gaussian random variable

$$\Phi_t = (\Phi_t^k)_{k=0}^n := \big(\int_0^t \varphi_s^k\, dW_s\big)_{k=0}^n.$$

Its covariance matrix $G_t$ satisfies

$$G_t = \big(\int_0^t \varphi_s^k \varphi_s^l\, ds\big)_{k,l=0}^n.$$

Note that $1 - G_t$ is positive definite for all $t \in [0, T[$. We shall denote the transpose (of a matrix) by $*$ and write $\varphi_t$ for the (column) vector

$(\varphi_t^0, \ldots, \varphi_t^n) \in \boldsymbol{R}^{n+1}$. For all $t \in [0, T[$ and $\omega \in \Omega$ we define the random variables $K_{t,\omega}, K'_{t,\omega} \in L^2(\Omega, \mathscr{F}_T, P)$ as

$$K_{t,\omega} := |\det(1-G_t)^{-1/2}|$$
$$\times \exp\{-\tfrac{1}{2}(\Phi_T - \Phi_t(\omega))^\star (1-G_t)^{-1}(\Phi_T - \Phi_t(\omega))\} \exp\{\tfrac{1}{2}\Phi_T^\star \Phi_T\},$$
$$K'_{t,\omega} := (\Phi_T - \Phi_t(\omega))^\star (1-G_t)^{-1} \varphi_t K_{t,\omega}.$$

Using the notation above, we can formulate the main result of this work as follows:

PROPOSITION 1. *For all* $t \in [0, T[$ *and* $\omega \in \Omega$:

$$Y_t^n(\omega) = E_P(K_{t,\omega} Y_T), \tag{4}$$

$$y_t^n(\omega) = E_P(K'_{t,\omega} Y_T). \tag{5}$$

Note that since $\Phi_t$ is $\mathscr{F}_t$-measurable, the value $\Phi_t(\omega) \in \boldsymbol{R}^{n+1}$ is observed at the time $t$. For this reason a calculation of (4) and of (5) involves only the evaluation of the mean value of the random variables $Y_T K_{t,\omega}$ and $Y_T K'_{t,\omega}$. This may be done numerically.

**2. The mathematical background.** Hilbert spaces used in this work are separable and inner products are linear on the right. The linear and the closed linear space spanned by a set $M$ are denoted by $\operatorname{lin} M$ and by $\overline{\operatorname{lin}}\, M$, respectively. All integrals of Hilbert space-valued functions are understood in the weak sense. The group of unitary operators on the Hilbert space $\mathscr{H}$ is denoted by $\mathscr{U}(\mathscr{H})$. The $\sigma$-algebra generated by a set $M$ of random variables is denoted by $\sigma(M)$. The space of continuous functions on a topological space $X$ is denoted by $C(X)$.

The wavelet analysis was introduced by Grossmann et al. in [7] and [8], and was motivated by applications in the signal processing. We recall some recent results from this theory.

Let $G$ be a locally compact group equipped with a left Haar measure $\mu$. A strongly continuous irreducible unitary representation $U$ of $G$ on the Hilbert space $\mathscr{H}$ is called *square integrable* if there exists a vector $v \in \mathscr{H}$ satisfying

$$v \neq 0 \quad \text{and} \quad \int_G |\langle U(g)v, v\rangle|^2 \mu(dg) < \infty. \tag{6}$$

Such a vector $v$ is called a *wavelet*. Given $G$, $U$, $\mathscr{H}$, and $v \in \mathscr{H}$ as above we introduce $V\colon \mathscr{H} \to C(G)$, $h \mapsto Vh$ by putting $Vh(g) := \langle U(g)v, h\rangle$ for all $g \in G$ and $h \in \mathscr{H}$. The mapping $V$ is called the *wavelet transform*. In [7] it is shown that the wavelet transform is, up to a positive constant, an isometric operator from $\mathscr{H}$ into $L^2(G, \mu)$. Let us choose the left Haar measure $\mu$ such that $V$ becomes isometric. The adjoint $V^*$ is given by

$$V^*\xi = \int_G \xi(g)\, U(g) v \mu(dg) \quad \text{for all } \xi \in L^2(G, \mu).$$

From $V^* V = 1_{\mathscr{H}}$ we infer the inversion formula of the wavelet transform:

(7) $$h = \int_G Vh(g)\, U(g)\, v\mu(dg) \quad \text{for all } h \in \mathscr{H}.$$

In the following we also need some special constructions of Hilbert spaces; for proofs and details we refer the reader to [14], p. 92. Let $M$ be any set. The map $k: M \times M \to \boldsymbol{C}$ is called a *positive definite kernel* on $M$ if the matrix $(k(x_i, x_j))_{i,j=1}^n$ is positive semidefinite for all $x_1, \ldots, x_n \in M$ and $n \in \boldsymbol{N}$. Let $k$ be a positive definite kernel on $M$. Then there exists a Hilbert space $K$ and a map $e: M \to K$ such that $\langle e(x), e(y)\rangle = k(x, y)$ for all $x, y \in M$ and the linear space $\mathrm{lin}\{e(x): x \in M\}$ is dense in $K$. The pair $(e, K)$ is called the *Kolmogoroff decomposition* of the positive definite kernel $k$. Let $(e_1, K_1)$ and $(e_2, K_2)$ be Kolmogoroff decompositions of $k$. Then there exists a unitary operator $\varrho: K_1 \to K_2$ such that $\varrho e_1(x) = e_2(x)$ holds for all $x \in M$. In this sense the Kolmogoroff decomposition is unique. The Kolmogoroff decomposition is useful in constructing new Hilbert spaces; for example, we obtain the direct sum $H_1 \oplus H_2$ of Hilbert spaces $H_1$ and $H_2$ by decomposing the kernel $k$ on $H_1 \times H_2$, given by

$$k: \big((h_1, h_2), (h_1', h_2')\big) \mapsto \langle h_1, h_1'\rangle + \langle h_2, h_2'\rangle.$$

In this case, $e$ is given as

$$e: (h_1, h_2) \mapsto h_1 \oplus h_2.$$

Similarly, the *symmetric Fock space* $\Gamma(H)$ over the Hilbert space $H$ is defined by the decomposition $(e, \Gamma(H))$ of the following kernel $k$ on $H$:

$$k(h, h') := e^{\langle h, h'\rangle} \quad \text{for all } h, h' \in H.$$

The vector $e(x) \in \Gamma(H)$ is called the *exponential vector* corresponding to $x \in H$. The exponential vectors are linearly independent and the map $e: H \to \Gamma(H)$ is continuous.

Remark 1. We consider one concrete realization of the symmetric Fock space over $L^2[0, T]$. For each $z \in L^2[0, T]$ and $t \in [0, T]$ we define the random variable $\mathscr{E}_t(z)$ on our probability space as

$$\mathscr{E}_t(z) := \exp\left\{\int_0^t z_s\, dW_s - \frac{1}{2}\int_0^t z_s^2\, ds\right\}.$$

A calculation shows that $E_P\big(\overline{\mathscr{E}_T(z)}\, \mathscr{E}_T(z')\big) = e^{\langle z, z'\rangle}$ holds for all $z, z' \in L^2[0, T]$. Using the fact that $\mathrm{lin}\{\mathscr{E}_T(z): z \in L^2[0, T]\}$ forms a dense subspace of $L^2(\Omega, \mathscr{F}_T, P)$ (see the proof of Lemma 5.36 of [9]), we conclude that $(\mathscr{E}_T(\cdot), L^2(\Omega, \mathscr{F}_T, P))$ defines a decomposition of the kernel $(z, z') \mapsto e^{\langle z, z'\rangle}$ on $L^2[0, T]$, and therefore $L^2(\Omega, \mathscr{F}_T, P)$ can be regarded as the symmetric Fock space over $L^2[0, T]$.

Let $K$ be a Hilbert space and $\boldsymbol{T} := \{\tau \in \boldsymbol{C}: |\tau| = 1\}$ be the one-dimensional torus. We endow the set $G_K := \boldsymbol{T} \times K$ with the multiplication $\circ$ as follows:

$$(\tau, z)\circ(\tau', z') := (\tau\tau' \exp\{-i\operatorname{Im}\langle z, z'\rangle\}, z+z') \quad \text{for all } (\tau, z), (\tau', z')\in G_K,$$

and we obtain a non-abelian topological group $(G_K, \circ)$. In the case where $K$ is a finite-dimensional Hilbert space, $(G_K, \circ)$ is a locally compact unimodular group, which is called the *Weyl–Heisenberg group*. The Haar measure $\omega_{G_K}$ of the Weyl–Heisenberg group $G_K$ is given by $\omega_{G_K} := \omega_{\boldsymbol{T}} \otimes \omega_K$, where $\omega_{\boldsymbol{T}}$ and $\omega_K$ denote the Haar measures of $\boldsymbol{T}$ and $K$, respectively.

Let $\Gamma(K)$ be the Fock space over $K$. For any $z\in K$ the *Fock–Weyl operator* $\mathscr{W}_K(z)$ is well defined by its action on all exponential vectors as

$$\mathscr{W}_K(z)\, e(z') = \exp\{-\tfrac{1}{2}\|z\|^2 - \langle z, z'\rangle\}\, e(z+z') \quad \text{for all } z, z'\in K.$$

It can be shown (see [14], p. 135) that each $\mathscr{W}_K(z)$ is a unitarity on $\Gamma(K)$ and that the Fock–Weyl operators obey the relation

$$\mathscr{W}_K(z)\,\mathscr{W}_K(z') = \exp\{-i\operatorname{Im}\langle z, z'\rangle\}\,\mathscr{W}_K(z+z') \quad \text{for all } z, z'\in K,$$

from which it follows that the map

$$U_K\colon G_K \to \mathscr{U}(\Gamma(K)), \quad (\tau, z)\mapsto \tau\,\mathscr{W}_K(z),$$

defines a unitary representation (the *Fock representation*) of $G_K$. This representation is strongly continuous and irreducible (see [14], p. 142).

Remark 2. In the case where $K$ is finite dimensional the Fock representation $U_K$ of the Weyl–Heisenberg group $G_K$ is square integrable and $e(0)$ can be chosen as a wavelet (compare with (6)), since

$$\begin{aligned}
&\int_{G_K} |\langle U_E(g)\, e(0), e(0)\rangle|^2\, \omega_{G_K}(dg) \\
&\quad = \int_K \int_T |\langle U_K((\tau, z))\, e(0), e(0)\rangle|^2\, \omega_{\boldsymbol{T}}(d\tau)\, \omega_K(dz) \\
&\quad = \int_K \int_T |\langle \tau \exp\{-\|z\|^2/2\}\, e(z), e(0)\rangle|^2\, \omega_{\boldsymbol{T}}(d\tau)\, \omega_K(dz) \\
&\quad = \omega_{\boldsymbol{T}}(\boldsymbol{T}) \int_K \exp\{-\|z\|^2\}\, \omega_K(dz) < \infty.
\end{aligned}$$

Consider a finite-dimensional subspace $E$ of a given Hilbert space $H$. Let $(e, \Gamma(H))$ be the decomposition defining the symmetric Fock space over $H$. We denote by $\Upsilon(E) \subset \Gamma(H)$ the space $\overline{\text{lin}}\{e(z)\colon z\in E\} \subset \Gamma(H)$ and by $P_{\Upsilon(E)}$ the orthogonal projector onto $\Upsilon(E)$. Let us also introduce the following transform:

$$\mathscr{L}\colon \Gamma(H) \to C(E), \quad \mathscr{L}h(z) := \langle e(z), h\rangle \text{ for all } z\in E.$$

From this definition we obtain $\mathscr{L} = \mathscr{L}P_{\Upsilon(E)}$. Let $\omega_E$ be the Haar measure of $E$ normalized as

$$\int_E \exp\{-\|z\|^2\}\, \omega_E(dz) = 1,$$

and $\gamma_E$ be the probability measure

$$\gamma_E(dz) := \exp\{-\|z\|^2\}\, \omega_E(dz).$$

PROPOSITION 2. *Using the above notation the following holds:*

(i) *$\mathscr{L}$ is a bounded operator mapping from $\Gamma(H)$ into $L^2(E, \gamma_E)$. Moreover, $\mathscr{L}^* \mathscr{L} = P_{\Upsilon(E)}$.*

(ii) *The adjoint $\mathscr{L}^*: L^2(E, \gamma_E) \to \Gamma(H)$ is given by*

$$\mathscr{L}^* \xi = \int_E \xi(z)\, e(z)\, \gamma_E(dz) \quad \textit{for all } \xi \in L^2(E, \gamma_E).$$

Proof. (i) It is obvious that the decomposition $(e|_E, \Upsilon(E))$ corresponds to the symmetric Fock space over $E$, and therefore we may identify $\Upsilon(E)$ with $\Gamma(E)$. The appropriate Fock–Weyl operators $\{\mathscr{W}_E(z): z \in E\} \subset \mathscr{U}(\Upsilon(E))$ are given by

$$\mathscr{W}_E(z) := P_{\Upsilon(E)} \mathscr{W}_H(z)|_{\Upsilon(E)} \quad \text{for all } z \in E.$$

From the second remark it follows that

$$\tilde{U}_E: G_E \to \mathscr{U}(\Upsilon(E)), \quad (\tau, z) \mapsto \tau\, \mathscr{W}_E(z),$$

defines a representation of $G_E$ which is unitarily equivalent to the Fock representation $U_E$. For this reason, $\tilde{U}_E$ is square integrable, and we may define the wavelet transform according to $\tilde{U}_E$ choosing $e(0)$ as the wavelet. This transform becomes isometric if the Haar measure $\omega_{G_E} = \omega_T \otimes \omega_E$ is normalized as

$$\omega_T(T) = \int_E \exp\{-\|z\|^2\}\, \omega_E(dz) = 1,$$

which implies for all $h \in \Upsilon(E)$

$$\begin{aligned} \|h\|^2 &= \int_{G_E} |\langle \tilde{U}_E(g)\, e(0), h\rangle|^2\, \omega_{G_E}(dg) \\ &= \int_T \int_E |\langle \tau \exp\{-\|z\|^2/2\}\, e(z), h\rangle|^2\, \omega_E(dz)\, \omega_T(d\tau) \\ &= \int_E |\langle e(z), h\rangle|^2\, \gamma_E(dz) = \|\mathscr{L} h\|^2. \end{aligned}$$

Hence $\|P_{\Upsilon(E)} h\|^2 = \|\mathscr{L} P_{\Upsilon(E)} h\|^2 = \|\mathscr{L} h\|^2$ for all $h \in \Gamma(H)$. Finally, we are led to $P_{\Upsilon(E)} = \mathscr{L}^* \mathscr{L}$ by polarizing $\|P_{\Upsilon(E)} h\|^2 = \|\mathscr{L} h\|^2$ for all $h \in \Gamma(H)$.

(ii) is straightforward. ∎

**3. Pricing contingent claims.** Let $\tilde{Y}_T \in L^2(\Omega, \mathscr{F}_T, P)$ be a contingent claim and $n \in N$. For the remainder of this work we identify $L^2(\Omega, \mathscr{F}_T, P)$ with the symmetric Fock space over $L^2[0, T]$ as described in the first remark. We also choose the finite-dimensional subspace

$$E^n := \text{lin}\{\varphi^0, \ldots, \varphi^n\} \subset L^2[0, T]$$

and consider the transform $\mathscr{L}: L^2(\Omega, \mathscr{F}_T, P) \to L^2(E^n, \gamma_{E^n})$ which corresponds to $E^n$ in this context.

Since pointwise arguments are important in this section, let us agree that for each $c = (c_0, \dots, c_n) \in \boldsymbol{C}^{n+1}$ and $t \in [0, T]$ the vector $\mathscr{E}_t(\sum_{k=0}^n c_k \varphi^k) \in L^2(\Omega, \mathscr{F}_T, P)$ is represented by the function

$$\Omega \to \boldsymbol{C}, \qquad \omega \mapsto \exp\left\{\sum_{k=0}^{n} c_k \Phi_t^k(\omega) - \frac{1}{2}\sum_{k,j=0}^{n} c_k c_j \int_0^t \varphi_s^j \varphi_s^k \, ds\right\}$$
$$= \exp\{c^\star \Phi_t(\omega) - \tfrac{1}{2} c^\star G_t c\}.$$

Proposition 3. *For all $t \in [0, T]$, $Y_t^n = \int_E \mathscr{L} Y_T(z) \mathscr{E}_t(z) \gamma_E(dz)$ holds in the weak sense and for all $t \in [0, T[$ pointwise.*

Proof. The identity $\Upsilon(E^n) := \overline{\text{lin}}\{\mathscr{E}_T(z) \colon z \in E^n\}$ holds by definition. Applying Lemma 5.36 of [9] we conclude that $\Upsilon(E^n) = L^2(\Omega, \sigma_n, P)$, which implies that the conditional expectation $Y_T^n = E_P(Y_T \mid \mathscr{F}_n)$ coincides with the projection $P_{\Upsilon(E^n)} Y_T$. From Proposition 2 it follows that

$$Y_T^n = P_{\Upsilon(E^n)} Y_T = \mathscr{L}^* \mathscr{L} Y_T = \int_{E^n} \mathscr{L} Y_T(z) \mathscr{E}_T(z) \gamma_{E^n}(dz)$$

holds in the weak sense for all $t \in [0, T]$. To show the pointwise representation it suffices to prove that for $(\omega, t) \in \Omega \times [0, T[$ the function $f_{t,\omega}\colon E^n \to \boldsymbol{C}$, $z \mapsto \mathscr{E}_t(z)(\omega)$ is contained in $L^2(E^n, \gamma_{E^n})$:

$$\int_{E^n} |f_{t,\omega}(z)|^2 \gamma_{E^n}(dz) = \int_{E^n} |\mathscr{E}_t(z)(\omega)|^2 \exp\{-\|z\|^2\} \omega_{E^n}(dz)$$
$$= \int_{C^{n+1}} |\exp\{c^\star \Phi_t(\omega) - \tfrac{1}{2} c^\star G_t c\}|^2 \exp\{-\bar{c}^\star c\} \frac{dc}{\pi^{n+1}}$$
$$= \int_{R^{n+1} \times R^{n+1}} \exp\{2u^\star \Phi_t(\omega) - u^\star G_t u + v^\star G_t v\} \exp\{-u^\star u - v^\star v\} \frac{du\,dv}{\pi^{n+1}}$$
$$= \int_{R^{n+1} \times R^{n+1}} \exp\{2u^\star \Phi_t(\omega) - u^\star (G_t + 1) u - v^\star (1 - G_t) v\} \frac{du\,dv}{\pi^{n+1}} < \infty.$$

The last inequality holds since the matrix $(1 - G_t)$ is positive definite for all $t \in [0, T[$. ∎

We are now able to prove (4). By Proposition 3, for all $(\omega, t) \in \Omega \times [0, T[$ we obtain

$$Y_t^n(\omega) = \int_{E^n} \mathscr{L} Y_T(z) f_{t,\omega}(z) \gamma_{E^n}(dz) = \langle \overline{f_{t,\omega}}, \mathscr{L} Y_T \rangle_{L^2(E^n, \gamma_{E^n})}$$
$$= \langle \mathscr{L}^* \overline{f_{t,\omega}}, Y_T \rangle_{L^2(\Omega, P)} = E_P(K_{t,\omega} Y_T),$$

where $\overline{K_{t,\omega}} = \mathscr{L}^* \overline{f_{t,\omega}}$. This random variable is calculated explicitly as:

$$\begin{aligned}\overline{K_{t,\omega}} &= \int_{E^n} \overline{f_{t,\omega}}(z)\,\mathscr{E}_T(z)\,\gamma_{E^n}(dz)\\ &= \int_{C^{n+1}} \exp\{\bar{c}^\star \Phi_t(\omega) - \tfrac{1}{2}\bar{c}^\star G_t \bar{c}\}\exp\{c^\star \Phi_T - \tfrac{1}{2} c^\star c\}\exp\{-\bar{c}^\star c\}\,\frac{dc}{\pi^{n+1}}\\ &= |\det(1-G_t)^{-1/2}|\exp\{-\tfrac{1}{2}(\Phi_T - \Phi_t(\omega))^\star (1-G_t)^{-1}(\Phi_T - \Phi_t(\omega))\}\exp\{\tfrac{1}{2}\Phi_T^\star \Phi_T\},\end{aligned}$$

which proves the statement (4).

**4. Hedging contingent claims.** To apply tools from the wavelet analysis to the hedging problem we consider the Hilbert space of predictable processes: $L^2(\Omega \times [0, T], \mathscr{A}, P\otimes\lambda)$. Here $\mathscr{A}$ denotes the $\sigma$-algebra of predictable sets and $P\otimes\lambda$ denotes the restriction to $\mathscr{A}$ of the measure-theoretic product of $P$ with the Lebesgue measure $\lambda$ on $[0, T]$. Then, by its construction, the stochastic integral

$$I\colon L^2(\Omega\times[0, T], \mathscr{A}, P\otimes\lambda) \to L^2(\Omega, \mathscr{F}_T, P), \qquad h = (h_t)_{t\in[0,T]} \mapsto \int_0^T h_s\,dW_s,$$

is an isometric operator. Each exponential vector $\mathscr{E}_T(z)$ is the terminal value of the martingale

$$\mathscr{E}(z) = (\mathscr{E}_t(z))_{t\in[0,T]} = \left(\exp\left\{\int_0^t z_s\,dW_s - \frac{1}{2}\int_0^t z_s^2\,ds\right\}\right)_{t\in[0,T]}.$$

Moreover, $\mathscr{E}(z)$ is the stochastic exponential of the martingale $(\int_0^t z_s\,dW_s)_{t\in[0,T]}$ and admits the representation

$$\mathscr{E}_T(z) - 1 = \int_0^T \varepsilon_t(z)\,dW_t,$$

where $\varepsilon(z) = (\varepsilon_t(z))_{t\in[0,T]} \in L^2(\Omega\times[0, T], \mathscr{A}, P\otimes\lambda)$ is given by

$$\varepsilon_t(z) = z_t\,\mathscr{E}_t(z) \qquad \text{for all } t\in[0, T].$$

Given the contingent claim $\tilde{Y}_T$ and $n\in N$ we are searching for predictable process $y^n = (y_t^n)_{t\in[0,T]}$ satisfying $Y_T^n - Y_0^n = \int_0^T y_t^n\,dW_t$. That means we wish to obtain $y^n \in L^2(\Omega\times[0, T], \mathscr{A}, P\otimes\lambda)$, which is uniquely determined by $Iy^n = Y_T^n - Y_0^n$ since $I$ is injective.

PROPOSITION 4. *The process $y^n$ satisfies $y^n := \int_{E^n} \mathscr{L} Y_T(z)\,\varepsilon(z)\,\gamma_{E^n}(dz)$, where the integral is understood in the weak sense. Moreover, for all $(\omega, t)\in\Omega\times[0, T[$,*

$$y_t^n(\omega) := \int_{E^n} \mathscr{L} Y_T(z)\,\varepsilon_t(z)(\omega)\,\gamma_{E^n}(dz).$$

Proof. It follows from Proposition 3 that

$$Y_T^n - Y_0^n = \int_{E^n} \mathscr{L} Y_T(z)\big(\mathscr{E}_T(z) - \mathscr{E}_0(z)\big)\,\gamma_{E^n}(dz)$$

in the weak sense. Using $\mathscr{E}_T(z)-\mathscr{E}_0(z)=I\varepsilon(z)$, we are led to

$$Y_T^n - Y_0^n = \int_{E^n} \mathscr{L} Y_T(z)\, I\varepsilon(z)\,\gamma_{E^n}(dz) = I \int_{E^n} \mathscr{L} Y_T(z)\,\varepsilon(z)\,\gamma_{E^n}(dz)$$

since $I$ is bounded. This proves the weak integral representation of $y^n$. To prove the second assertion it suffices to show that for each $(\omega, t)\in\Omega\times[0, T[$ the function

$$f'_{t,\omega}\colon E^n \to \boldsymbol{C}, \qquad z \mapsto \varepsilon_t(z)(\omega),$$

is contained in $L^2(E^n, \gamma_{E^n})$. This is true since $(1-G_t)$ is positive definite for $t\in[0, T[$:

$$\int_{E^n} |f'_{t,\omega}|^2\,\gamma_{E^n}(dz) = \int_{E^n} |z_t|^2\,|\mathscr{E}_t(z)(\omega)|^2\,\gamma_{E^n}(dz)$$

$$= \int_{\boldsymbol{C}^{n+1}} |\varphi_t^\star c|^2\,|\exp\{c^\star \Phi_t(\omega) - \tfrac{1}{2} c^\star G_t c\}|^2 \exp\{-\bar{c}^\star c\}\,\frac{dc}{\pi^{n+1}}$$

$$= \int_{\boldsymbol{R}^{n+1}\times\boldsymbol{R}^{n+1}} |\varphi_t^\star(u+iv)|^2 \exp\{2u^\star \Phi_t(\omega)$$

$$-u^\star(G_t+1)\,u - v^\star(1-G_t)\,v\}\,\frac{du\,dv}{\pi^{n+1}} < \infty. \ \blacksquare$$

Now we are able to show (5) by evaluating

$$\int_{E^n} \mathscr{L} Y_T(z)\,\varepsilon_t(z)(\omega)\,\gamma_{E^n}(dz), \qquad \text{where } (\omega, t)\in\Omega\times[0, T[.$$

We get

$$y_t^n(\omega) = \int_{E^n} \mathscr{L} Y_T(z)\, f'_{t,\omega}(z)\,\gamma_{E^n}(dz) = \langle \overline{f'_{t,\omega}}, \mathscr{L} Y_T\rangle_{L^2(E^n,\gamma_{E^n})}$$

$$= \langle \mathscr{L}^* \overline{f'_{t,\omega}}, Y_T\rangle_{L^2(\Omega,P)} = E_P(K'_{t,\omega}\, Y_T^n),$$

where $\overline{K'_{t,\omega}} = \mathscr{L}\overline{f'_{t,\omega}}$. This random variable is calculated explicitly as:

$$\overline{K'_{t,\omega}} = \int_{E^n} \overline{f'_{t,\omega}}(z)\,\mathscr{E}_T(z)\,\gamma_{E^n}(dz)$$

$$= \int_{\boldsymbol{C}^{n+1}} \bar{c}^\star \varphi_t \exp\{\bar{c}^\star \Phi_t(\omega) - \tfrac{1}{2}\bar{c}^\star G_t \bar{c}\}$$

$$\times \exp\{c^\star \Phi_T - \tfrac{1}{2} c^\star c\} \exp\{-\bar{c}^\star c\}\,\frac{dc}{\pi^{n+1}}$$

$$= (\Phi_T - \Phi_t(\omega))^\star (1-G_t)^{-1} \varphi_t\,|\det(1-G_t)^{-1/2}|$$

$$\times \exp\{-\tfrac{1}{2}(\Phi_T - \Phi_t(\omega))^\star (1-G_t)^{-1}(\Phi_T - \Phi_t(\omega))\} \exp\{\tfrac{1}{2}\Phi_T^\star \Phi_T\},$$

which proves (5).

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