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LOCAL INVARIANCE PRINCIPLE FOR MARKOV CHAINS

BY

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Abstract. We consider stationary homogeneous Markov chains and the polygonal processes defined by a usual way using such chains. There are many results about invariance principles of those processes. In this paper, we prove that under additional conditions, a stronger assertion (in some sense) is true. Indeed, we establish the convergence in variation for the distributions of the functionals of such a process, that is a local invariance principle. We study also the particular case of positive Harris recurrent Markov chains. Finally, we prove that the invariance principle and the local invariance principle remain valid when the initial chain is homogeneous but not stationary.

1. INTRODUCTION

The origin of this paper is work done by Y. A. Davydov, who proved a local invariance principle (IP) for a sequence of independent and identically distributed random variables (i.i.d.r.v.'s). The method used by Y. A. Davydov consists at first in estimation of the total variation (denoted by $\|\cdot\|$) between the distribution of a sequence of i.i.d.r.v.'s and its translate.

Indeed, let $\xi = (\xi_k)_{k\in N^*}$ be a sequence of i.i.d.r.v.'s defined on some probability space $(\Omega, \mathscr{A}, P)$ and taking values in $(R, \mathscr{B}(R), \lambda)$, where $\mathscr{B}(R)$ is the σ-field of Borel subsets of R, and λ is the Lebesgue measure. For all $k\in N^*$, the r.v. ξ_k has the density p, assumed to be absolutely continuous (a.c.). Let $I(p)$ denote the Fisher quantity associated with p, defined by

$$I(p) = \int_R \left[\frac{p'}{p}(x)\right]^2 p(x)\,dx.$$

Let $a = (a_k)_{k\in N^*}$ be a real sequence and $n\in N^*$. We denote by $\mathscr{P}_n$ and $\mathscr{P}_n^a$ the distributions of the vector $\bar{\xi} = (\xi_1, \ldots, \xi_n)$ and its translate $\bar{\xi}+\bar{a} = (\xi_1+a_1, \ldots, \xi_n+a_n)$, respectively. In this case, if $I(p) < \infty$, we obtain

$$\|\mathscr{P}_n^a - \mathscr{P}_n\| \leqslant \sqrt{I(p)}\sqrt{\sum_{k=1}^{n} a_k^2}. \tag{1.1}$$

Then, using (1.1), Y. A. Davydov proved a local IP for the stochastic functionals in the IP of Donsker–Prokhorov. Here, the sequence ξ is such that, for all $k \in N^*$, $E(\xi_k) = 0$ and $\mathrm{Var}(\xi_k) = \sigma^2 < \infty$. Let us put $S_0 = 0$ and, for all $k \in N^*$, $S_k = \xi_1 + \ldots + \xi_k$. Let $n \in N^*$. We can construct the polygonal process ζ_n defined for all $t \in [0; 1]$ and all $\omega \in \Omega$ by

$$\zeta_n(t, \omega) = \frac{1}{\sigma\sqrt{n}}\left[S_{[nt]}(\omega) + (nt - [nt])\,\xi_{[nt]+1}(\omega)\right], \tag{1.2}$$

where $[x]$ is the integer part of x. Let P_n denote the distribution of ζ_n. The IP of Donsker–Prokhorov ([2], Theorem 10.1) states that $P_n \Rightarrow W$, where W is the measure of the Wiener process $\{w(t);\ t \in [0; 1]\}$, and $\Rightarrow$ is the notation for the weak convergence. Let $C[0; 1]$ denote the space of continuous functions on $[0; 1]$. Thus, if φ is a functional defined on $C[0; 1]$, W-almost everywhere (a.e.) continuous, we have

$$P_n \varphi^{-1} \Rightarrow W\varphi^{-1}.$$

Then Y. A. Davydov proved, by imposing more stringent conditions on the common distribution of the r.v. ξ_k and by restricting the class of functionals, that it is possible to obtain a stronger assertion, that is to say

$$P_n \varphi^{-1} \xrightarrow[n \to \infty]{\mathrm{var}} W\varphi^{-1}.$$

The aim of this paper is to use the same method to show a local IP for stationary (or not) homogeneous Markov chains (m.c.'s).

2. NOTATION AND INEQUALITY

Let $\xi = (\xi_k)_{k \in N^*}$ be a stationary homogeneous m.c., defined on $(\Omega, \mathscr{A}, P)$ and taking values in $(R, \mathscr{B}(R), \lambda)$. Let Π denote its stationary probability distribution with density π assumed to be a.c. and let P denote the probability transition kernel with the family of transition densities $\{p(x, \cdot);\ x \in R\}$. For all $x \in R$, we suppose that $p(x, \cdot)$ is a.c. The quantity π' is the derivative of π, and p'_x, p'_y are the partial derivatives of p.

Let us denote by q the transition density for a time-reversed (with respect to Π) chain. This transition density is given by

$$q(y, \cdot) = \frac{\pi(\cdot)\, p(\cdot, y)}{\pi(y)}.$$

Now, let us give the following notion of I-regularity defined by Ibragimov and Has'minskii in [12].

Definition 2.1. Let $O \subseteq R$ be an open set. The family $\{p(\cdot, \theta);\ \theta \in O\}$ of probability densities is said to be *I-regular* (or *information regular*) if the map from O to $L^2(R, \lambda)$, sending $\theta \in O$ to $p^{1/2}(\cdot, \theta)$, is continuously differentiable (in the sense of standard Hilbert norms on R and $L^2(R, \lambda)$).

Remark 2.1. This implies that, for all $\theta \in O$,

$$\frac{\partial}{\partial\theta} p^{1/2}(\cdot, \theta) \in L^2(\boldsymbol{R}, \lambda).$$

Let us denote by $I(\theta)$ the *Fisher quantity* defined by

$$I(\theta) = 4 \left\| \frac{\partial}{\partial\theta} p^{1/2}(\cdot, \theta) \right\|^2_{L^2(\boldsymbol{R},\lambda)}.$$

Here, $I(\theta)$ can also be written in its usual form:

$$I(\theta) = \int_{\boldsymbol{R}} \left[\frac{(\partial/\partial\theta)\, p(x, \theta)}{p(x, \theta)} \right]^2 p(x, \theta)\, dx.$$

So, our regularity assumptions are the following:

(**R**) The family $\{\pi(\cdot + t);\ t \in \boldsymbol{R}\}$ is I-regular.

The corresponding Fisher quantity does not depend on t and can be written as

$$I(\pi) = \int_{\boldsymbol{R}} \left[\frac{\pi'}{\pi}(x) \right]^2 \pi(x)\, dx.$$

Set $J = \{x \in \boldsymbol{R} \mid \pi(x) > 0\}$.

($\mathbf{R}^+$) The family $\{p(x, \cdot);\ x \in J\}$ is I-regular.

We denote by $I^+(x)$ the corresponding Fisher quantity defined for $x \in J$ as

$$I^+(x) = \int_{\boldsymbol{R}} \left[\frac{p'_x}{p}(x, y) \right]^2 p(x, y)\, dy.$$

($\mathbf{R}^-$) The family $\{q(y, \cdot);\ y \in J\}$ is I-regular.

We denote by $I^-(y)$ the corresponding Fisher quantity defined for $y \in J$ as

$$I^-(y) = \int_{\boldsymbol{R}} \left[\frac{p'_y}{p}(x, y) - \frac{\pi'}{\pi}(y) \right]^2 q(y, x)\, dx.$$

Moreover, we suppose that

$$I^+ = \int_J I^+(x)\, \pi(x)\, dx < \infty \quad \text{and} \quad I^- = \int_J I^-(y)\, \pi(y)\, dy < \infty. \tag{2.1}$$

Let us put $I = I(\pi) + I^+ + I^-$. Finally, let $a = (a_k)_{k \in N^*}$ be a real sequence. As in Section 1, we denote by $\mathscr{P}_n$ and $\mathscr{P}_n^a$ the distributions of the vectors $\bar{\xi}$ and $\bar{\xi} + \bar{a}$, respectively.

THEOREM 2.1. *If* (**R**), ($\mathbf{R}^+$), ($\mathbf{R}^-$) *and* (2.1) *are satisfied, then*

$$\|\mathscr{P}_n^a - \mathscr{P}_n\| \leqslant \sqrt{2I} \sqrt{\sum_{k=1}^{n} a_k^2}. \tag{2.2}$$

Remark 2.2. This theorem is proved in [15]. Moreover, other inequalities and applications are presented, in particular in the case of a sequence of i.i.d.r.v.'s and a random translation. In fact, all those inequalities allow us to prove absolute continuity between a sequence and its translate.

3. LOCAL INVARIANCE PRINCIPLE FOR STATIONARY HOMOGENEOUS MARKOV CHAINS

Let f be a real-valued function defined on $\boldsymbol{R}$. Let $S_0 = 0$ and, for all $k \in \boldsymbol{N}^*$, $S_k = f(\xi_1)+ \ldots +f(\xi_k)$. Let $n \in \boldsymbol{N}^*$. Then the process ζ_n is defined for all $t \in [0; 1]$ and for all $\omega \in \Omega$ by

$$(3.1)\qquad \zeta_n(t, \omega) = \frac{1}{\sqrt{n}}\left[S_{[nt]}(\omega)+(nt-[nt])\, f\left(\xi_{[nt]+1}(\omega)\right)\right].$$

As before, P_n is the distribution of ζ_n. First, we define a new class of functionals smaller than $\boldsymbol{C}[0; 1]$. Then, if φ belongs to this class and if we have the weak convergence for P_n, we prove the convergence in variation. Finally, we apply this result to the Harris recurrent positive m.c.

3.1. Definition of the class $\mathcal{M}_P$. We can find the notions introduced here in Sections 13 and 14 in [5]. Let $(X, \mathcal{B}_X, P)$ be a complete separable metric space. Each element l of X generates a group of shifts $G^l = \{G_c^l;\, c \in \boldsymbol{R}\}$ with $G_c^l(x) = x+cl$ for $x \in X$.

Definition 3.1. A vector l is an *admissible shift* for P if $P^l = P(G_1^l)^{-1} \ll P$. Moreover, l defines an *admissible direction* for P if $P^{cl} \ll P$ for all $c \in \boldsymbol{R}$. The set of all vectors defining admissible directions for P will be denoted by H_P.

The set $\Delta = \{x+cl;\, c \in [a, b]\}$, $x, l \in X$, $a, b \in \boldsymbol{R}$, is called a *segment parallel* to the vector l. In this case, we use the notation $\Delta \| l$. The segments $\Delta_n = \{x_n+cl_n;\, c \in [a_n; b_n]\}$ *converge* to Δ if $x_n \to x$, $l_n \to l$, $a_n \to a$ and $b_n \to b$ (we use the notation $\Delta_n \to \Delta$). Finally, for a segment Δ and a real-valued functional φ defined on X, we set for all $c \in [a; b]$

$$\varphi_\Delta(c) = \varphi(x+cl).$$

Definition 3.2. We say that the functional φ *belongs to the class* $\mathcal{M}(x, l, (l_n))$, where $x, l \in X$ and $l_n \to l$, if there exists a neighborhood V of the point x such that for P-almost all $y \in V$ and any segment $\Delta = \{x+cl;\, c \in [a; b]\}$, $\Delta \subset V$,

$$\left.\begin{array}{l} \Delta_n \| l_n \\ \Delta_n \to \Delta \end{array}\right\} \Rightarrow \lambda\varphi_{\Delta_n}^{-1} \xrightarrow[n \to \infty]{\text{var}} \lambda\,\varphi_\Delta^{-1}.$$

Definition 3.3. We say that φ *belongs to the class* $\mathcal{M}_P$ if for P-almost all $x \in X$ there is a vector $l \in H_P$ such that $\varphi \in \mathcal{M}(x, l, (l_n))$ for any sequence (l_n) converging to l.

In the following, we are especially interested in the class $\mathcal{M}_W$. Let X be the space $C[0; 1]$, and P be the Wiener measure W.

Remark 3.1. We know that

$$H_W = \{l \in C[0; 1] \mid l \text{ a.c., } l(0) = 0 \text{ and } l' \in L^2([0; 1])\}.$$

Remark 3.2. The classes $\mathcal{M}_P$ and $\mathcal{M}_W$ are very large. For example, we can assume that X is a separable Banach space. We denote by X^* the dual space to X equipped with the weak*-topology, and by $\langle \cdot, \cdot \rangle$ we denote the bilinear form expressing the duality of X and X^*. We say that a functional φ is *continuously Fréchet differentiable* in a region $G \subset X$ if the mapping $x \mapsto D\varphi(x)$ from G to X^* is continuous in the weak*-topology. If we suppose that for P-almost all x there exists a neighborhood V_x of the point x in which φ is continuously Fréchet differentiable and if $Df(x)(H_P) \neq \{0\}$, then $\varphi \in \mathcal{M}_P$ ([5], Theorem 13.7).

Now, we cite some examples of functionals which belong to $\mathcal{M}_W$.

EXAMPLE 3.1. Assume that there exists $t_0 \in [0; 1]$. The following functional belongs to $\mathcal{M}_W$:

$$\varphi_1\colon C[0; 1] \to \boldsymbol{R},$$
$$x \mapsto \varphi_1(x) = x(t_0).$$

EXAMPLE 3.2. The following functional belongs to $\mathcal{M}_W$:

$$\varphi_2\colon C[0; 1] \to \boldsymbol{R},$$
$$x \mapsto \varphi_2(x) = \sup_{t \in [0;1]} x(t).$$

EXAMPLE 3.3. The following functional belongs to $\mathcal{M}_W$:

$$\varphi_3\colon C[0; 1] \to \boldsymbol{R},$$
$$x \mapsto \varphi_3(x) = \sup_{t \in [0;1]} |x(t)|.$$

EXAMPLE 3.4. Let us consider the following integral functional:

$$\varphi_4\colon C[0; 1] \to \boldsymbol{R},$$
$$x \mapsto \varphi_4(x) = \int_0^1 q(x(t))\,\mu(dt),$$

where μ is a finite measure on $\mathcal{B}(C[0; 1])$ and q is some measurable function on $\boldsymbol{R}$. In view of Theorem 14.4 of [5], if we suppose that for all $\varepsilon > 0$ there exists an open interval $J \subset (-\varepsilon; \varepsilon)$ on which q' is continuous and non-zero, then $\varphi_4 \in \mathcal{M}_W$.

Remark 3.3. Assume that there exists $\sigma > 0$. Let W_σ denote the distribution of the process $\{\sigma w(t);\, t \in [0; 1]\}$. It is clear that $\mathcal{M}_{W_\sigma}$ and $\mathcal{M}_W$ are the same.

Other examples of functionals which belong to $\mathcal{M}_W$ or, more generally, to $\mathcal{M}_P$ are available in [5] and [6].

3.2. Local invariance principle. The polygonal process ζ_n is defined at the beginning of this section by (3.1) and the distribution of ζ_n is P_n.

THEOREM 3.1. *If the following assumptions are satisfied:*

(1) ($\mathbf{R}$), ($\mathbf{R}^+$), ($\mathbf{R}^-$) *and* (2.1);

(2) *there exist* $\delta > 0$ *and* $M \geqslant 0$ *such that, for all* $x \in \boldsymbol{R}$, $f'(x) \geqslant \delta$ *and* $|f''(x)| \leqslant M$;

(3) $P_n \Rightarrow W_\sigma$ *for some constant* $\sigma > 0$;

(4) $\varphi \in \mathcal{M}_W$,

then

$$P_n \varphi^{-1} \xrightarrow[n \to \infty]{\text{var}} W_\sigma \varphi^{-1}.$$

Remark 3.4. We shall see in the following section that if we add more assumptions on the m.c. ξ and on the function f, then there exists an IP for the process ζ_n, that is to say that assumption (3) will hold true for some constant σ.

3.3. Harris recurrent positive Markov chains. Now, let us remind IP for m.c. There are different versions of the central limit theorem (CLT) and IP for m.c. Some of them are stated in terms of distribution of some return times or in terms of some auxiliary objects like "atoms" (see, for example, Theorem 7.6 in [14]). We prefer here the approach based on solvability of an equation in the Hilbert space $L^2(J, \Pi)$ with respect to the stationary probability distribution Π. Gordin proved a CLT for general stationary sequences in [8]. Then, IP for general stationary processes was proved in [9]. Later, in [10], this approach was specialized to prove the CLT for general stationary m.c. A bit of time later, under the same assumptions, the IP was announced in [11]. In the same way, IP was proved in [13] for Harris recurrent positive m.c. Since we are going to restrict ourselves to this case of m.c., [13] can be considered as one of appropriate references. However, Maigret [13] considers also functions on the path space of the chain which depend on two successive values of the chain. Here, we restrict ourselves to functions depending on the current state of the chain only.

Let ξ be a stationary homogeneous Harris recurrent m.c. on J with stationary probability distribution Π. Let f be a function which belongs to $L^2(J, \Pi)$ such that

(C) $Pf = Pg - g$ with some $g \in L^2(J, \Pi)$.

Remark 3.5. Representation (C) for a function Pf is possible if and only if f can be represented in the same form:

(C′) $f = Ph - h$ with some $h \in L^2(J, \Pi)$.

Remark 3.6. Let us observe that $\int_J f(x)\,\Pi(dx) = 0$ is an obvious necessary condition of the solvability of equation (C).

Let us put

$$L_0^1(J, \Pi) = \{f \in L^1(J, \Pi) \mid \int_J f(x)\Pi(dx) = 0\}.$$

Remark 3.7. Equation (C) has a measurable solution for any $f \in L_0^1(J, \Pi)$. This is noticed in [13] and follows from Theorem 5.1 of [14]. Moreover, if $|Pf|$ is a so-called special function (see [14] and [16]) and if $f \in L_0^1(J, \Pi)$, then we have $Pf = Pg - g$ with $g \in L^\infty(J, \Pi)$, which is the space of measurable bounded functions.

The assumption in the form (C) has some advantages for a Harris recurrent m.c. because Pf can be a "better" function than f. On the other hand, the formula for the limiting variance looks simpler having been written in terms of h from (C′).

Let us denote by $\|\ \|_2$ the norm in $L^2(J, \Pi)$.

THEOREM 3.2. *If ξ is a Harris recurrent positive m.c. and if $f \in L^2(J, \Pi)$ satisfies* (C), *then*

$$P_n \Rightarrow W_\sigma,$$

where $\sigma^2 = \|g-f\|_2^2 - \|P(f-g)\|_2^2 = \|h\|_2^2 - \|Ph\|_2^2$.

This theorem is a compilation of results in [10], [11] and [13]. We shall see that assumption (3) of Theorem 3.1 is satisfied with this constant σ.

Then the previous result and Theorem 3.1 allow us to give the following corollary.

COROLLARY 3.1. *If the following assumptions are satisfied:*

(1) (**R**), (**R**$^+$), (**R**$^-$) *and* (2.1);

(2) $f \in L^2(J, \Pi)$ *and there exist $\delta > 0$ and $M \geq 0$ such that, for all $x \in \boldsymbol{R}$, $f'(x) \geq \delta$ and $|f''(x)| \leq M$ and f satisfies* (C) (*or* (C′)), *with*

$$\sigma^2 = \|g-f\|_2^2 - \|P(f-g)\|_2^2 = \|h\|_2^2 - \|Ph\|_2^2 > 0;$$

(3) *ξ is indecomposable*;

(4) $\varphi \in \mathcal{M}_W$,

then

$$P_n \varphi^{-1} \xrightarrow[n \to \infty]{\text{var}} W_\sigma \varphi^{-1}.$$

Remark 3.8. In Theorem 3.2, that is to say for the weak convergence, we have not to assume that $\sigma^2 > 0$, but this is necessary in the previous corollary for strong convergence.

Remark 3.9. We shall see in the following section that assumptions (1) and (3) allow us to conclude that the m.c. ξ is Harris recurrent positive on J. Using this and assumption (C), we have the IP (Theorem 3.2) and assumption (3) of Theorem 3.1 is satisfied.

4. PROOFS

First, using Theorem 2.1, we prove another inequality. Then we prove the local IP, that is to say Theorem 3.1. Finally, we study the case of Harris recurrent positive m.c.

4.1. Inequality. In Theorem 2.1, we have an inequality for total variation between the distribution $\mathscr{P}_n$ of $(\xi_1, \ldots, \xi_n)$ and the distribution $\mathscr{P}_n^a$ of $(\xi_1+a_1, \ldots, \xi_n+a_n)$. In the following, we shall estimate the total variation between the distribution of $(f(\xi_1), \ldots, f(\xi_n))$ and the distribution of $(f(\xi_1)+a_1, \ldots, f(\xi_n)+a_n)$ denoted by $\hat{\mathscr{P}}_n$ and $\hat{\mathscr{P}}_n^a$, respectively.

THEOREM 4.1. *Let $f\colon \boldsymbol{R} \to \boldsymbol{R}$ satisfy the following assumptions:*
(i) *there exists $\delta > 0$ such that, for all $x \in \boldsymbol{R}$, $f'(x) \geqslant \delta$;*
(ii) *there exists $M \geqslant 0$ such that, for all $x \in \boldsymbol{R}$, $|f''(x)| \leqslant M$.*
If (R), (R$^+$), (R$^-$) *and* (2.1) *are satisfied, then*

$$\|\hat{\mathscr{P}}_n^a - \hat{\mathscr{P}}_n\| \leqslant \sqrt{2\hat{I}}\sqrt{\sum_{k=1}^{n} a_k^2}, \quad \text{where } \hat{I} = \frac{1}{\delta^2} I + \frac{2M^2}{\delta^4}. \tag{4.1}$$

Remark 4.1. In view of assumptions, it is clear that $\hat{I} < \infty$.

Proof. f is a bijection (strictly increasing) on $\boldsymbol{R}$. Therefore, $(f(\xi_k))_{k\in \boldsymbol{N}^*}$ remains a stationary homogeneous m.c. Let $\hat{\Pi}$ denote the stationary probability distribution defined by the density

$$\hat{\pi}(u) = \frac{\pi[f^{-1}(u)]}{f'[f^{-1}(u)]}.$$

Moreover, let $\{\hat{p}(u, \cdot);\ u \in \boldsymbol{R}\}$ denote the family of transition densities defined by

$$\hat{p}(u, v) = \frac{p[f^{-1}(u), f^{-1}(v)]}{f'[f^{-1}(v)]}.$$

In view of (i) and (ii), we can prove that $\hat{\pi}$ and $\hat{p}(u, \cdot)$ for all $u \in \boldsymbol{R}$ are a.c. Let $\{\hat{q}(v, \cdot);\ v \in \boldsymbol{R}\}$ denote the transition densities for a time-reversed chain. Those transition densities are given by

$$\hat{q}(v, u) = \frac{\hat{\pi}(u)\hat{p}(u, v)}{\hat{\pi}(v)} = \frac{\pi[f^{-1}(u)]\, p[f^{-1}(u), f^{-1}(v)]}{f'[f^{-1}(u)]\, \pi[f^{-1}(v)]}.$$

We shall estimate the following total variation:

$$\|\hat{\mathscr{P}}_n^a - \hat{\mathscr{P}}_n\| = \int_{\boldsymbol{R}^n} \Big|\hat{\pi}(u_1+a_1) \prod_{k=2}^{n} \hat{p}(u_{k-1}+a_{k-1}, u_k+a_k) - \hat{\pi}(u_1) \prod_{k=2}^{n} \hat{p}(u_{k-1}, u_k)\Big|\, d\bar{u}.$$

We can apply inequality (2.2) to the m.c. $(f(\xi_k))_{k\in \boldsymbol{N}^*}$. For this, we calculate $I(\hat{\pi})$, $\hat{I}^+$ and $\hat{I}^-$. In view of (i) and (ii) and using $(a-b)^2 \leqslant 2a^2+2b^2$, we obtain

$$(4.2)\qquad I(\hat{\pi}) = \int_{R} \left[\frac{\hat{\pi}'}{\hat{\pi}}(u)\right]^2 \hat{\pi}(u)\,du \leqslant \frac{2}{\delta^2} I(\pi) + \frac{2M^2}{\delta^4}.$$

In the same way, we obtain

$$(4.3)\qquad \hat{I}^+ = \int_{R}\int_{R} \left[\frac{\hat{p}'_u}{\hat{p}}(u, v)\right]^2 \hat{p}(u, v)\,dv\,\hat{\pi}(u)\,du \leqslant \frac{1}{\delta^2} I^+,$$

$$(4.4)\qquad \hat{I}^- = \int_{R}\int_{R} \left[\frac{(\partial/\partial v)\,\hat{q}(v, u)}{\hat{q}(v, u)}\right]^2 \hat{q}(v, u)\,du\,\hat{\pi}(v)\,dv \leqslant \frac{1}{\delta^2} I^-.$$

In view of (4.2)–(4.4), we have the announced result. ∎

4.2. Proof of Theorem 3.1. The idea of the proof is to use inequality (4.1) and a local limit theorem for functionals of random processes (Theorem 1 in [3]). Indeed, we consider a sequence of probability measures $(P_n)_{n\in N^*}$ defined on the σ-algebra $\mathscr{B}_X$ of Borel subsets of a complete separable metric space (X, ϱ). Let $Z(X)$ denote the Banach space of finite measures on $\mathscr{B}_X$ with the total variation as norm. The restriction of the Lebesgue measure to $\mathscr{B}([a; b])$ is $\lambda_{[a;b]}$. Let φ be a measurable mapping from X to R or, briefly, a functional. Theorem 1 of [3] is the following

THEOREM 4.2. *Suppose that $P_n \Rightarrow P_\infty$ and, for P_∞-almost all x, there exists an open ball B with center at x, a number $\varepsilon > 0$, and $(G_{c,n};\ c\in(0;\varepsilon];\ n\in N^*)$ a family of transformations of X such that*

(i) *for each $c\in(0;\varepsilon)$, $G_{c,n}x \to G_{c,\infty}x$, $n\to\infty$, in measure P_n;*

(ii) *$G_{c,\infty}$ is continuous for each $c\in(0;\varepsilon)$ and for each ball S,*

$$d(S, c) = \sup_{z\in S} \varrho(z, G_{c,\infty}) \to 0, \qquad c\to 0;$$

(iii) $\lim_{c\to 0} \limsup \|P_n G_{c,n}^{-1} - P_n\| = 0$;

(iv) *for each $\delta\in(0;\varepsilon)$,*

$$\int_{B} \|\lambda_{[0;\delta]}\varphi_{z,n}^{-1} - \lambda_{[0;\delta]}\varphi_{z,\infty}^{-1}\|\,P_n(dz) \to 0, \qquad n\to\infty,$$

where $\varphi_{z,n}(c) = \varphi(G_{c,n}(z))$, $c\in(0;\varepsilon]$, $n\in N^*$;

(v) *for each $\delta\in(0;\varepsilon)$, the mapping*

$$B \to Z(R),$$

$$z \mapsto \lambda_{[0;\delta]}\varphi_{z,\infty}^{-1},$$

is continuous P_∞-a.e.

Then

$$P_n\varphi^{-1} \xrightarrow[n\to\infty]{\text{var}} P_\infty\varphi^{-1}.$$

In our case, $X = C[0; 1]$ is a complete separable metric space and the distance ϱ is defined for $f, g \in C[0; 1]$ by

$$\varrho(f, g) = \sup_{t \in [0;1]} |f(t) - g(t)|.$$

In view of assumption (3) of Theorem 3.1, we have $P_\infty = W_\sigma$. Let $n \in N^*$. We denote by F_n the space of polygonal lines constructed on the points $(k/n, x(k/n))$ for all $0 \leqslant k \leqslant n$ and all $x \in C[0; 1]$. Therefore, F_n has a finite dimension and P_n is concentrated on F_n. Let Π_n denote the mapping from $C[0; 1]$ to F_n, which to all points x assigns the polygonal line constructed on the points $(k/n, x(k/n))$ for all $0 \leqslant k \leqslant n$.

Let φ be a functional which belongs to $\mathscr{M}_W$. Let X_0 denote the set considered in the definition of the class $\mathscr{M}_W$, that is to say $W(X_0) = 1$. Let $x \in X_0$. Then there exists $l \in H_W$ and a neighborhood V of x such that for W-almost all $y \in V$ and any segment $\Delta = \{y + cl;\ c \in [a; b]\} \subset V$:

$$\left.\begin{matrix} \Delta_n \parallel l_n \\ \Delta_n \to \Delta \end{matrix}\right\} \Rightarrow \lambda\varphi_{\Delta_n}^{-1} \xrightarrow[n \to \infty]{\text{var}} \lambda\varphi_\Delta^{-1}$$

for any (l_n) converging to l. Therefore, l and V are fixed by the previous definition. Let us choose an open ball B with center at x such that $B \subset V$ and let us choose $\varepsilon > 0$ such that, for W-almost all $y \in V$, any segment $\Delta = \{y + cl;\ c \in [0; \varepsilon]\} \subset V$. For all $c \in [0; \varepsilon]$ and all $y \in B$, we put

$$G_{c,n}\, y = y + c\Pi_n(l) \quad \text{and} \quad G_{c,\infty}\, y = y + cl.$$

The aim is now to establish five assumptions of Theorem 4.2.

(i) For each $c \in (0; \varepsilon)$:

$$G_{c,n}\, x = x + c\Pi_n(l) \quad \text{and} \quad G_{c,\infty}\, x = x + cl.$$

But $\Pi_n(l) \to l$ when $n \to \infty$. Consequently, $G_{c,n}\, x \to G_{c,\infty}\, x$ in measure P_n.

(ii) For each $c \in (0; \varepsilon)$, $G_{c,\infty}\, y = y + cl$ is continuous. For each ball S, we have

$$d(S, c) = \sup_{z \in S} \varrho(z, G_{c,\infty}\, z) = c \sup_{t \in [0;1]} |l(t)| \to 0, \qquad c \to 0.$$

(iii) Let T_n be the mapping from F_n to R^n, which to all polygonal lines constructed on the points $(k/n, x_k)$ assigns $(\sqrt{n}\, x_1, \sqrt{n}(x_2 - x_1), \ldots, \sqrt{n}(x_{n-1} - x_n))$. Then T_n is linear bijective. Let us remind that ζ_n is the polygonal line constructed on the points $(k/n, S_k/\sqrt{n})$. Therefore

$$T_n(\zeta_n) = (f(\xi_1), \ldots, f(\xi_n))$$

and $P_n T_n^{-1} = \hat{\mathscr{P}}_n$. Moreover, $\zeta_n + c\Pi_n(l)$ is the polygonal line constructed on the points $(k/n, S_k/\sqrt{n} + cl(k/n))$. Let us remind that $S_0 = 0$ et $l(0) = 0$ because

$l \in H_W$. Consequently,

$$T_n(\zeta_n + c\Pi_n(l)) = \big(f(\xi_1) + c\sqrt{n}\, l(1/n), \ldots, f(\xi_n) + c\sqrt{n}\big(l(1) - l(1-1/n)\big)\big).$$

If we put for all $0 \leqslant k \leqslant n-1$

$$a_{k+1} = \sqrt{n}\left[l\left(\frac{k+1}{n}\right) - l\left(\frac{k}{n}\right)\right],$$

then

$$T_n(\zeta_n + c\Pi_n(l)) = \big(f(\xi_1) + ca_1, \ldots, f(\xi_n) + ca_n\big).$$

But $\zeta_n + c\Pi_n(l) = G_{c,n}\zeta_n$. Then $P_n G_{c,n}^{-1} T_n^{-1} = \hat{\mathscr{P}}_n^{ca}$. Finally, we have

$$\|P_n G_{c,n}^{-1} - P_n\| = \|P_n G_{c,n}^{-1} T_n^{-1} - P_n T_n^{-1}\| = \|\hat{\mathscr{P}}_n^{ca} - \hat{\mathscr{P}}_n\|.$$

In view of assumptions (1) et (2) of Theorem 3.1, we can apply Theorem 4.1 and inequality (4.1) and we obtain

$$\|P_n G_{c,n}^{-1} - P_n\| \leqslant c\sqrt{2\hat{f}}\sqrt{\sum_{k=1}^{n} a_k^2}.$$

But

$$\sum_{k=1}^{n} a_k^2 = \sum_{k=1}^{n} n\left[l\left(\frac{k}{n}\right) - l\left(\frac{k-1}{n}\right)\right]^2 \leqslant \|l'\|_2^2.$$

Therefore, we have

$$\|P_n G_{c,n}^{-1} - P_n\| \leqslant c\sqrt{2\hat{f}}\, \|l'\|_2.$$

In view of Remarks 3.1 and 4.1, we know that $\hat{f} < \infty$ and $l' \in L^2([0;\, 1])$. Consequently, we can conclude that

$$\lim_{c \to 0} \limsup \|P_n G_{c,n}^{-1} - P_n\| = 0.$$

(iv) For each $\delta \in (0;\, \varepsilon)$, for all $c \in (0;\, \varepsilon]$, and all $z \in B$, we have

$$\varphi_{z,n}(c) = \varphi(G_{c,n} z) = \varphi\big(z + c\Pi_n(l)\big), \qquad \varphi_{z,\infty}(c) = \varphi(G_{c,\infty} z) = \varphi(z + cl).$$

We know that, for W-almost all $z \in B$, $\Delta = \{z + cl;\ c \in [0;\, \varepsilon]\} \subset \bar{V}$. Let us put

$$\Delta_n = \{z_n + c\Pi_n(l);\ c \in [0;\, \varepsilon]\} \| \Pi_n(l).$$

We have $\Pi_n(l) \to l$. If $z_n \to z$, then $\Delta_n \to \Delta$. Therefore, in view of the definition of the class $\mathscr{M}_W$, we obtain

$$\lambda\varphi_{\Delta_n}^{-1} \xrightarrow[n \to \infty]{\text{var}} \lambda\varphi_{\Delta}^{-1},$$

where, for $c \in (0;\, \varepsilon)$,

$$\varphi_{\Delta_n}(c) = \varphi\big(z_n + \Pi_n(l)\big) = \varphi_{z_n,n}(c), \qquad \varphi_{\Delta}(c) = \varphi(z + cl) = \varphi_{z,\infty}(c).$$

Therefore, for W-almost all $z \in B$, we have

$$\lambda\varphi_{z_n,n}^{-1} \xrightarrow[n\to\infty]{\text{var}} \lambda\varphi_{z,\infty}^{-1}, \qquad \lambda_{[0;\delta]}\,\varphi_{z_n,n}^{-1} \xrightarrow[n\to\infty]{\text{var}} \lambda_{[0;\delta]}\,\varphi_{z,\infty}^{-1},$$

(4.5)

$$\|\lambda_{[0;\delta]}\,\varphi_{z_n,n}^{-1} - \lambda_{[0;\delta]}\,\varphi_{z,\infty}^{-1}\| \to 0, \qquad n\to\infty.$$

Moreover, if we put $\Delta_n = \{z_n + cl;\ c \in [0;\ \varepsilon]\}$, then if $z_n \to z$, $\Delta_n \to \Delta$ and in view of the same definition, we obtain

$$\lambda\varphi_{\Delta_n}^{-1} \xrightarrow[n\to\infty]{\text{var}} \lambda\varphi_{\Delta}^{-1},$$

where, for $c \in (0;\ \varepsilon)$,

$$\varphi_{\Delta_n}(c) = \varphi(z_n + cl) = \varphi_{z_n,\infty}(c), \qquad \varphi_{\Delta}(c) = \varphi(z+cl) = \varphi_{z,\infty}(c).$$

Thus, for W-almost all $z \in B$, we have

$$\|\lambda_{[0;\delta]}\,\varphi_{z_n,\infty}^{-1} - \lambda_{[0;\delta]}\,\varphi_{z,\infty}^{-1}\| \to 0, \qquad n\to\infty. \tag{4.6}$$

In the sequel, we need the following result which is a consequence of Theorem 5.5 of [2].

LEMMA 4.1. *Let h_n and h be bounded functions. If $P_n \Rightarrow P$ and if*

$$D = \{z \mid \forall (z_n),\ z_n \to z,\ h_n(z_n) \to h(z)\}$$

is such that $P(D) = 1$, then

$$\int h_n\,dP_n \to \int h\,dP.$$

Here, $P_n \Rightarrow W_c$ and let us write

$$h_n(z_n) = \|\lambda_{[0;\delta]}\,\varphi_{z_n,n}^{-1} - \lambda_{[0;\delta]}\,\varphi_{z_n,\infty}^{-1}\| \quad \text{and} \quad h(z) = 0.$$

But

$$h_n(z_n) \leqslant \|\lambda_{[0;\delta]}\,\varphi_{z_n,n}^{-1} - \lambda_{[0;\delta]}\,\varphi_{z,\infty}^{-1}\| + \|\lambda_{[0;\delta]}\,\varphi_{z_n,\infty}^{-1} - \lambda_{[0;\delta]}\,\varphi_{z,\infty}^{-1}\|.$$

In view of (4.5) and (4.6), we know that for W-almost all $z \in B$, if $z_n \to z$, then $h_n(z_n) \to 0$. Thus, $W(D) = 1$ and, in view of Lemma 4.1, we have

$$\int_B \|\lambda_{[0;\delta]}\,\varphi_{z,n}^{-1} - \lambda_{[0;\delta]}\,\varphi_{z,\infty}^{-1}\|\,P_n(dz) \to 0, \qquad n\to\infty.$$

(v) For each $\delta \in (0;\ \varepsilon)$, let us remind that $Z(\boldsymbol{R})$ is the Banach space of finite measures on $\mathscr{B}(\boldsymbol{R})$ with the total variation as norm. For $z \in B$, let $(z_n)_{n\in N^*}$ be a sequence such that $z_n \in B$ and $z_n \to z$. In view of (4.6), we have

$$\|\lambda_{[0;\delta]}\,\varphi_{z_n,\infty}^{-1} - \lambda_{[0;\delta]}\,\varphi_{z,\infty}^{-1}\| \to 0, \qquad n\to\infty.$$

That is to say, the mapping

$$B \to Z(\boldsymbol{R}),$$

$$z \mapsto \lambda_{[0;\delta]}\,\varphi_{z,\infty}^{-1},$$

is continuous W-a.e. Now, we can apply Theorem 4.2 and the result is

$$P_n\varphi^{-1} \xrightarrow[n\to\infty]{\text{var}} W_\sigma\varphi^{-1}. \ \blacksquare$$

4.3. Harris recurrent positive Markov chains. In this section, we shall study the relations between assumptions of I-regularity and assumptions required for IP, that is to say, assumption (3) of Theorem 3.1. The following definitions and results come mainly from [16]. Let $\xi = (\xi_k)_{k \in N^*}$ be the m.c. defined in Section 2.

DEFINITION 4.1. A non-empty set F is said to be *transition closed for the kernel P* if $P(x, F^c) = 0$ for all $x \in F$. The chain is *indecomposable* if there are no two disjoint transition closed subsets.

We assume that ξ is indecomposable. For a measurable set $B \in \mathscr{B}(\boldsymbol{R})$ and an initial state x, $h_B^\infty(x)$ denotes the probability having starting at x, to visit B infinitely many times.

DEFINITION 4.2. The chain ξ is called *recurrent with respect to a measure μ on the state space* if, for any measurable set B with $\mu(B) > 0$, $h_B^\infty(x) > 0$ for all x and $h_B^\infty = 1$ except for a certain μ-null set. If $h_B^\infty \equiv 1$, the chain is said to be *Harris recurrent.*

Remark 4.2. If a chain is recurrent with respect to some non-trivial μ, there exists a measure called *maximal*, which has the least possible collection of null sets among other measures with respect to which the chain is recurrent (Proposition 2.4 in [16]). Any stationary measure is maximal by the same proposition (assuming that the chain is recurrent with respect to some measure).

LEMMA 4.2. *If an m.c. is indecomposable and has a stationary probability measure, then it is recurrent.*

Proof. According to Theorem 3.6 of [16], any indecomposable chain is either dissipative or recurrent. Let us remind that the chain is said to be *dissipative* (Definition 3.3 in [16]) if $Gg = \sum_{k \geq 0} P^k g < \infty$ everywhere on the state space for some measurable strictly positive function g. For a dissipative chain, such a g can be chosen so that $Gg \leq 1$ everywhere on the state space (Proposition 3.9 in [16]). Integrating Gg for such a g with respect to the stationary probability, we come to $\infty \leq 1$. Hence the chain is recurrent. ∎

LEMMA 4.3. *The transition kernel $x \mapsto P(x, \cdot)$ is continuous in variation in any open interval $J_0 \subseteq J$.*

Proof. Let us remind that $J = \{x \in \boldsymbol{R} \mid \pi(x) > 0\}$. For all $B \in \mathscr{B}(\boldsymbol{R})$

$$P(x, B) = \int_B p(x, y)\, dy.$$

Let us put, for $n \in N^*$, $J_n = \{x \in \boldsymbol{R} \mid \pi(x) \geq 1/n\}$. Then J_n is open because π is a.c. Moreover, $J = \bigcup_{n \in N^*} J_n$. Let $x_1, x_2 \in J_0$ be such that $x_1 < x_2$. Then there exists $n \in N^*$ such that $J_0 \subset J_n$ and we have

$$\int_{\boldsymbol{R}} |p(x_2, y) - p(x_1, y)|\, dy \leq \int_{\boldsymbol{R}} \int_{x_1}^{x_2} |p'_x(x, y)|\, dx\, dy$$

$$\leqslant \int_{x_1}^{x_2} [I^+(x)]^{1/2}\,dx \leqslant |x_2-x_1|^{1/2}\Big(\int_{J_0} I^+(x)\,dx\Big)^{1/2}$$

$$\leqslant c^{-1/2}|x_2-x_1|^{1/2}\Big(\int_{J_0} I^+(x)\,\pi(x)\,dx\Big)^{1/2} \leqslant \sqrt{n}\,|x_2-x_1|^{1/2}\sqrt{I^+}.$$

But in view of (2.1) the lemma is proved. ∎

LEMMA 4.4. *The set J is transition closed.*

Proof. Let $x_0 \in J$ and let A be a set such that $A \cap J = \varnothing$ and $P(x_0, A) > 0$. As a consequence of Lemma 4.3, we have

$$P(\cdot, A) \geqslant \tfrac{1}{2}P(x_0, A)$$

in a neighborhood N of x_0, $N \subseteq J$. Then, in view of the stationarity of Π, we have

$$\Pi(A) = \int_A \pi(y)\,dy = \int_A \int_J p(x, y)\,\pi(x)\,dx\,dy$$

$$= \int_J P(x, A)\,\pi(x)\,dx \geqslant \int_N P(x, A)\,\pi(x)\,dx$$

$$\geqslant \frac{1}{2}P(x_0, A)\int_N \pi(x)\,dx = \frac{1}{2}P(x_0, A)\,\Pi(N).$$

But $N \subseteq J$; then $\Pi(N) > 0$ and $P(x_0, A) > 0$. Therefore, $\Pi(A) > 0$. On the other hand, $A \cap J = \varnothing$ implies $\Pi(A) = 0$, which contradicts the previous assertion. ∎

According to Proposition 3.13 of [16], for a recurrent chain ξ, there exists such a transition closed subset H on the state space which has stationary probability 1 and such that the restriction $\xi|_H$ of ξ is Harris recurrent. Such an H is said to be a *Harris set.*

LEMMA 4.5. *J is a Harris set for ξ.*

Proof. Let us remind that a function f is said to be *harmonic* (respectively, *superharmonic*) for a transition kernel P if $Ph = h$ (respectively, $Ph \leqslant h$). Let us prove first that any bounded harmonic function for $\xi|_H$ is a constant.

Let h be such a harmonic function defined on J and $h \leqslant K$ in J for a constant K. Set

$$\tilde{h}(x) = \begin{cases} h(x) & \text{if } x \in J,\\ K & \text{if } x \notin J.\end{cases}$$

But J is transition closed. Then, for all $x \in J$, $P(x, J^c) = 0$. Let $x \in J$. We know that h is harmonic for P on J, so

$$P\tilde{h}(x) = \int_{\mathbf{R}} \tilde{h}(y)\,P(x, dy) = \int_J \tilde{h}(y)\,P(x, dy)$$

$$= \int_J h(y)\,P(x, dy) = Ph(x) = h(x) = \tilde{h}(x).$$

Let $x \notin J$. Therefore

$$P\tilde{h}(x) = \int_{R} \tilde{h}(y) P(x, dy) \leqslant K \int_{R} P(x, dy) = K = \tilde{h}(x).$$

Then $\tilde{h}$ is superharmonic. By Proposition 3.13 of [16], the recurrence of ξ implies that any superharmonic function is a constant Π-a.e. Hence h is a constant λ-a.e. on J. Further, the right-hand side of the identity $h = Ph$ is continuous (as a function on J) in view of Lemma 4.3 and the boundedness of h. This allows us to conclude that h is a constant. Consequently, we proved that any bounded harmonic function for $\xi|_H$ is a constant. According to Theorem 3.8 of [16], this implies that $\xi|_H$ is either dissipative or Harris recurrent. However, it cannot be dissipative, because of the existence of a stationary probability measure (see the proof of Lemma 4.2), which completes the proof. ■

Remark 4.3. According to Theorem 5.2 of [16], when the stationary measure is finite, the chain is said to be *positive recurrent*.

Therefore, if ξ is a stationary homogeneous indecomposable m.c. and if ξ satisfies (**R**), ($\mathbf{R}^+$), ($\mathbf{R}^-$) and (2.1), then ξ is Harris recurrent positive on J. Thus, we can apply the IP, that is to say Theorem 3.2, which proves Corollary 3.1.

4.4. Applications. We can apply Corollary 3.1 to the examples announced in Section 3.1. Let us remind that $S_0 = 0$ and, for all $k \in N^*$, $S_k = f(\xi_1) + \ldots + f(\xi_k)$. Let $n \in N^*$; P_n is the distribution of ζ_n and W_σ is the distribution of the process $\{\sigma w(t);\ t \in [0; 1]\}$.

EXAMPLE 4.1.

$$\varphi_1 \colon C[0; 1] \to R,$$
$$x \mapsto \varphi_1(x) = x(1).$$

Therefore, $P_n \varphi_1^{-1}$ is the distribution of $\zeta_n(1) = S_n/\sqrt{n}$ and $W_\sigma \varphi_1^{-1}$ is the distribution of $\sigma w(1)$, that is to say, $\mathcal{N}(0, \sigma^2)$. Consequently, we have

$$\mathcal{L}\left(\frac{S_n}{\sqrt{n}}\right) \xrightarrow[n \to \infty]{\text{var}} \mathcal{N}(0, \sigma^2).$$

Remark 4.4. This is a local limit theorem for the chain $(f(\xi_k))_{k \in N^*}$.

EXAMPLE 4.2.

$$\varphi_2 \colon C[0; 1] \to R,$$
$$x \mapsto \varphi_2(x) = \sup_{t \in [0;1]} x(t).$$

Therefore,

$$\mathcal{L}\left(\frac{1}{\sqrt{n}} \max_{0 \leqslant k \leqslant n} S_k\right) \xrightarrow[n \to \infty]{\text{var}} \mathcal{L}\left(\sigma \sup_{t \in [0;1]} w(t)\right).$$

EXAMPLE 4.3.

$$\varphi_3\colon C[0;1] \to \boldsymbol{R},$$

$$x \mapsto \varphi_3(x) = \sup_{t\in[0;1]} |x(t)|.$$

Here we have

$$\mathscr{L}\left(\frac{1}{\sqrt{n}} \max_{0\leqslant k\leqslant n} |S_k|\right) \xrightarrow[n\to\infty]{\text{var}} \mathscr{L}\left(\sigma \sup_{t\in[0;1]} |w(t)|\right).$$

5. LOCAL INVARIANCE PRINCIPLE FOR NON-STATIONARY HOMOGENEOUS MARKOV CHAINS

5.1. Notation and inequality. In this section, we are interested in a non-stationary homogeneous m.c., that is to say, the initial distribution Γ is not the stationary probability distribution Π. Let γ denote the initial density assumed to be a.c. and let $\{p(x,\cdot);\ x\in\boldsymbol{R}\}$ denote the family of transition densities, where for all $x\in\boldsymbol{R}$, $p(x,\cdot)$ is assumed to be a.c. For $k\geqslant 2$, f_k is the density of the r.v. ξ_k. We assume that $\gamma>0$ λ-a.e. and $p>0$ λ^2-a.e. We consider the transition density q_{k-1} for a time-reversed chain. Our regularity assumptions are the following:

(**R**) The family $\{\gamma(\cdot+t);t\in\boldsymbol{R}\}$ is I-regular.

(**R**$^+$) The family $\{p(x,\cdot);\ x\in\boldsymbol{R}\}$ is I-regular.

(**R**$_k$) For all $k\geqslant 2$, the family $\{f_k(\cdot+t);\ t\in\boldsymbol{R}\}$ is I-regular.

(**R**$_k^-$) For all $k\geqslant 2$, the family $\{q_{k-1}(y,\cdot);\ y\in\boldsymbol{R}\}$ is I-regular.

The corresponding Fisher quantities are $I(\gamma)$, $I^+(x)$, $I(f_k)$ and $I_{k-1}^-(y)$. We assume that for all $k\geqslant 2$

$$(5.1)\qquad I_k^+ = \int_{\boldsymbol{R}} I^+(x) f_{k-1}(x)\,dx < \infty \quad\text{and}\quad I_k^- = \int_{\boldsymbol{R}} I_{k-1}^-(y) f_k(y)\,dy < \infty.$$

Let $n\in\boldsymbol{N}^*$. Set

$$I_1 = I(\gamma)+I(f_2)+I_2^+ +I_2^-,$$

$$I_k = I(f_k)+I(f_{k+1})+I_k^+ +I_{k+1}^+ +I_k^- +I_{k+1}^- \quad\text{for all } 2\leqslant k\leqslant n-1,$$

$$I_n = I_n^- +I_n^+ +I(f_n).$$

Finally, $\mathscr{Q}_n^a$ and $\mathscr{Q}_n$ denote the distributions of $\bar{\xi}+\bar{a}$ and $\bar{\xi}$, respectively.

THEOREM 5.1. *If* (**R**), (**R**$^+$), (**R**$_k$), (**R**$_k^-$) *and* (5.1) *for all* $k\geqslant 2$ *are satisfied, then*

$$(5.2)\qquad \|\mathscr{Q}_n^a-\mathscr{Q}_n\| \leqslant \sqrt{\frac{3}{2}\sum_{k=1}^{n} a_k^2 I_k}.$$

The proof of this inequality is similar to the proof of Theorem 2.1.

5.2. Local invariance principle. Let us remind that $S_0 = 0$ and, for all $k \in N^*$, $S_k = f(\xi_1) + \ldots + f(\xi_k)$. The polygonal process ζ_n is defined for all $\omega \in \Omega$ and all $t \in [0; 1]$ by

$$\zeta_n(t, \omega) = \frac{1}{\sqrt{n}} \left[S_{[nt]}(\omega) + (nt - [nt]) f\left(\xi_{[nt]+1}(\omega)\right) \right].$$

Here, Q_n denote the distribution of ζ_n. As in Section 3.2, we can announce a local invariance principle.

THEOREM 5.2. *If the following assumptions are satisfied:*

(1) (**R**), (**R**$^+$), (**R**$_k$), (**R**$_k^-$) *and* (5.1) *for all* $k \geqslant 2$;

(2) *there exist* $\delta > 0$ *and* $M \geqslant 0$ *such that, for all* $x \in \boldsymbol{R}$, $f'(x) \geqslant \delta$ *and* $|f''(x)| \leqslant M$;

(3) $Q_n \Rightarrow W_\sigma$ *for some constant* $\sigma > 0$;

(4) $\varphi \in \mathcal{M}_W$,

then

$$Q_n \varphi^{-1} \xrightarrow[n \to \infty]{\text{var}} W_\sigma \varphi^{-1}.$$

The proof is the same as the proof of Theorem 3.1. We can write an inequality for the total variation between the distributions of the vectors $(f(\xi_1), \ldots, f(\xi_n))$ and $(f(\xi_1)+a_1, \ldots, f(\xi_n)+a_n)$. Then this inequality is used when we establish assumption (iii) of Theorem 4.2.

5.3. Harris recurrent positive Markov chains. Here, ξ is a Harris recurrent positive m.c., Γ denotes the initial probability distribution, Π is the stationary probability distribution. We use the subscript Γ if Γ is the initial probability distribution and we omit it if Π is the initial probability distribution. The distribution of ζ_n is P_n or Q_n if the initial probability distribution is Π or Γ, respectively. In view of Theorem 3.2, we know that $P_n \Rightarrow W_\sigma$.

THEOREM 5.3. *If ξ is Harris recurrent positive and if $f \in L^2(\boldsymbol{R}, \Pi)$ satisfies* (**C**), *then*

$$Q_n \Rightarrow W_\sigma.$$

Proof. The idea of the proof comes from the method used by Billingsley in [1] and [2], but also from a method used by Gizbrekht in [7]. In view of [1], there exists a sequence of natural integers $(k_n)_{n \in N^*}$ going to infinity so slowly that for all $\varepsilon > 0$

$$\lim_{n \to \infty} \boldsymbol{P}\{\max_{k \leqslant k_n} |S_k| \geqslant \varepsilon\sqrt{n}\} = 0 \quad \text{and} \quad \lim_{n \to \infty} \boldsymbol{P}_\Gamma\{\max_{k \leqslant k_n} |S_k| \geqslant \varepsilon\sqrt{n}\} = 0. \tag{5.3}$$

Let ζ'_n be the following process defined for all $t \in [0; 1]$ by

$$\zeta'_n(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant k_n/n, \\ \zeta_n(t) - S_{k_n}/\sqrt{n} & \text{if } k_n/n \leqslant t \leqslant 1. \end{cases}$$

We denote by P'_n and Q'_n the distributions of ζ'_n when the initial probability distribution is Π and Γ, respectively. Set

$$\delta_n = \sup_{0 \leqslant t \leqslant 1} |\zeta_n(t) - \zeta'_n(t)|.$$

It is clear that

$$\delta_n \leqslant \frac{1}{\sqrt{n}} \max_{k \leqslant k_n} |S_k|,$$

that is, by (5.3) we have

$$\lim_{n \to \infty} P\{\delta_n \geqslant \varepsilon\} = 0 \quad \text{and} \quad \lim_{n \to \infty} P_\Gamma\{\delta_n \geqslant \varepsilon\} = 0.$$

Moreover, the total variation is

$$\begin{aligned} \|P'_n - Q'_n\| &= \|P_{(f(\xi_{k_n+1}),\ldots,f(\xi_n))} - Q_{(f(\xi_{k_n+1}),\ldots,f(\xi_n))}\| \\ &= \|P_{(\xi_{k_n+1},\ldots,\xi_n)} - Q_{(\xi_{k_n+1},\ldots,\xi_n)}\|, \end{aligned}$$

where $P_{(\xi_{k_n+1},\ldots,\xi_n)}$ (respectively, $Q_{(\xi_{k_n+1},\ldots,\xi_n)}$) is the distribution of $(\xi_{k_n+1}, \ldots, \xi_n)$ when the initial probability distribution is Π (respectively Γ). Consequently,

$$\begin{aligned} \|P'_n - Q'_n\| = \int_{\boldsymbol{R}^{n-k_n}} &|\Pi(dx_{k_n+1}) P(x_{k_n+1}, dx_{k_n+2}) \ldots P(x_{n-1}, dx_n) \\ &- \Gamma P^{k_n+1}(dx_{k_n+1}) P(x_{k_n+1}, dx_{k_n+2}) \ldots P(x_{n-1}, dx_n)|, \end{aligned}$$

where ΓP^{k_n+1} is the distribution of ξ_{k_n+1} when Γ is the initial probability distribution. Therefore

$$\|P'_n - Q'_n\| = \int_{\boldsymbol{R}} |\Pi(dx) - \Gamma P^{k_n+1}(dx)| = \|\Pi - \Gamma P^{k_n+1}\|.$$

But, in view of Theorem 2.2 of [16] and the definition of J, we know that ξ is aperiodic. In view of Corollary 6.7 of [16], if ξ is aperiodic Harris recurrent positive, then for all initial probability distributions Γ we have

$$\lim_{n \to \infty} \|\Gamma P^n - \Pi\| = 0.$$

Therefore

$$\lim_{n \to \infty} \|P'_n - Q'_n\| = 0.$$

By Theorem 4.1 of [2], we obtain

$$\left.\begin{array}{l} P_n \Rightarrow W_\sigma \\ \delta_n \xrightarrow{P} 0 \end{array}\right\} \Rightarrow P'_n \Rightarrow W_\sigma.$$

Moreover, $\|P'_n - Q'_n\| \to 0$, $n \to \infty$, so $Q'_n \Rightarrow W_\sigma$. Applying Theorem 4.1 of [1], we have

$$\left.\begin{array}{l} Q_n' \Rightarrow W_\sigma \\ \delta_n \xrightarrow{P_\xi} 0 \end{array}\right\} \Rightarrow Q_n \Rightarrow W_\sigma.$$

This completes the proof of Theorem 5.3. ∎

Finally, we obtain a corollary using Theorems 5.2 and 5.3.

COROLLARY 5.1. *If the following assumptions are satisfied*:

(1) (**R**), ($\mathbf{R}^+$), ($\mathbf{R}_k$), ($\mathbf{R}_k^-$) *and* (5.1) *for all* $k \geqslant 2$;

(2) $f \in L^2(\boldsymbol{R}, \Pi)$ *and there exist* $\delta > 0$ *and* $M \geqslant 0$ *such that, for all* $x \in \boldsymbol{R}$, $f'(x) \geqslant \delta$ *and* $|f''(x)| \leqslant M$, *and* f *satisfies* (**C**) *with* $\sigma^2 > 0$;

(3) ξ *is indecomposable*;

(4) $\varphi \in \mathscr{M}_W$,

then

$$Q_n \varphi^{-1} \xrightarrow[n \to \infty]{\text{var}} W_\sigma \varphi^{-1}.$$

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REFERENCES

[1] P. Billingsley, *The invariance principle for dependent random variables*, Trans. Amer. Math. Soc. 83, No. 1 (1956), pp. 250–268.

[2] — *Convergence of Probability Measures*, Wiley, New York 1968.

[3] Y. A. Davydov, *Local limit theorems for functionals of random processes*, Theory Probab. Appl. 33, No. 4 (1988), pp. 732–738.

[4] — *A variant of infinite-dimensional local limit theorem* (in Russian), LOMI seminaries 177 (1989), pp. 46–50.

[5] — and M. A. Lifshits, *Fibering method in some probabilistic problems*, J. Soviet. Math. 31 (1985), pp. 2796–2858.

[6] — and N. V. Smorodina, *Local properties of distributions of stochastic functionals* (in Russian), American Mathematical Society, New York 1998.

[7] N. V. Gizbrekht, *On the invariance principle for the sum of random variables that are defined on a homogeneous Markov chain*, Siberian Math. J. 31, No. 1 (1990), pp. 159–162.

[8] M. J. Gordin, *On the central limit theorem for stationary processes*, J. Soviet. Math. 10, No. 5 (1969), pp. 1174–1176.

[9] — *Some questions of the theory of stationary random processes*, Ph. Doctor Thesis, Leningrad State University, 1970.

[10] — and B. A. Lifshits, *The central limit theorem for stationary random processes*, J. Soviet. Math. 19, No. 2 (1978), pp. 392–394.

[11] — *Invariance principle for stationary random processes*, ibidem 23, No. 4 (1978), pp. 865–866.

[12] I. A. Ibragimov and R. Z. Has'minskii, *Statistic Estimation: Asymptotic Theory*, Springer, Berlin–New York 1981.

[13] N. Maigret, *Théorème limite central fonctionnel pour une chaîne récurrente au sens de Harris et positive*, Ann. Inst. H. Poincaré 14, No. 4 (1978), pp. 425–440.

[14] J. Neveu, *Potentiel markovien récurrent des chaînes de Harris*, Ann. Inst. Fourier 22 (1972), pp. 85–130.
[15] C. Noquet, *Inequalities for the total variation between the distributions of a sequence and its translate*, Theory Probab. Appl. (1998).
[16] E. Nummelin, *General irreductible Markov chains and non-negative operators*, Cambridge University Press, London 1984.

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