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INTERACTING PARTICLE APPROXIMATION FOR NONLOCAL QUADRATIC EVOLUTION PROBLEMS

BY

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Abstract. The existence of McKean's nonlinear jump Markov processes and related Monte Carlo type approximation schemes by interacting particle systems (propagation of chaos) are studied for a class of multidimensional doubly nonlocal evolution problems with a fractional power of the Laplacian and a quadratic nonlinearity involving an integral operator. Asymptotically, these equations model the evolution of density of mutually interacting particles with anomalous (fractal) Lévy diffusion.

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1. Introduction. The present paper can be viewed as a continuation of our article [8], where we studied global and exploding solutions for equations of the form

$$u_t = -(-\Delta)^{\alpha/2} u + \nabla \cdot \big(uB(u)\big). \tag{1.1}$$

Here $u\colon \Omega \times (0, T) \subset \mathbf{R}^d \times \mathbf{R}^+ \to \mathbf{R}$, $(-\Delta)^{\alpha/2}$ is a fractional power of the minus Laplacian in $\mathbf{R}^d$, $0 < \alpha \leqslant 2$, and

$$B(u)(x) = \int_{\mathbf{R}^d} b(x, y)\, u(y)\, dy$$

is a linear $\mathbf{R}^d$-valued integral operator with the kernel $b\colon \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^d$. The dimension is restricted to the physically interesting cases $d = 1, 2$, or 3. The goal here is to establish the existence of McKean's nonlinear diffusions, and

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related interacting particle approximation schemes (propagation of chaos in a wide sense) for the same class of equations.

Equations (1.1) describe various physical phenomena involving diffusion and interaction of pairs of particles when suitable assumptions are made on the possibly singular integral operator B. Since our main interest is in u as a description of the density of particles in $\boldsymbol{R}^d$, we will only consider nonnegative solutions to (1.1).

In the case of *classical Brownian diffusion*, i.e., $\alpha = 2$, a deterministic study of these models in [7] was initially motivated by the Fokker–Planck type parabolic equations with nonlocal nonlinearity and we studied them mostly in bounded domains of $\boldsymbol{R}^d$, supplemented with suitable (nonlinear) boundary conditions. For instance, if

$$b(x, y) = c(x-y)|x-y|^{-d}, \tag{1.2}$$

then the equation (1.1) models the diffusion of charge carriers ($c < 0$) in electrolytes, semiconductors or plasmas interacting via Coulomb forces. If $c > 0$, it describes gravitational interaction of particles in a cloud, or galaxies in a nebula.

Related equations and parabolic systems appear in mathematical biology where they are used to model chemotaxis phenomena (see [3]). There, we have been mainly interested in the possibility of the continuation of local in time solutions of (1.1) up to $T = +\infty$. The answer to this question depends strongly on the type of interaction. For instance, for Newtonian attraction of particles or chemotactic attraction of cells, finite time collapse of solutions is possible (see [2] and [3]), while for the Coulomb forces global in time existence of solutions is guaranteed (cf. [5]).

Further, for the Biot–Savart kernel

$$b(x, y) = (2\pi)^{-1}(x_2-y_2, y_1-x_1)|x-y|^{-2} \tag{1.3}$$

in $\boldsymbol{R}^2$, the equation (1.1) with $\alpha = 2$ is equivalent to the vorticity formulation of the Navier–Stokes equations. Its solutions are global in time. Also, formally, the singular kernel $b(x, y) = c\delta(x-y)$ leads to the classical Burgers equation

$$u_t = u_{xx} + c(u^2)_x. \tag{1.4}$$

A new important ingredient of a more general class of model problems (1.1) in [8], studied in the whole space $\boldsymbol{R}^d$, was the *anomalous Lévy α-stable diffusion* described by a fractional power of the Laplace operator in $\boldsymbol{R}^d$. In the physical literature such fractal diffusions have been vigorously studied in the context of statistical mechanics, hydrodynamics, acoustics, relaxation phenomena and biology, see e.g. [1] and [29]. They also appear in nonlinear models of interfacial growth which involve hopping and trapping effects [25].

In probabilistic terms, replacing the Laplacian by its fractional power leads to interesting questions of extension of results for Brownian motion driven stochastic equations to those driven by Lévy α-stable flights; the latter, of course, having discontinuous sample paths. Linear equations with α-stable processes have been considered e.g. in [19], [20], and [31].

In fact, the probabilistic theory of interacting particle systems and theory of McKean's diffusions have been our immediate theoretical inspiration for [8]. McKean's processes and "propagation of chaos" results connect the detailed Liouvillean picture of the evolution of diffusing and interacting particles and the reduced hydrodynamic description. We cite only a few of references that deal with different aspects of this connection in the case of classical Brownian diffusion: [9], [10], [16], [21], [26], [28], [32], and [35].

The analogous interacting particle system approximation questions for the "fractal" Burgers equation with α-stable processes

$$u_t = -(-\Delta)^{\alpha/2} u + a \cdot \nabla (u^r) \tag{1.5}$$

have been dealt with in [14] for $d = 1$ and $r = 2$, see also [35]. Based on various estimates of solutions to the deterministic Burgers equation with fractal diffusion in [4], theorems in the "propagation of chaos" spirit have been recently proved in [14]. This paper relates to [8] as [14] to [4]. Also, in [6] we studied the first and the second order asymptotics of equations similar to (1.5).

Let us note that a direct numerical approach to equations like (1.1) or (1.5) is extremely difficult because of the doubly nonlocal character of these equations. First, the linear operator $(-\Delta)^{\alpha/2}$ for $0 < \alpha < 2$ is no longer a differential operator but an integro-differential one. Second, the nonlinearity of $uB(u)$ involves integrals over the whole space $\boldsymbol{R}^d$. However, if the "propagation of chaos" property is established even in a wide sense considered below, then an efficient numerical analysis of these equations via Lévy α-stable Monte Carlo simulations becomes available; cf. the references [9], [15], [27], [33] for analogous aspects of the numerical analysis of classical PDE's.

The original propagation of chaos property does not seem to hold because insufficient regularity is gained from the fractional Laplacian (see remarks in Section 5).

The composition of the paper is as follows: Section 2 recalls and extends some results from [8] on local and global in time solvability of equations (1.1). A construction of McKean's nonlinear diffusion is provided in Section 3. Here we also formulate our main results on the stochastic particle approximation scheme for a smoothed version of (1.1), including some error estimates; the proofs are in Section 4. The notion of propagation of chaos in a wide sense is introduced in Section 5 and results pertaining to the equation (1.1) can be found there, as well as some comments relating the class of equations (1.1) to the fractal Burgers equation (1.5). Our probabilistic constructions rely on papers [26], [11], [12], [19], [20] and [32]. The first reference started with a study of (1.1) with $\alpha = 2$ and B defined by a Lipschitz kernel b, see also [32]. Our kernels motivated by the above-mentioned applications are far more singular. As a general reference for PDE theory we cite [23] together with a brief note [17] used for various interpolation inequalities.

Notation. $|u|_p$ stands for the Lebesgue $L^p(\mathbf{R}^d)$-norm of the function u, $\|u\|_{k,p}$ — for the Sobolev $W^{k,p}(\mathbf{R}^d)$-norm, and $\|u\|_k$ is the $H^k \equiv W^{k,2}$-norm. Inessential constants will be denoted generically by C, even if they vary from line to line.

2. Local and global existence of solutions. In this section we provide existence results for the local and global in time (weak) solutions of the initial value problem for (1.1). We consider in the sequel only the simplest case of $\Omega = \mathbf{R}^d$, although most of results in this section extends to u defined on an open subset Ω of $\mathbf{R}^d$ and satisfying suitable boundary conditions on $\partial\Omega$.

We restrict ourselves to the case of convolution operators B in (1.1), the most important in the applications, so that from now on $b(x, y) = b(x-y)$. Moreover, we assume that b satisfies potential estimates like either

(2.1) $$|b(x)| \leqslant C|x|^{\beta-d}$$

or

(2.2) $$|Db(x)| \leqslant C|x|^{\gamma-d}$$

for some $0 < \beta < d$, $0 < \gamma < d$, which is motivated by the examples (1.2) and (1.3). Formally, (1.4) corresponds to the limit case $\beta = 0$ but, of course, the operator $B(u) = cu$, $0 \neq c \in \mathbf{R}^d$, is not an integral one. In fact, assumptions (2.1) and (2.2) can be weakened as, e.g., in Section 2 of [8], but we prefer to keep the potential character and smoothing properties of B clear. This permits us to obtain some extensions of results in [8].

By the fractional power of the minus Laplacian in $\mathbf{R}^d$ we mean the Fourier multiplier

$$(-\Delta)^{\alpha/2} v(x) \equiv D^\alpha v(x) = \mathscr{F}^{-1}\big(|\xi|^\alpha \hat{v}(\xi)\big)(x),$$

which has also the representation

$$-(-\Delta)^{\alpha/2} v(x) = K \int_{\mathbf{R}^d} \big(v(x+y) - v(x) - \nabla v(x)\cdot y(1+|y|^2)^{-1}\big)|y|^{-d-\alpha}\,dy$$

for the range of parameter α, $\alpha \in (0, 2)$, we are interested in. Here $K = K_{\alpha,d}$ is a constant.

Now, we recall results from [8] on the local in time existence of solutions to (1.1) with the initial condition

(2.3) $$u(x, 0) = u_0(x),$$

under the assumption (2.1) or (2.2) specified to the case $d \leqslant 3$, in order to use the framework of Hilbertian Sobolev spaces $H^k(\mathbf{R}^d)$. By a solution we mean a *weak* one, i.e. a function $u \in L^2((0, T); H^{\alpha/2}(\boldsymbol{R}^d))$ such that the integral identity

$$\int_{\boldsymbol{R}^d} u(x, t)\eta(x, t)\,dx - \int_0^t ds \int_{\boldsymbol{R}^d} u\eta_s\,dx + \int_0^t ds \int_{\boldsymbol{R}^d} \left(D^{\alpha/2} u D^{\alpha/2} \eta + uB(u)\cdot \nabla\eta\right) dx = \int_{\boldsymbol{R}^d} u_0(x)\eta(x, 0)\,dx$$

holds for every test function $\eta \in H^1(\boldsymbol{R}^d \times (0, T))$, cf. [8], Section 2.

THEOREM 2.1. *Suppose that* $\alpha+\beta > d/2+1$ *in* (2.1), $\alpha \in (0, 2]$, $\beta \in (0, d)$, $d = 1, 2, 3$, *and the initial condition is* $0 \leqslant u_0 \in L^2(\boldsymbol{R}^d) \cap L^1(\boldsymbol{R}^d)$. *Then there exist* $T > 0$ *and a weak solution* $u \geqslant 0$ *of the Cauchy problem* (1.1), (2.3). *Moreover,* $|u(t)|_1 = |u_0|_1$ *for all* $t \in (0, T)$.

The above theorem contains Theorems 2.1 and 2.2 in [8], and improves over those results for some $0 < \beta < 1$ and for $d = 1$ not considered there.

Proof. We give only a crucial *a priori* estimate of $u(t)$ in L^2 referring to [8] for a description of the construction of u. Observe that

$$\frac{d}{dt}|u|_2^2 + 2|D^{\alpha/2}u|_2^2 = -2\int_{\boldsymbol{R}^d} uB(u)\cdot\nabla u\,dx \tag{2.4}$$

and the right-hand side of (2.4) can be transformed into

$$-\int_{\boldsymbol{R}^d} \nabla(u^2)\cdot B(u)\,dx = \int_{\boldsymbol{R}^d} u^2\,\nabla\cdot B(u)\,dx.$$

Then we estimate, from the Schwarz inequality and the condition (2.1) which assure smoothing properties of B,

$$\Big|\int_{\boldsymbol{R}^d} u^2\,\nabla\cdot B(u)\,dx\Big| \leqslant |u^2|_2\,\|B(u)\|_1 \leqslant C|u|_4^2\,\|u\|_{1-\beta}. \tag{2.5}$$

Note that the assumptions (2.1) and (2.2) on the potential nature of the kernel b are stricter than those imposed in [8], thus permitting stronger estimates than L^p-estimates in that paper.

Next, by interpolation we get

$$\Big|\int_{\boldsymbol{R}^d} u^2\,\nabla\cdot B(u)\,dx\Big| \leqslant C\|u\|_{\alpha/2}^{d/\alpha+2(1-\beta)/\alpha}|u|_2^{3-d/\alpha-2(1-\beta)/\alpha} \leqslant \|u\|_{\alpha/2}^2 + C|u|_2^m \tag{2.6}$$

for some $m > 0$ if $1-\beta \leqslant \alpha/2$ and $d/\alpha + 2(1-\beta)/\alpha < 2$, which is the assumption in Theorem 2.1. Now, (2.4) and (2.6) lead to the differential inequality

$$\frac{d}{dt}|u|_2^2 + |D^{\alpha/2}u|_2^2 \leqslant C(|u|_2^2 + |u|_2^m) \tag{2.7}$$

which implies a local bound $|u(t)|_2 \leqslant C(T) < \infty$ for some $T = T(|u_0|_2) > 0$ and all $t \in (0, T)$.

Note that for $\beta \geqslant 1$ the proof of Theorem 2.2 in [8] involved another reasoning based on the Hardy–Littlewood–Sobolev inequality.

The positivity and total mass preserving properties of (1.1) are the consequences of those properties of Lévy and Gauss semigroups

$$\exp\left(-t(-\Delta)^{\alpha/2}\right) = \mathscr{F}^{-1}\left(\exp\left(-t\,|\xi|^{\alpha}\right)\mathscr{F}\right)$$

of *probability* measures corresponding to the cases $0 < \alpha < 2$ and $\alpha = 2$, respectively. Moreover, weak solutions to (1.1) enjoy some supplementary regularity properties, due to parabolic smoothing by $(-\Delta)^{\alpha/2}$, see Section 2 in [7], Sections 2 and 3 in [5], and [8]. ∎

Remark 2.1. Although the calculations above are not directly applicable to the Burgers equation (1.4), the assumption $\alpha+\beta > d/2+1$ gives a correct result. This guarantees even the global existence of solutions if $d = 1$, $\beta = 0$, $u_0 \in H^1(\boldsymbol{R})$, so that $\alpha > 3/2$ (see [4], Theorem 2.1). Concerning the higher dimensional quadratic Burgers equation (1.5) with $r = 2$, the condition $\alpha+\beta > d/2+1$ may suggest that no weak solutions exist for $d \geqslant 2$ and $\alpha \in (0, 2]$. This can serve as an heuristic motivation for the study of another kind of solutions, namely *mild* ones in [4], Section 6.

The theorem below recalls sufficient conditions for the global in time existence of solutions, see [8], Section 3.

THEOREM 2.2. *Suppose that* $\alpha+\beta > d+1$ *in* (2.1), $\alpha \in (0, 2]$, $\beta \in (0, d)$, $d = 1, 2, 3$. *Then any local solution to the Cauchy problem* (1.1), (2.3) *with* $u_0 \in L^2(\boldsymbol{R}^d) \cap L^1(\boldsymbol{R}^d)$ *can be continued to the whole half-line* $(0, \infty)$.

Proof. The right-hand side of the energy identity (2.4) can be estimated as in (2.5) for $0 < \beta \leqslant 1$. After interpolation of norms this quantity is bounded by

$$C\,\|u\|_{\alpha/2}^{3d/(\alpha+d)+(d+2-2\beta)/(\alpha+d)}\,|u|_1^m$$

with some $m > 0$. Our assumption shows that the exponent of $\|u\|_{\alpha/2}$ above is strictly less than 2. Hence, (2.4) implies that

$$\frac{d}{dt}|u|_2^2 + |D^{\alpha/2}u|_2^2 \leqslant C\left(|u|_2^2 + |u|_1^M\right),$$

so a locally uniform estimate of $|u(t)|_2$ follows, and by the results of Theorem 2.1 $u(t)$ has a continuation to $(0, \infty)$.

For $\beta > 1$ we apply to the second factor on the right-hand side of (2.7) the Hardy–Littlewood–Sobolev inequality, and then the interpolation to obtain

$$\|B(u)\|_1 \leqslant C\,|u|_q \leqslant C\,\|u\|_{\alpha/2}^{k}\,|u|_1^{1-k}$$

with $1/2 = 1/q-(\beta-1)/d$ and $k = (d+2\beta-2)/(\alpha+d)$. The conclusion follows now as before when $0 < \beta \leqslant 1$. ∎

For $\alpha = 2$ we recover Theorem 3.1 in [8] where $\beta > d-1$. This result is sharp as Section 4 in [8] showed. Namely, if $\beta \leqslant d-1$, there exist equations of type (1.1) and their particular solutions u that cannot be continued beyond an interval $[0, T)$ with a finite time $0 < T < \infty$.

Note that under essentially the same growth assumption $\gamma > d-\alpha$ in (2.2) (since $\beta = \gamma - 1$ for smooth kernels b satisfying (2.2)) Theorem 2.2 for $0 < \alpha < 2$ has been proved in [8], Theorem 3.2.

3. Nonlinear Markov processes and approximating particle systems for regularized equations. We begin this section with the construction of a nonlinear Markov process for which the equation (1.1) serves as the Fokker–Planck–Kolmogorov equation. The assumption $\alpha \in (1, 2)$ permits us to use freely the expectations of the α-stable processes involved in the construction.

Let $u \geqslant 0$ be a (local in time) solution of (1.1). Without loss of generality we can assume that u is bounded, i.e.

$$\sup_{x \in \boldsymbol{R}^d, t \in [0,T]} |u(x, t)| < \infty. \tag{3.1}$$

This is a property similar to that in Theorem 2.1 (iii) in [8] where the case $\alpha = 2$ was considered. Whenever a local solution u can be defined, the parabolic regularization property of $(-\Delta)^{\alpha/2}$, $\alpha \in (1, 2]$, leads to an instantaneous smoothing of u to a locally bounded function. Indeed, by standard arguments of Moser's type ([5], Theorem 3) it can be proved that $u \in L^\infty_{\mathrm{loc}}((0, T); L^\infty(\boldsymbol{R}^d))$. Here, the key estimate is an L^p-analog of (2.4). The full exposition of this line of reasoning is omitted; the details are laborious and essentially repeat step-by-step, with appropriate adjustment of exponents, the proofs of Section 2 in [5].

Shrinking, if necessary, the time interval of existence of the solution (or assuming from the beginning that $u_0 \in L^\infty(\boldsymbol{R}^d) \cap L^1(\boldsymbol{R}^d)$ is regular enough), we obtain (3.1). Moreover, since we are working with $(L^1 \cap L^\infty)$-solutions, the estimate

$$\sup_{x \in \boldsymbol{R}^d, t \in [0,T]} \left|B\big(u(t)\big)(x)\right| < \infty \tag{3.2}$$

follows from the potential estimate (2.1), the Sobolev embedding theorem and (3.1).

Consider a solution $X(t)$ of the stochastic differential equation

$$dX(t) = dS(t) - B\big(u(t)\big)\big(X(t)\big)\,dt, \tag{3.3}$$

where u is a given (bounded) solution of (1.1), $X(0) \sim u(x, 0)\,dx$ in law, and $S(t)$ is a standard α-stable spherically symmetric process with its values in $\boldsymbol{R}^d$. Recall that it has the structure

$$S(1) \sim (A^{1/2} G_1, \ldots, A^{1/2} G_d),$$

where A is an $(\alpha/2)$-stable, totally asymmetric positive random variable, and $G_1, \ldots, G_d$ are independent identically distributed Gaussian random variables. So, conditionally on A, $S(1)$ is Gaussian with characteristic function $\exp(-|\xi|^2)$.

Since the coefficient $B(u)$ in (3.3) is bounded, based on the work [20], we infer that the stochastic differential equation (3.3) has a unique solution X. The measure-valued function

$$v(dx, t) \equiv P\big(X(t) \in dx\big) \tag{3.4}$$

satisfies the weak forward equation

$$\frac{d}{dt}\langle v(t), \eta\rangle = \langle v(t), \mathscr{L}_{u(t)}\eta\rangle \tag{3.5}$$

for all $\eta \in \mathscr{S}(\boldsymbol{R}^d)$, the Schwartz class of functions on $\boldsymbol{R}^d$, with the initial condition $v(0) = u(x, 0)\,dx$, and the operator

$$\mathscr{L}_u = -(-\Delta)^{\alpha/2} - B(u)\cdot\nabla, \quad u = u(x).$$

PROPOSITION 3.1. *Let $1 < \alpha < 2$ and u be a solution of* (1.1) *satisfying* (3.1). *The process $X(t)$ in* (3.3) *is the McKean process* (*nonlinear Markov process*) *corresponding to* (1.1), *that is, it satisfies the relation*

$$P(X(t) \in dx) = u(x, t)\,dx.$$

Proof. From the results of [11] (see [12]), the following two statements are equivalent:

- The martingale problem for the operator $\mathscr{L}_{u(t)}$ is well posed.
- The existence and uniqueness theorem holds for the corresponding *linear* weak forward equation (3.5).

Here, the martingale problem associated with (3.3) is well posed. However, $u(dx, t) \equiv u(x, t)\,dx$ is also a solution of (3.5) since

$$\frac{d}{dt}\langle u(t), \eta\rangle = \langle -(-\Delta)^{\alpha/2}u + \nabla\cdot(uB(u)), \eta\rangle = \langle u, (-(-\Delta)^{\alpha/2} - B(u)\cdot\nabla)\eta\rangle.$$

Since the coefficients of the *linear* equation (3.5) are regular ($B(u) \in L^\infty$), the problem

$$w_t = -(-\Delta)^{\alpha/2}w - B(u)\cdot\nabla w, \quad w(0) = 0,$$

has the unique solution $w \equiv 0$. This can be easily seen from the energy estimates as in the proof of Theorem 2.1. Now, the uniqueness for (3.5) implies that $v(dx, t) = u(dx, t)$, which yields Proposition 3.1. ∎

The classical *propagation of chaos* result for the partial differential equation (1.1) would show that the empirical distribution

$$\tilde{X}^n(t) = \frac{1}{n}\sum_{i=1}^{n}\delta(X^{i,n}(t)) \tag{3.6}$$

of n interacting particles with positions $\{X^{i,n}(t)\}_{i=1,\ldots,n}$, whose dynamics is described by the system of stochastic differential equations

$$dX^{i,n}(t) = dS^i(t) - \frac{1}{n}\sum_{j\neq i} b(X^{i,n}(t), X^{j,n}(t))\,dt, \tag{3.7}$$

is close to the distribution of the McKean process $X(t)$ in the sense that

$$\tilde{X}^n(t) \Rightarrow u(x, t)\,dx \text{ in probability} \quad \text{as } n \to \infty, \tag{3.8}$$

where $\Rightarrow$ denotes the weak convergence of measures. In our situation $\{S^i(t)\}_{i=1,\ldots,n}$ are independent copies of symmetric Lévy α-stable processes with the common infinitesimal generator $-(-\Delta)^{\alpha/2}$.

Results in this spirit, when S is replaced by a more familiar Wiener process, have been proved in various situations after the pioneering work [26]. We have chosen some classical as well as new references containing reformulations, extensions and generalizations of the above scheme for various evolution problems of physical origin: [16], [10], [21], [28], [15], [32], [35], [9], [27], [33]. Besides a purely mathematical interest, they give also reasonably well-working tools for the numerical approximation of solutions, especially when convergence rates can be found.

The recent paper [14] deals with the first, to the best of our knowledge, "propagation of chaos" result for Lévy α-stable processes driven stochastic differential equations associated with the fractal Burgers equation. As we mentioned in the Introduction, because of a rather weak parabolic regularization effect of $(-\Delta)^{\alpha/2}$, a preliminary step involving the replacement of $X^{i,n}$ by solutions of regularized stochastic differential equations ((3.10) below) seems to be necessary in order to have an analogue of (3.8).

Let us consider a standard smoothing kernel

$$\delta_\varepsilon(x) = (2\pi\varepsilon)^{-d/2}\exp\left(-|x|^2/(2\varepsilon)\right), \qquad \varepsilon > 0, \tag{3.9}$$

and the system of regularized equations (3.7):

$$dX^{i,n,\varepsilon}(t) = dS^i(t) - \frac{1}{n}\sum_{j\neq i} b_\varepsilon\left(X^{i,n,\varepsilon}(t) - X^{j,n,\varepsilon}(t)\right)dt, \tag{3.10}$$

where $b(x, y) = b(x-y)$, $b_\varepsilon = b * \delta_\varepsilon$. Then define random empirical measures

$$Y^{n,\varepsilon}(t) = \frac{1}{n}\sum_{i=1}^{n}\delta\left(X^{i,n,\varepsilon}(t)\right),$$

instead of previously considered $\tilde{X}^n$ in (3.6).

First, we prove the "propagation of chaos" property for a regularized version (3.14) (below) of the equation (1.1), including an error estimate. In Section 5 we will prove a weaker property "propagation of chaos in a wide sense" for the original equation (1.1), which is, however, a satisfactory basis for an approximation scheme for numerical solving of that equation. The extension will rely on purely analytic estimates of solutions u^ε of (3.14).

THEOREM 3.1. *Let the conditions of Theorem* 2.1 *ensuring the local in time existence of solutions to* (1.1) *on* $\boldsymbol{R}^d \times (0, T)$ *be satisfied. Moreover, assume that*

$$|\hat{b}(\xi)| \leqslant C(1+|\xi|^{-\beta}) \tag{3.11}$$

(*which is, of course, compatible with the potential estimate* (2.1)) *and that the* *initial conditions* $\{X^{i,n,\varepsilon}(0)\}_{i=1,\dots,n}$ *satisfy*

$$\sup_n \sup_{\lambda\in\mathbf{R}^d} n^{1-1/\alpha}(1+|\lambda|^a)^{-1} E[\langle Y^{n,\varepsilon}(0)-u^\varepsilon(x,0),\chi_\lambda\rangle] < \infty \tag{3.12}$$

for some $a \geq 0$ *and all the characters* $\chi_\lambda(x) = e^{i\lambda x}$. *Then:*

(i) *For each* $\varepsilon > 0$ *the empirical process is weakly convergent*

$$Y^{n,\varepsilon}(t) \Rightarrow u^\varepsilon(x,t)\,dx \text{ in probability} \quad \text{as } n \to \infty. \tag{3.13}$$

The limit density $u^\varepsilon = u^\varepsilon(x,t)$, $x\in\mathbf{R}^d$, $t\in(0,T)$, *solves the regularized equation* (1.1):

$$u_t^\varepsilon = -(-\Delta)^{\alpha/2}u^\varepsilon + \nabla\cdot(u^\varepsilon B_\varepsilon(u^\varepsilon)) \tag{3.14}$$

with $B_\varepsilon = \delta_\varepsilon * B$ *defined by the kernel* $b_\varepsilon = \delta_\varepsilon * b$.

(ii) *For each* $\varepsilon > 0$, *there exists a constant* C_ε *such that for any* $\phi\in\mathscr{S}(\mathbf{R}^d)$

$$E|\langle Y^{n,\varepsilon}(t)-u^\varepsilon(t),\phi\rangle| \leq C_\varepsilon n^{1/\alpha-1}\int_{\mathbf{R}^d}(1+|\lambda|^a)|\hat\phi(\lambda)|\,d\lambda. \tag{3.15}$$

(iii) *Under the assumptions of Theorem* 2.2 *guaranteeing the global in time existence of solutions to* (1.1), *the conclusions* (i) *and* (ii) *are valid for all* $t\in(0,\infty)$.

We will prove Theorem 3.1 in the next section. The case $\alpha = 2$ is, of course, classical (see the approach in [32]), but it can be also handled using the scheme of proof in Section 4.

4. Proof of Theorem 3.1. The proof of Theorem 3.1 needs some representation formulas for the α-stable process $S(t)$. Recall a decomposition of the process $S(t)$:

$$S(t) = \int_0^{t+}\int_{0<|y|<1} y\tilde N(ds\,dy) + \int_0^{t+}\int_{|y|\geq 1} yN(ds\,dy), \tag{4.1}$$

where $N(ds\,dy)$ is a Poisson point process with intensity $\hat N(ds\,dy) = ds\,\nu(dy)$, $\nu(dy) = K|y|^{-d-\alpha}dy$ is the Lévy measure, and $\tilde N(ds\,dy) = N(ds\,dy)-\hat N(ds\,dy)$, see, e.g., [18].

Proof of Theorem 3.1. The assertion (3.13) follows essentially from (3.15) since $\alpha > 1$. Hence, it is sufficient to prove the rate of convergence in (3.15). The following decomposition lemma prepares the quantities in (3.15) to be estimated easily.

LEMMA 4.1. *Let us define*

$$\vartheta^n(t) \equiv Y^{n,\varepsilon}(t)-u^\varepsilon(t). \tag{4.2}$$

Then for each $\phi\in C_b^\infty(\mathbf{R}^d)$ *the identity*

$$\begin{aligned}\langle\vartheta^n(t),\phi\rangle-\langle\vartheta^n(0),\phi\rangle \\ = m^n(\phi,t)+\int_0^t\langle\vartheta^n(s),-(-\Delta)^{\alpha/2}\phi\rangle\,ds+\int_0^t q^n(\phi,s)\,ds\end{aligned} \tag{4.3}$$

holds, with

$$(4.4)\quad m^n(\phi, t) = \frac{1}{n}\sum_{i=1}^{n}\int_0^{t+}\int_{\mathbf{R}^d}\left(\phi\left(X^{i,n,\varepsilon}(s-)+y\right)-\phi\left(X^{i,n,\varepsilon}(s-)\right)\right)\tilde{N}^i(ds\,dy),$$

and

$$(4.5)\quad \begin{aligned} q^n(\phi, t) = &-\langle \vartheta^n(dx, t)\,\vartheta^n(dy, t),\, b_\varepsilon(x-y)\cdot\nabla\phi(x)\rangle \\ &-\langle u^\varepsilon(dx, t)\,\vartheta^n(dy, t)+\vartheta^n(dx, t)\,u^\varepsilon(dy, t),\, b_\varepsilon(x-y)\cdot\nabla\phi(x)\rangle \\ &+\frac{1}{n}\,b_\varepsilon(0)\cdot\langle Y^{n,\varepsilon}(t), \nabla\phi\rangle. \end{aligned}$$

Here $\tilde{N}^i = N^i - \hat{N}$, *and* $N^i = N^i(ds\,dy)$ *are independent Poisson point processes with identical intensity* $\hat{N}(ds\,dy) = K|y|^{-d-\alpha}\,ds\,dy$.

Proof. First note that

$$\langle \vartheta^n(t), \phi\rangle = \frac{1}{n}\sum_{i=1}^{n}\phi\left(X^{i,n,\varepsilon}(t)\right)-\langle u^\varepsilon(t), \phi\rangle.$$

Then, by the decomposition (4.1) of $S(t)$ and the Itô formula for α-stable processes [18],

$$\begin{aligned} &\langle \vartheta^n(t), \phi\rangle - \langle \vartheta^n(0), \phi\rangle \\ &\quad= \frac{1}{n}\sum_{i=1}^{n}\int_0^{t+}\int_{\mathbf{R}^d}\left(\phi\left(X^{i,n,\varepsilon}(s-)+y\right)-\phi\left(X^{i,n,\varepsilon}(s-)\right)\right)\tilde{N}^i(ds\,dy) \\ &\qquad+\frac{1}{n}\sum_{i=1}^{n}\int_0^{t}\Big(-(-\Delta)^{\alpha/2}\phi\left(X^{i,n,\varepsilon}(s)\right) \\ &\qquad-\frac{1}{n}\sum_{j\neq i} b_\varepsilon\left(X^{i,n,\varepsilon}(s)-X^{j,n,\varepsilon}(s)\right)\cdot\nabla\phi\left(X^{i,n,\varepsilon}(s)\right)\Big)\,ds \\ &\qquad-\int_0^{t}\left\langle -(-\Delta)^{\alpha/2}u^\varepsilon(s)+\nabla\cdot\left(u^\varepsilon(s)\,B_\varepsilon\left(u^\varepsilon(s)\right)\right), \phi\right\rangle ds \\ &\quad= m^n(\phi, t)+\int_0^{t}\langle \vartheta^n(s), -(-\Delta)^{\alpha/2}\phi\rangle\,ds+\int_0^{t} r^n(\phi, s)\,ds, \end{aligned}$$

where the last term is defined as

$$\begin{aligned} r^n(\phi, s) \equiv -n^{-2}\sum_{i=1}^{n}\sum_{j\neq i} b_\varepsilon\left(X^{i,n,\varepsilon}(s)-X^{j,n,\varepsilon}(s)\right)\cdot\nabla\phi\left(X^{i,n,\varepsilon}(s)\right) \\ +\langle u^\varepsilon(s), B_\varepsilon\left(u^\varepsilon(s)\right)\cdot\nabla\phi\rangle. \end{aligned}$$

Since

$$\begin{aligned} \frac{1}{n}\sum_{j\neq i} b_\varepsilon\left(X^{i,n,\varepsilon}(s)-X^{j,n,\varepsilon}(s)\right) &= \langle Y^{n,\varepsilon}(s), b_\varepsilon\left(X^{i,n,\varepsilon}(s)-.\right)\rangle-\frac{1}{n}\,b_\varepsilon(0) \\ &= \langle \vartheta^n(s), b_\varepsilon\left(X^{i,n,\varepsilon}(s)-.\right)\rangle+u^\varepsilon(s)*b_\varepsilon\left(X^{i,n,\varepsilon}(s)\right)-\frac{1}{n}\,b_\varepsilon(0), \end{aligned}$$

we obtain

$$r^n(\phi, s) = \frac{-1}{n}\sum_{i=1}^{n}\Big(\langle \vartheta^n(s), b_\varepsilon(X^{i,n,\varepsilon}(s)-.)\rangle$$
$$+u^\varepsilon(s)*b_\varepsilon(X^{i,n,\varepsilon}(s))-\frac{1}{n}b_\varepsilon(0)\Big)\cdot\nabla\phi(X^{i,n,\varepsilon}(s))+\langle u^\varepsilon(s), B_\varepsilon(u^\varepsilon(s))\cdot\nabla\phi\rangle$$
$$= -\langle Y^{n,\varepsilon}(dx, s)\,\vartheta^n(dy, s), b_\varepsilon(x-y)\cdot\nabla\phi(x)\rangle-\langle Y^{n,\varepsilon}(s), u^\varepsilon(s)*b_\varepsilon\cdot\nabla\phi\rangle$$
$$+\frac{1}{n}b_\varepsilon(0)\cdot\langle Y^{n,\varepsilon}(s), \nabla\phi\rangle+\langle u^\varepsilon(s), B_\varepsilon(u^\varepsilon(s))\cdot\nabla\phi\rangle$$
$$= -\langle \vartheta^n(dx, s)\,\vartheta^n(dy, s), b_\varepsilon(x-y)\cdot\nabla\phi(x)\rangle$$
$$-\langle u^\varepsilon(dx, s)\,\vartheta^n(dy, s), b_\varepsilon(x-y)\cdot\nabla\phi(x)\rangle$$
$$-\langle \vartheta^n(s), u^\varepsilon(s)*b_\varepsilon\cdot\nabla\phi(x)\rangle+\frac{1}{n}b_\varepsilon(0)\cdot\langle Y^{n,\varepsilon}(s), \nabla\phi\rangle$$
$$= q^n(\phi, s),$$

as claimed. ∎

Proof of (3.15). Now, we shall prove estimates for the quantities $\vartheta^n(t)$ defined in (4.2). We apply the decomposition Lemma 4.1 with $\phi = \chi_\lambda$ and $\chi_\lambda(x) = e^{i\lambda x}$ to get

$$\langle \vartheta^n(t), \chi_\lambda\rangle+c\,|\lambda|^\alpha\int_0^t\langle \vartheta^n(s), \chi_\lambda\rangle\,ds = \langle \vartheta^n(0), \chi_\lambda\rangle+m^n(\chi_\lambda, t)+\int_0^t q^n(\chi_\lambda, s)\,ds,$$

since $(-\Delta)^{\alpha/2}\chi_\lambda = c\,|\lambda|^\alpha\chi_\lambda$, where $c = c(d, \alpha)$ is a positive constant. Observe that

$$d\big(\exp(c\,|\lambda|^\alpha t)\langle \vartheta^n(t), \chi_\lambda\rangle\big) = \exp(c\,|\lambda|^\alpha t)\big(dm^n(\chi_\lambda, t)+q^n(\chi_\lambda, t)\,dt\big),$$

so that

$$\langle \vartheta^n(t), \chi_\lambda\rangle \equiv I_1+I_2+I_3, \tag{4.6}$$

where $I_j = I_j^{n,\lambda}(t)$ are defined as follows:

$$I_1 = \exp(-c\,|\lambda|^\alpha t)\langle \vartheta^n(0), \chi_\lambda\rangle,\qquad I_2 = \int_0^{t+}\exp\big(-c\,|\lambda|^\alpha(t-s)\big)\,dm^n(\chi_\lambda, s),$$
$$I_3 = \int_0^t\exp\big(-c\,|\lambda|^\alpha(t-s)\big)\,q^n(\chi_\lambda, s)\,ds.$$

We will show a uniform estimate for the quantity

$$g^n(t) = \sup_{\lambda\in\mathbf{R}^d}(1+|\lambda|^a)^{-1}n^{1-1/\alpha}E\,[|\langle \vartheta^n(t), \chi_\lambda\rangle|],\qquad t\in[0, T], \tag{4.7}$$

remembering that the assumption on I_1, i.e. on the initial positions of $X^{i,n,\varepsilon}(0)$, was $\sup_n g^n(0) < \infty$ for some $a \geqslant 0$. In the next two lemmas we will handle the integrals I_2 and I_3 separately.

LEMMA 4.2.

$$\sup_n \sup_{\lambda \in \mathbf{R}^d} \sup_{t \in [0,T]} n^{1-1/\alpha} E[|I_2^{n,\lambda}(t)|] < \infty.$$

Proof. Since the L^1-norm is dominated by the weak L^α-norm, $\alpha > 1$, we can write

$$E|n^{1-1/\alpha} I_2| \leqslant n^{1-1/\alpha} \sup_{z>0} z\{P(|I_2| > z)\}^{1/\alpha}$$
$$\leqslant C\{E[\int_0^t \sum_{i=1}^n |\exp(c|\lambda|^\alpha (s-t)) F^i|^\alpha ds]\}^{1/\alpha},$$

where

$$F^i = n^{-1/\alpha} \sup_{s \leqslant t, S^i(s) \neq S^i(s-)} \left| \frac{\chi_\lambda(X^{i,n,\varepsilon}(s-)+S^i(s)-S^i(s-)) - \chi_\lambda(X^{i,n,\varepsilon}(s-))}{S^i(s)-S^i(s-)} \right|.$$

The second inequality above is a consequence of the finite-dimensional vector version

$$\sup_{z>0} z^\alpha P(\sup_{s \leqslant t} \int_0^s F(r)\cdot dS(r) > z) \leqslant c_\alpha E[\int_0^t |F(s)|^\alpha ds]$$

of Theorem 9.5.3 from [22]. Finally, since $|F^i|^\alpha \leqslant n^{-1}|\lambda|^\alpha$, Lemma 4.2 is proved. ■

In order to bound I_3 we begin with

LEMMA 4.3.

$$|q^n(\chi_\lambda, t)| \leqslant |\lambda| \Big(\int_{\mathbf{R}^d} (3|\langle \vartheta^n(t), \chi_{-\xi}\rangle| + |\langle \vartheta^n(t), \chi_{\lambda+\xi}\rangle|) |\hat{b}_\varepsilon(\xi)| d\xi + n^{-1}|b_\varepsilon(0)| \Big).$$

Proof. We shall estimate the three terms in the definition (4.5) of q^n. For the first summand we have

$$|\langle \vartheta^n(dx, t)\vartheta^n(dy, t), b_\varepsilon(x-y)\cdot \nabla\chi_\lambda(x)\rangle|$$
$$= \Big| \int_{\mathbf{R}^d} \hat{b}_\varepsilon(\xi)\cdot \langle \vartheta^n(dx, t)\vartheta^n(dy, t), e^{i\xi(x-y)} i\lambda e^{i\lambda x}\rangle d\xi \Big|$$
$$\leqslant |\lambda| \int_{\mathbf{R}^d} |\hat{b}_\varepsilon(\xi)| |\langle \vartheta^n(t), \chi_{-\xi}\rangle| |\langle \vartheta^n(t), \chi_{\lambda+\xi}\rangle| d\xi \leqslant 2|\lambda| \int_{\mathbf{R}^d} |\hat{b}_\varepsilon(\xi)| |\langle \vartheta^n(t), \chi_{-\xi}\rangle| d\xi,$$

because $|\langle \vartheta^n(t), \chi_{\lambda+\xi}\rangle| \leqslant 2$.

For the second term we obtain

$$|\langle u^\varepsilon(dx, t)\vartheta^n(dy, t), b_\varepsilon(x-y)\cdot \nabla\chi_\lambda(x)|$$
$$= \Big| \int_{\mathbf{R}^d} \hat{b}_\varepsilon(\xi)\cdot \langle u^\varepsilon(dx, t)\vartheta^n(dy, t), e^{i\xi(x-y)}\cdot i\lambda e^{i\lambda x}\rangle d\xi \Big|$$
$$\leqslant |\lambda| \int_{\mathbf{R}^d} |\hat{b}_\varepsilon(\xi)| |\langle \vartheta^n(t), \chi_{-\xi}\rangle| |\langle u^\varepsilon(t), \chi_{\lambda+\xi}\rangle| d\xi \leqslant |\lambda| \int_{\mathbf{R}^d} |\hat{b}_\varepsilon(\xi)| |\langle \vartheta^n(t), \chi_{-\xi}\rangle| d\xi,$$

and, similarly,

$$|\langle \vartheta^n(dx, t) u^\varepsilon(dy, t), b_\varepsilon(x-y)\cdot \nabla\chi_\lambda(x)\rangle| \leqslant |\lambda| \int_{\mathbf{R}^d} |\hat{b}_\varepsilon(\xi)| |\langle \vartheta^n(t), \chi_{\lambda+\xi}\rangle| d\xi.$$

The third term is easy; remember that $\nabla\chi_\lambda = i\lambda\chi_\lambda$. ■

Thus, we arrive at

$$(4.8)\qquad E|I_3| \leqslant \int_0^t |\lambda| \exp\left(c|\lambda|^\alpha (s-t)\right) \int_{R^d} E\left[|\langle \vartheta^n(s), \chi_\xi\rangle|\right] \left(3|\hat{b}_\varepsilon(\xi)| + |\hat{b}_\varepsilon(\xi-\lambda)|\right) d\xi\, ds$$

$$+ \int_0^t |\lambda| \exp\left(c|\lambda|^\alpha (s-t)\right) n^{-1} |b_\varepsilon(0)|\, ds.$$

Now we are ready to derive an integral inequality for $g^n(t)$ defined in (4.7). Combining Lemmas 4.2, 4.3, and the assumption on the initial data we get

$$(4.9)\qquad g^n(t) \leqslant g^n(0) + C + C\int_0^t (t-s)^{-1/\alpha} g^n(s)\, ds + C,$$

where the constants C may depend on α, ε, T. Indeed,

$$|\lambda| \exp\left(-c|\lambda|^\alpha t\right) \leqslant Ct^{-1/\alpha}$$

holds for all $\lambda \in R^d$ and $t \in (0, T]$. Moreover, the integral

$$J = \int_{R^d} (1+|\lambda|^a)^{-1} (1+|\xi|^a) \left(3|\hat{b}_\varepsilon(\xi)| + |\hat{b}_\varepsilon(\xi-\lambda)|\right) d\xi$$

is uniformly bounded in $\lambda \in R^d$ for each fixed $a \geqslant 0$. To see this, note that $\hat{b}_\varepsilon = \hat{\delta}_\varepsilon \hat{b}$ has an exponential decay at $|\xi| = \infty$ and (by (3.11)) an integrable singularity at $\xi = 0$. Its integrand is majorized, e.g., by $C_\varepsilon(|\hat{b}_{\varepsilon/2}(\xi)| + |\hat{b}_{\varepsilon/2}(\xi-\lambda)|)$, $\hat{\delta}_{\varepsilon/2}(\xi) = \exp(-\varepsilon|\xi|^2/4)$ compensates polynomial factors in J, so J is uniformly bounded; cf. (4.12) in [14]. Using the Gronwall lemma, from the integral inequality (4.9) we obtain

$$n^{1-1/\alpha} E\left[|\langle \vartheta^n(t), \chi_\lambda\rangle|\right] \leqslant C_\varepsilon (1+|\lambda|^a).$$

Reconstructing ϕ, $\phi(x) = c\int \hat{\phi}(\lambda)\chi_\lambda(x)\, d\lambda$, we see that $\langle \vartheta, \phi\rangle = \int \langle \vartheta, \chi_\lambda\rangle \times \hat{\phi}(\lambda)\, d\lambda$, so finally we obtain

$$n^{1-1/\alpha} E\left[|\langle \vartheta^n(t), \phi\rangle|\right] \leqslant n^{1-1/\alpha} \int_{R^d} E\left[|\langle \vartheta^n(t), \chi_\lambda\rangle|\right] |\hat{\phi}(\lambda)|\, d\lambda$$

$$\leqslant C \int_{R^d} (1+|\lambda|^a) |\hat{\phi}(\lambda)|\, d\lambda,$$

which completes the proof of the convergence rate (3.15). Note that the regularization b_ε of b was useful only in obtaining uniform bounds for J. ∎

5. Approximations for nonregularized equations. In this section we will prove the "propagation of chaos in a wide sense" for the equation (1.1), by which we mean that given any sequence of regularizations (3.14) with $\varepsilon \to 0$, the family of empirical distributions $\{Y^{n,\varepsilon}(t)\}$ contains a subsequence weakly convergent to a solution $u(t)$ of (1.1).

THEOREM 5.1. *Let the general conditions of Theorem* 3.1 *be satisfied. Assume that* $u^\varepsilon(t)$ *are solutions of the regularized equation* (3.14) *such that their initial conditions satisfy* $|u^\varepsilon(0) - u(0)|_2 \to 0$ *as* $\varepsilon \to 0$ *for some* $u(0) \in L^2(R^d)$. *Then given any sequence* $\varepsilon_k \to 0$ *as* $k \to \infty$, *there exists a sequence* $n_k \to \infty$ *and a weak* *solution* $u(t)$ *of* (1.1) *such that for each* $\phi \in C_0^\infty(\boldsymbol{R}^d)$ *and all* $t \in (0, T)$

$$E\,|\langle Y^{n_k,\varepsilon_k}(t) - u(t),\ \phi\rangle| \to 0. \tag{5.1}$$

Moreover, under the assumptions of Theorem 3.1 (iii), *the convergence* (5.1) *can be strengthened to the global in time convergence for all* $t \in (0, \infty)$.

Proof. The proof requires only the following purely analytic weak convergence result to be demonstrated below in Propositions 5.1 and 5.2:

$$|\langle u^{\varepsilon_k}(t) - u(t),\ \phi\rangle| \to 0 \tag{5.2}$$

as $\varepsilon_k \to 0$ for each ϕ in a suitable function class containing $C_0^\infty(\boldsymbol{R}^d)$. Indeed, (3.15) combined with (5.2) shows that

$$E\,[|\langle Y^{n_k,\varepsilon_k}(t) - u(t),\ \phi\rangle|] \to 0$$

for some sequence $\varepsilon_k \to 0$ and suitably large $n_k \to \infty$ as $k \to \infty$. ∎

Remark 5.1. Under fairly general assumptions of Theorem 3.1 (i)–(ii), when only local in time solutions exist (and it may actually happen that they blow up in a finite time), (5.2) is a rather weak result. When stronger assumptions in Theorem 3.1 (iii) guarantee the global in time existence of solutions, convergence of solutions of regularized equations (3.14) to those of the original one (1.1) will be, of course, stronger. To obtain those convergence properties (the proofs of Propositions 5.1 and 5.2 below state them), we will show compactness of the family of approximating solutions using either the Aubin–Lions or the Ascoli–Arzelà criteria for vector-valued functions.

Remark 5.2. Note that so far the issue of uniqueness of solutions to (1.1) was not addressed in this paper. For $\alpha = 2$ the uniqueness of weak solutions holds true, see [8] and Remark 5.4 below. For $1 < \alpha < 2$, we can only prove the uniqueness of more regular solutions in $L^\infty((0, T); H^1(\boldsymbol{R}^d))$. However, we do not develop this issue here because, although the convergence in (5.2) would then be improved to all $\varepsilon \to 0$, in (5.1) we still would need to select a subsequence $n_k \to \infty$. We suspect that the solutions to (1.1) with sifficiently regular initial data are unique, but they are not necessarily unique in general. In such a case our interacting jump Markov processes approximation selects a solution of (1.1) similarly to the way the viscosity method selects a unique, so-called *viscosity*, solution of conservation laws (cf. [30]).

Remark 5.3. Unlike the case of the one-dimensional fractal Burgers equation in [14], the estimates of u^ε leading to (5.2) (gaining extra information from the degree of approximation of $\delta_\varepsilon * u^\varepsilon - u^\varepsilon$) will be similar to those of u in the existence Theorems 2.1 and 2.2. It seems that in the higher dimensional case, $d \geqslant 2$, we cannot obtain results in the same spirit for the fractal Burgers equation (1.5) with $r > 1$, because the diffusion operator $(-\Delta)^{\alpha/2}$, $\alpha < 2$, is too weak compared to the nonlinear term; see also the remark after the formulation of Theorem 2.1.

PROPOSITION 5.1. *Assume all the hypotheses of Theorem* 2.1 *as well as the hypothesis* $|u^\varepsilon(0)-u_0|_2 \to 0$ *as* $\varepsilon \to 0$ *are satisfied. For the family* $\{u^\varepsilon\}_{\varepsilon>0}$ *of solutions of regularized equations* (3.14) *approximating* (1.1) *and any sequence* $\varepsilon_k \to 0$, *there exist a subsequence (still denoted by* $\varepsilon_k \to 0$) *and a function* $u \in L^2((0, T); H^{\alpha/2}(\mathbf{R}^d))$ *solving* (1.1) *such that* $|\langle u^{\varepsilon_k}(t)-u(t), \phi\rangle| \to 0$ *for each* $t \in (0, T)$ *and* $\phi \in C_0^\infty(\mathbf{R}^d)$.

Proof. Recall the energy relation (2.4), now for $u = u^\varepsilon$, and observe that we obtain the inequality (2.6) with a constant C good for all the approximating equations (3.14). Indeed,

$$B_\varepsilon(u) = \delta_\varepsilon * b * u = b * (\delta_\varepsilon * u), \qquad |\delta_\varepsilon|_1 \equiv 1.$$

In such a manner (2.7) implies a local bound

$$|u(t)|_2^2 + \int_0^t |D^{\alpha/2} u(s)|_2^2 \, ds \leqslant C(T) \tag{5.3}$$

(since $|u^\varepsilon(0)|_2$ is bounded in ε) for all $t \in [0, T]$, $u = u^\varepsilon$ with some T *independent* of ε. The inequality (5.3) means that the set $\{u^\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty((0, T); L^2(\mathbf{R}^d)) \cap L^2((0, T); H^{\alpha/2}(\mathbf{R}^d))$.

To get some regularity of $u = u^\varepsilon$ with respect to t, take any test function $\psi \in H^1(\mathbf{R}^d)$, $\|\psi\|_1 \leqslant 1$, and estimate $\langle u_t, \psi\rangle = \langle D^{\alpha/2} u, D^{\alpha/2}\psi\rangle - \langle uB_\varepsilon(u), \nabla\psi\rangle$. We have

$$|\langle u_t, \psi\rangle| \leqslant |D^{\alpha/2} u|_2 + |u|_4 |B_\varepsilon(u)|_4 \leqslant |D^{\alpha/2} u|_2 + |u|_4 \|B_\varepsilon(u)\|_1 \leqslant 2\|u\|_{\alpha/2} + C|u|_2^M,$$

since $L^4(\mathbf{R}^d) \subset H^1(\mathbf{R}^d)$ for $d \leqslant 4$, and $\|B_\varepsilon(u)\|_1$ is estimated as in (2.5). This, together with (5.3), gives $\{u_t^\varepsilon\}_{\varepsilon>0}$ bounded in $L^2((0, T); H^{-1}(\mathbf{R}^d))$. It is almost all we need to apply the Aubin–Lions lemma (see [24], Ch.I, Sec. 5.2), except for the lack of compact embedding of Sobolev spaces over the whole space $\mathbf{R}^d$. A standard remedy in this issue is to consider like in [14] weighted Sobolev spaces

$$H_\lambda^\varrho \equiv \{v\colon ve^{-\lambda\theta} \in H^\varrho\} \supset C_0^\infty, \tag{5.4}$$

where $0 \leqslant \theta$ is a C^∞-function such that $\theta(x) = |x|$ for $|x| \geqslant 1$, $\varrho, \lambda \in \mathbf{R}$, as introduced in, e.g., [13] (where they have been used, however, for quite different purposes). Now, if $\sigma < \varrho$ and $\mu > \lambda$, H_λ^ϱ is compactly embedded in H_μ^σ. This suffices to conclude that each sequence $\{u_k^\varepsilon\}$ contains a subsequence converging to a function $u \in L^2((0, T); H_\lambda^0(\mathbf{R}^d))$ for each $\lambda > 0$, which is a solution of (1.1) with u_0 as the initial data. ■

PROPOSITION 5.2. *Assume all the hypotheses of Theorem* 2.2 *as well as the hypothesis* $|u^{\varepsilon_k}(0)-u(0)|_2 \to 0$ *as* $\varepsilon \to 0$ *are satisfied. For the family* $\{u^\varepsilon\}_{\varepsilon>0}$ *of solutions to regularized equations* (3.14) *approximating* (1.1) *and any* $\varepsilon_k \to 0$, *there exist a subsequence (still denoted by* $\varepsilon_k \to 0$) *and a function* $u \in L^2((0, T); H^{\alpha/2}(\mathbf{R}^d))$ *solving* (1.1) *such that* $|\langle u^{\varepsilon_k}(t)-u(t), \phi\rangle| \to 0$ *for each* $t \in (0, \infty)$ *and all* $\phi \in C_0^\infty(\mathbf{R}^d)$.

Proof. Using essentially the same argument as in the proof of Theorem 2.2, modified as in the demonstration of Proposition 5.1 to adopt it to the approximating solutions $u = u^\varepsilon$, we have a locally uniform bound on $|u(t)|_2$, $t \in [0, T]$ for each $T < \infty$, i.e.

$$u \in L^\infty((0, T); L^2(\boldsymbol{R}^d)) \cap L^2((0, T); H^{\alpha/2}(\boldsymbol{R}^d)).$$

Next we derive a stronger estimate of $\|u(t)\|_{\alpha/2}$. Multiply (3.14) with $u = u^\varepsilon$ by $2(-\Delta)^{\alpha/2} u$ to obtain

$$\frac{d}{dt} |D^{\alpha/2} u|_2^2 + 2|D^\alpha u|_2^2 = 2 \int_{\boldsymbol{R}^d} \nabla \cdot (u B_\varepsilon(u))(-\Delta)^{\alpha/2} u \, dx. \tag{5.5}$$

The right-hand side of (5.5) can be bounded from above by

$$|D^\alpha u|_2^2 + \|u B_\varepsilon(u)\|_1^2 = |D^\alpha u|_2^2 + |u B_\varepsilon(u)|_2^2 + |\nabla u \cdot B_\varepsilon(u)|_2^2 + |u \nabla \cdot B_\varepsilon(u)|_2^2.$$

It suffices to consider the last two terms since the first one is absorbed in (5.5), and the second one is easier to manipulate with than others.

Thus we begin with

$$\begin{aligned} |\nabla u \cdot B_\varepsilon(u)|_2^2 &\leqslant |\nabla u|_{2p}^2 |B_\varepsilon(u)|_{2q}^2 \leqslant C |\nabla u|_{2p}^2 |u|_r^2 \\ &\leqslant C \|u\|_\alpha^{2(1+d-d/(2p)+d-d/(2q)-\beta)/(\alpha+d/2)} |u|_1^m \leqslant \tfrac{1}{2}\|u\|_\alpha^2 + C|u|_1^M, \end{aligned}$$

where $1/p + 1/q = 1$, $1/(2q) = 1/r - \beta/d$, m, $M > 0$, are suitably chosen. The last inequality follows from $(1 + 3d/2 - \beta)/(\alpha + d/2) < 1$, since the assumption in Theorem 2.2 is just $\alpha + \beta > d + 1$.

The last term is bounded by

$$\begin{aligned} |u \nabla \cdot B_\varepsilon(u)|_2^2 &\leqslant |u|_{2p}^2 |\nabla \cdot B_\varepsilon(u)|_{2q}^2 \leqslant C |u|_{2p}^2 |u|_s^2 \\ &\leqslant C \|u\|_\alpha^{2(d-d/(2r)+d-d/(2s)-\beta+1)/(\alpha+d/2)} |u|_1^m \leqslant \tfrac{1}{2}\|u\|_\alpha^2 + C|u|_1^M, \end{aligned}$$

where again $1/p + 1/q = 1$, $1/(2q) = 1/s - (\beta - 1)/d$, m, $M > 0$, but they are not necessarily the same as previously. The last inequality is a consequence of $(3d/2 - \beta + 1)/(\alpha + d/2) < 1$, which is exactly the assumption $\alpha + \beta > d + 1$.

These computations lead to

$$\frac{d}{dt} \|u\|_{\alpha/2}^2 + \|u\|_\alpha^2 \leqslant C|u|_2^2 + C|u|_1^M.$$

After the integration we get

$$u \in L^\infty((0, T); H^{\alpha/2}(\boldsymbol{R}^d)) \cap L^2((0, T); H^\alpha(\boldsymbol{R}^d)) \quad \text{for each } T < \infty.$$

Now, our goal is deriving an estimate for the time derivative of u^ε. Differentiating (3.14) with respect to t and multiplying by u_t we get

$$\begin{aligned} \frac{d}{dt} |u_t|_2^2 + 2|D^{\alpha/2} u_t|_2^2 &= -2 \int_{\boldsymbol{R}^d} u_t B_\varepsilon(u) \cdot \nabla u_t \, dx - 2 \int_{\boldsymbol{R}^d} u_t u B_\varepsilon(u_t) \, dx \\ &= \int_{\boldsymbol{R}^d} u_t^2 \nabla \cdot B_\varepsilon(u) \, dx - 2 \int_{\boldsymbol{R}^d} u_t u B_\varepsilon(u_t) \, dx \\ &\leqslant |u_t|_4^2 \|B_\varepsilon(u)\|_1 + |u_t|_4 |u|_p |B_\varepsilon(u_t)|_q \end{aligned} \tag{5.6}$$

with p, q satisfying $1/p+1/q = 3/4$. Then the right-hand side of (5.6) is less than

$$C\|u_t\|_{\alpha/2}^{d/\alpha}|u_t|_2^{2-d/\alpha}\|u\|_{\alpha/2}^{(2+d-2\beta)/(\alpha+d)}|u|_1^m + C\|u_t\|_{\alpha/2}^{d/(2\alpha)}|u_t|_2^{2-d/(2\alpha)}\|u\|_{\alpha/2}^{(d/2+2d/q)/(\alpha+d)}|u|_1^M \leqslant \|u_t\|_{\alpha/2}^2 + C|u_t|_2^2$$

for some $m, M > 0$. This gives us

$$\frac{d}{dt}|u_t|_2^2 + \|u_t\|_{\alpha/2}^2 \leqslant C|u_t|_2^2 + C,$$

and after the integration we obtain

$$u_t \in L^\infty((0, T); L^2(\mathbf{R}^d)) \cap L^2((0, T); H^{\alpha/2}(\mathbf{R}^d)),$$

whenever $u_0 \in H^\alpha(\mathbf{R}^d)$ as $u_t(0) = -(-\Delta)^{\alpha/2}u_0 + \nabla\cdot(u_0 B_\varepsilon(u_0))$.

Having established bounds on u^ε in Proposition 5.2 we can arrive at the conclusion of the proof of the second part of Theorem 5.1. $\{u^\varepsilon(t)\}_{\varepsilon>0}$ is equicontinuous as a set of $H^{\alpha/2}(\mathbf{R}^d)$-valued functions on $[0, T]$, since

$$\|u(t_2)-u(t_1)\|_{\alpha/2} \leqslant \int_{t_1}^{t_2}\|u_t(s)\|_{\alpha/2}\,ds \leqslant (t_2-t_1)^{1/2}\Big(\int_0^T\|u_t(s)\|_{\alpha/2}^2\,ds\Big)^{1/2}$$

for all $0 \leqslant t_1 < t_2 \leqslant T$, $0 < T < \infty$. The $H^{\alpha/2}$-bound

$$\sup_{\varepsilon>0}\sup_{t\in[0,T]}\|u(t)\|_{\alpha/2} < \infty$$

and the compact embedding

$$H^{\alpha/2}(\mathbf{R}^d) \subset H^\varrho_\lambda(\mathbf{R}^d) \quad \text{for any } \varrho < \alpha/2,\ \lambda > 0$$

(cf. (5.4)) show that $\{u^\varepsilon\}_{\varepsilon>0}$ is relatively compact in $C([0, T]; H^\varrho_\lambda(\mathbf{R}^d))$. The limit of each convergent sequence $\{u^{\varepsilon_k}\}$ satisfies the original equation (1.1). Therefore $\langle u^{\varepsilon_k}(t)-u(t), \psi\rangle \to 0$ as $\varepsilon_k \to 0$ for each $\psi \in H^{-\varrho}_{-\lambda} \supset C_0^\infty$ and arbitrary $\lambda > 0$, which completes the proof. ∎

Remark 5.4. The case $\alpha = 2$ is substantially different (and easier to treat) than that of $\alpha < 2$. Namely, the global in time solutions to (1.1) are expected (by, e.g., [8]) to satisfy a Gaussian bound in the space variable. Then the use of $H^\varrho_{-\mu}$-estimates, $\varrho < 1$, $\mu > 0$ (like in [13]), would lead to a much stronger approximation result in Theorem 5.1, in particular, for all ϕ satisfying the condition

$$\int(1+|\lambda|^a)|\hat{\phi}(\lambda)|\,d\lambda < \infty.$$

Indeed, $(H^\varrho_{-\mu})^* = H^{-\varrho}_\mu$ contains those functions ϕ. However, we cannot expect such an exponential decay of solutions to (1.1) if $\alpha < 2$. Even for linear equations, in particular for the Lévy semigroup, the best one can obtain is an algebraic decay rate $|x|^{-d-\alpha}$, see [19] and Lemma 5.3 in [8]. This is a heuristic explanation of seemingly very weak convergence properties obtained in Theorem 5.1.

Moreover, by the uniqueness of solutions to (1.1) for $\alpha = 2$ (see [8], Theorem 2.1 (ii) and its proof as in [7], Theorem 1), the whole sequence $\{u^{\varepsilon_k}\}$ converges to the solution u for *any* sequence $\varepsilon_k \to 0$.

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