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EXIT TIME AND GREEN FUNCTION OF CONE FOR SYMMETRIC STABLE PROCESSES

BY

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Abstract. We obtain estimates of the harmonic measure and the expectation of the exit time of a bounded cone for symmetric α-stable processes X_t in $\boldsymbol{R}^d$ ($\alpha \in (0, 2)$, $d \geqslant 3$). This enables us to study the asymptotic behaviour of the corresponding Green function of both bounded and unbounded cones. We also apply our estimates to the problem concerning the exit time τ_V of the process X_t from the unbounded cone V of angle $\lambda \in (0, \pi/2)$. We namely obtain upper and lower bounds for the constant $p_0 = p_0(d, \alpha, \lambda)$ such that for all $x \in V$ we have $E^x(\tau_V^p) < \infty$ for $0 \leqslant p < p_0$ and $E^x(\tau_V^p) = \infty$ for $p > p_0$.

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1. Introduction. In recent years several results concerning potential theory of symmetric α-stable processes have been established (see e.g. [7], [8], [11]). They extend classical potential theory of Brownian motion. Among the new results there are sharp estimates of the harmonic measure and Green function of symmetric α-stable processes for open bounded sets with $C^{1,1}$ boundary (see [15], [10]). These estimates show that the asymptotic behaviour of Green function and the harmonic measure of these smooth sets is the same as for a ball.

On more general Lipschitz domains the situation is more complicated. In [7] the boundary Harnack principle as well as some absolute estimates of α-harmonic functions in bounded Lipschitz domains have been obtained. The absolute estimates of [7] (Lemmas 3 and 5) do not give a full description

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of asymptotics of a harmonic measure and may be further elaborated for particular Lipschitz domains. We find it instructive to investigate the case of the cone in more detail. The first aim of this paper is to study asymptotics of Green function and harmonic measure of symmetric α-stable processes X_t in $\mathbf{R}^d$ for the cone, especially near its vertex ($\alpha \in (0, 2)$, $d \geqslant 3$). One of the main questions is how this asymptotics depends on the opening of the cone. We mainly focus on the case when the opening is arbitrarily narrow. This complements [7] where the emphasis is put on comparing α-harmonic functions when the domain is fixed.

We first investigate the harmonic measure and the expectation of the exit time for bounded cones and then we obtain estimates of Green function of the unbounded and bounded cones. We recall that similar problems have been investigated for the Brownian motion (see e.g. [1]). We have to point out that, in contrast with the case of open bounded sets with $C^{1,1}$ boundary, our results are not sharp. For further discussion we refer to the end of Section 3.

Another problem we investigate concerns the exit time $\tau_V = \inf\{t > 0: X_t \notin V\}$ of the process from the unbounded cone V of angle λ. We obtain some lower and upper bounds for the critical value $p_0 = p_0(d, \alpha, \lambda)$ such that for all $x \in V$ we have $E^x(\tau_V^p) < \infty$ for $0 \leqslant p < p_0$ and $E^x(\tau_V^p) = \infty$ for $p > p_0$. The problem of finding the constant p_0 has been extensively studied for the Brownian motion. In that case Burkholder [9] proved that for $p \geqslant 0$ and $x \in V$ we have $E^x(\tau_V^p) < \infty$ if and only if $p < p_0$, where $p_0 = p_0(d, \lambda)$ is defined in terms of a certain hypergeometric function. Earlier, Spitzer [17] showed that $p_0 = \pi/(4\lambda)$ for dimension $d = 2$. This problem has also been studied for generalized cones and conditioned Brownian motion (see [2], [12]). For the two-dimensional symmetric α-stable process DeBlassie [13] expressed p_0 in terms of some rather complicated differential operator and obtained some estimates of p_0. However, the estimates do not seem to provide information about the behaviour of p_0 when λ tends to 0. In our paper we treat the case $d \geqslant 3$ and use completely different methods than those used in [13]. While we do not give an exact expression for p_0, we are able to describe its asymptotics in $\lambda \to 0$.

2. Preliminaries. For $x \in \mathbf{R}^d$, $r > 0$ we put $B(x, r) = \{y \in \mathbf{R}^d: |y-x| < r\}$ and $S(x, r) = \{y \in \mathbf{R}^d: |y-x| = r\}$. The surface area of the $(d-1)$-dimensional sphere $S(0, 1) \subset \mathbf{R}^d$ will be denoted by $\omega_d = 2\pi^{d/2}/\Gamma(d/2)$. For any subset $A \subset \mathbf{R}^d$, we denote its complement by A^c, its closure by $\bar{A}$, its interior by $\operatorname{int}(A)$ and its boundary by ∂A. For $t > 0$ and $A \subset \mathbf{R}^d$ we write $tA = \{tx: x \in A\}$ and $A/t = t^{-1}A$. Furthermore, we write $\operatorname{dist}(A, B) = \inf\{|x-y|: x \in A, y \in B\}$, $\operatorname{diam}(A) = \sup\{|x-y|: x, y \in A\}$ and $\delta_A(x) = \operatorname{dist}(x, \partial A)$ for $A, B \subset \mathbf{R}^d$ and $x \in \mathbf{R}^d$. We write $m(A)$ for the d-dimensional Lebesgue measure of the set $A \subset \mathbf{R}^d$. $\mathscr{B}(\mathbf{R}^d)$ denotes the Borel σ-field of $\mathbf{R}^d$.

The notation $c = c(x, y, z)$ means that c is a constant depending only on x, y, z. Constants are always numbers in $(0, \infty)$.

For the rest of the paper let $\alpha \in (0, 2)$ and $d \geqslant 3$. By (X_t, P^x) we denote the standard [5] rotation invariant ("symmetric") α-stable, $\boldsymbol{R}^d$-valued Lévy process (i.e. homogeneous, with independent increments), with index of stability α and the characteristic function of the form

$$E^0 \exp\{i\xi X_t\} = \exp\{-t|\xi|^\alpha\}, \quad \xi \in \boldsymbol{R}^d, \, t \geqslant 0.$$

As usual, E^x denotes the expectation with respect to the distribution P^x of the process starting from $x \in \boldsymbol{R}^d$. We always assume that sample paths of X_t are right-continuous and have left-hand limits almost surely. (X_t, P^x) is a Markov process with transition probabilities given by $P_t(x, A) = P^x(X_t \in A)$ and is strong Markov with respect to the so-called "standard filtration" and quasi--left-continuous on $[0, \infty)$ (see [5]). The transition density of X_t will be denoted by $p(t, x, y)$. For the sake of brevity we will refer to this process as to "symmetric α-stable". The Lévy measure ν of this process is of the form

$$\nu(dx) = C_{d,\alpha}|x|^{-\alpha-d}\,dx, \quad \text{where } C_{d,\alpha} = \alpha 2^{\alpha-1}\Gamma((d+\alpha)/2)\pi^{-d/2}\Gamma^{-1}(1-(\alpha/2)).$$

For $A \in \mathscr{B}(\boldsymbol{R}^d)$, we define $\tau_A = \inf\{t > 0\colon X_t \in A^c\}$, the first exit time from A. It is well known that τ_{tA} and $t^\alpha \tau_A$ (for $t > 0$) have the same law under P^0.

Let $f \geqslant 0$ be a Borel measurable function on $\boldsymbol{R}^d$. We say that f is α-*harmonic* in an open set $D \subset \boldsymbol{R}^d$ if

$$f(x) = E^x f(X(\tau_A)), \quad x \in A,$$

for every bounded open set A with the closure $\bar{A}$ contained in D.

We define the harmonic measure ω_D^x (for D, in x, with respect to X) by the formula $\omega_D^x(A) = P^x(X(\tau_D) \in A)$, where $x \in \boldsymbol{R}^d$; $A, D \in \mathscr{B}(\boldsymbol{R}^d)$. When D is unbounded, by the usual convention, $P^x(X(\tau_D) \in A)$ is understood as $P^x(X(\tau_D) \in A;\, \tau_D < \infty)$. It is clear that $\operatorname{supp}(\omega_D^x) \subset \overline{D^c}$.

The *Riesz kernel* is defined by

$$u(x, y) = \int_0^\infty p(t, x, y)\,dt, \quad x, y \in \boldsymbol{R}^d.$$

According to [5] we have $u(x, y) = A_{d,\alpha}|x-y|^{\alpha-d}$, where

$$A_{d,\alpha} = 2^{-\alpha}\pi^{-d/2}\Gamma((d-\alpha)/2)(\Gamma(\alpha/2))^{-1}.$$

Let D be an open set. We define

$$G_D(x, y) = u(x, y) - \int_{D^c} u(z, y)\,d\omega_D^x(z), \quad x, y \in D,$$

and call $G_D(x, y)$ the *Green function* for D. Additionally, we set $G_D(x, y) = 0$ if $x \in D^c$ or $y \in D^c$. It is well known that $G_D(x, y) \geqslant 0$ and $G_D(x, y) = G_D(y, x)$ for

$x, y \in \mathbf{R}^d$. If $A \subset D$ is an open bounded set, then we have

$$G_D(x, y) = \int_{A^c} G_D(u, y)\, d\omega_A^x(u) \quad \text{for } x \in A,\ y \notin \bar{A}. \tag{2.1}$$

In particular, $G_D(\cdot, y)$ is α-harmonic in $D \backslash \{y\}$. It satisfies also the following scaling property:

$$G_{tD}(tx, ty) = t^{\alpha-d} G_D(x, y), \quad t > 0,\ x, y \in D.$$

$G_D(\cdot, \cdot)$ is continuous in the extended sense as a mapping from $D \times D$ into $[0, \infty]$. If $D_1 \subset D_2$ are open sets, then we have $G_{D_1}(x, y) \leqslant G_{D_2}(x, y)$ for every $x, y \in \mathbf{R}^d$.

By P_t^D we denote the semigroup generated by the process (X_t) killed on exiting D. For $t > 0$ and $x, y \in D$ let $p_D(t, x, y)$ be the transition density for the process, i.e.

$$P_t^D f(x) = E^x\big(f(X_t);\ t < \tau_D\big) = \int_D f(y)\, p_D(t, x, y)\, dy, \quad x \in D,\ t > 0,$$

for any nonnegative Borel f. We have $p_D(t, x, y) = p_D(t, y, x)$ and $p_D(t, x, y) \leqslant p(t, x, y)$ for all $t > 0$ and all $x, y \in D$. We also have

$$G_D(x, y) = \int_0^\infty p_D(t, x, y)\, dt, \quad x, y \in D.$$

We denote the Green operator for D by G_D. We have

$$G_D f(x) = E^x\Big(\int_0^{\tau_D} f(X_t)\, dt\Big) = \int_D G_D(x, y) f(y)\, dy, \quad x \in D,$$

for nonnegative Borel functions f.

The potential theory of symmetric α-stable processes is enriched by the fact that the density of the harmonic measure for a ball is given by an explicit formula. Let $x \in B(0, r)$. The harmonic measure $\omega_{B(0,r)}^x$ for $B(0, r)$ has the density function $P_r(x, \cdot)$ (with respect to the Lebesgue measure) given by the formula

$$P_r(x, y) = \begin{cases} c_\alpha^d \left(\dfrac{r^2 - |x|^2}{|y|^2 - r^2} \right)^{\alpha/2} |x - y|^{-d} & \text{for } |y| > r, \\ 0 & \text{for } |y| \leqslant r, \end{cases}$$

where $c_\alpha^d = \Gamma(d/2)\pi^{-d/2-1} \sin(\pi\alpha/2)$ (see e.g. [16] or [6]).

More generally, it is proved in [7] that if D is a bounded open set with the outer cone property and $x \in D$, then the harmonic measure ω_D^x is concentrated on $\operatorname{int}(D^c)$ and is absolutely continuous with respect to the Lebesgue measure on D^c. The corresponding density function will be denoted by $f_D^x(z)$, $x \in D$, $z \in \operatorname{int}(D^c)$. According to [7], $f_D^x(z)$ is C^∞ in $(x, z) \in D \times \operatorname{int}(D^c)$; see also (2.2) below. Similarly to the Green function, also $f_D^x(z)$ satisfies a scaling property. We have

$$f_{tD}^x(z) = t^{-d} f_D^{x/t}(z/t), \qquad t > 0,\ x \in tD,\ z \in \operatorname{int}(tD^c).$$

Assume now that $D \subset \mathbf{R}^d$ is an open nonempty bounded set, $E \in \mathscr{B}(\mathbf{R}^d)$ and $\operatorname{dist}(E, D) > 0$. The following formula [14], exhibiting the relation between Green function, Lévy measure and harmonic measure, will be useful in our further considerations:

$$P^x(X(\tau_D) \in E) = \int_D G_D(x, y) \int_E \frac{C_{d,\alpha}}{|y-z|^{d+\alpha}} dz\, dy, \qquad x \in D. \tag{2.2}$$

We note that if $\omega_D^x(\partial D) = 0$ for $x \in D$, then we can replace the assumption $\operatorname{dist}(E, D) > 0$ by $E \subset D^c$. In particular, this applies to open bounded sets D with the outer cone property.

We now briefly recall known estimates of Green function and harmonic measure for bounded open sets with a $C^{1,1}$ boundary. At first, let us recall the definition of these sets (cf. [18]).

A function F: $\mathbf{R}^d \to \mathbf{R}$ is of *class* $C^{1,1}$ if its derivative F' satisfies $|F'(x) - F'(y)| \leqslant \lambda |x-y|$, $x, y \in \mathbf{R}^d$, with a constant λ. We say that a bounded open set $D \subset \mathbf{R}^d$ has a $C^{1,1}$ *boundary* if for each $x \in \partial D$ there are: a $C^{1,1}$ function F_x: $\mathbf{R}^{d-1} \to \mathbf{R}$ (with $\lambda = \lambda(D)$), an orthonormal coordinate system CS_x and a constant $\eta = \eta(D)$ such that if $y = (y_1, \ldots, y_n)$ in CS_x coordinates, then

$$D \cap B(x, \eta) = \{y:\ y_n > F_x(y_1, \ldots, y_{n-1})\} \cap B(x, \eta).$$

It is clear that a $C^{1,1}$ set D satisfies the outer cone property, and thus its harmonic measure has the density function on $\operatorname{int}(D^c)$.

It is proved in [15] that there exist constants $c_1 = c_1(D, \alpha)$ and $c_2 = c_2(D, \alpha)$ such that for any $x, y \in D$

$$c_1 \min\left(\frac{1}{|x-y|^{d-\alpha}}, \frac{\delta_D^{\alpha/2}(x)\,\delta_D^{\alpha/2}(y)}{|x-y|^d}\right) \leqslant \frac{G_D(x, y)}{A_{d,\alpha}} \leqslant \min\left(\frac{1}{|x-y|^{d-\alpha}}, c_2 \frac{\delta_D^{\alpha/2}(x)\,\delta_D^{\alpha/2}(y)}{|x-y|^d}\right). \tag{2.3}$$

This result and formula (2.2) give the following estimates of the harmonic measure for D. There exist constants $c_1 = c_1(D, \alpha)$ and $c_2 = c_2(D, \alpha)$ such that for any $x \in D$ and $z \in \operatorname{int}(D^c)$

$$\frac{c_1\, \delta_D^{\alpha/2}(x)}{\delta_D^{\alpha/2}(z)\big(1 + \delta_D^{\alpha/2}(z)\big)|x-z|^d} \leqslant f_D^x(z) \leqslant \frac{c_2\, \delta_D^{\alpha/2}(x)}{\delta_D^{\alpha/2}(z)\big(1 + \delta_D^{\alpha/2}(z)\big)|x-z|^d}. \tag{2.4}$$

This inequality is proved in [10].

As an application of (2.3) one also obtains the inequality

$$c_1\,\delta^{\alpha/2}(x) \leqslant E^x(\tau_D) \leqslant c_2\,\delta^{\alpha/2}(x), \qquad x\in D, \tag{2.5}$$

where $c_1 = c_1(D, \alpha)$ and $c_2 = c_2(D, \alpha)$ (see [15]).

In this paper we give some analogues of the inequalities (2.3)–(2.5) for bounded cones.

The paper is organized as follows. In Section 3 we obtain estimates for the harmonic measure and the expectation of the exit time for bounded cones. This is the most technical part of the work. The results proved in the rest of the paper are consequences of the estimates from Section 3. The most important results in Section 3 are upper and lower bounds for the harmonic measure (Theorems 3.12 and 3.18) and corresponding upper and lower bounds for the expectation of the exit time (Theorems 3.13 and 3.17).

In Section 4 we get estimates of Green function of the unbounded and bounded cones (Theorems 4.6 and 4.7). Of independent interest may be Proposition 4.4 which may be treated as the local version of the upper bound estimate in the inequality (2.3). Section 5 concerns the exit time τ_V from the unbounded cone V of angle λ. In the section we obtain bounds for the critical value $p_0 = p_0(d, \alpha, \lambda)$ such that for all $x\in V$ we have $E^x(\tau_V^p) < \infty$ for $0 \leqslant p < p_0$ and $E^x(\tau_V^p) = \infty$ for $p > p_0$.

3. Harmonic measure. Let us introduce spherical coordinates $(\varrho, \varphi_1, \ldots, \varphi_{d-1})$ with origin 0, where $\varrho\in[0, \infty)$, $\varphi_1, \ldots, \varphi_{d-2}\in[0, \pi]$, and $\varphi_{d-1}\in[0, 2\pi)$. For the rest of the paper fix $\lambda\in(0, \pi/2)$ and $r > 0$. We define the unbounded cone

$$V = \{x = (\varrho, \varphi_1, \ldots, \varphi_{d-1}):\ \varrho > 0,\ \varphi_1\in[0, \lambda)\}$$

and the bounded cone

$$C = \{x = (\varrho, \varphi_1, \ldots, \varphi_{d-1}):\ \varrho\in(0, r),\ \varphi_1\in[0, \lambda)\}$$

and we call λ the *angle* of C and V. It is clear that C is an open bounded set with the outer cone property so, as pointed out in the Preliminaries, for $x\in C$ the harmonic measure ω_C^x is concentrated on $\operatorname{int}(C^c)$ and has there a density function $f_C^x(z)$ which is C^∞ in $(x, z)\in C\times\operatorname{int}(C^c)$.

Our main aim in this section is to obtain estimates of $f_C^x(z)$ (when $x\in C/2$, $z\in V\setminus\overline{C}$) and to see how they depend on the opening of the cone. As mentioned in the Introduction we are especially interested in the case when the opening of the cone is "narrow". Our methods elaborate those used in [7] (see Lemmas 3 and 5 therein). By introducing the measures $q_{i_1,\ldots,i_k}(x, B)$ and $p_k(x, B)$ below we are able to obtain upper bound estimates for $f_C^x(\cdot)$ which in case of narrow cones are more precise than those which can be deduced from [7].

For $n\in\mathbf{Z}$ let us put

$$C_n = 2^n C = \{x = (\varrho, \varphi_1, \ldots, \varphi_{d-1}):\ \varrho\in(0, 2^n r),\ \varphi_1\in[0, \lambda)\}$$

and

$$A_n = C_n \setminus C_{n-1} = \{x = (\varrho, \varphi_1, \ldots, \varphi_{d-1}): \varrho \in [2^{n-1} r, 2^n r), \varphi_1 \in [0, \lambda)\}.$$

B and B_n ($n \in \boldsymbol{Z}$) will denote sets belonging to $\mathscr{B}(\boldsymbol{R}^d)$.

For $m \in \boldsymbol{Z}$ and $x \in A_m$ let us define

$$p(x, B) = P^x\big(X(\tau_{C_{m+1}}) \in B\big).$$

We write $dp(x, y)$ for $p(x, dy)$. $p(x, B)$ is the probability of the event that the process starting from $x \in A_m$ jumps directly to B when leaving C_{m+1}. We can think of $p(x, B)$ as the probability of a "single jump". We shall mainly be interested in sets B, $B_n \subset V$. This is reflected in the following definition.

For $i \in \boldsymbol{Z}$ and $x \in V$ we set $q_i(x, B) = p(x, B \cap A_i)$ and by induction, for $k \in \boldsymbol{N}$, $i_1, \ldots, i_k \in \boldsymbol{Z}$ and $x \in V$, we set

$$q_{i_1, \ldots, i_k}(x, B) = \int_{A_{i_1}} q_{i_2, \ldots, i_k}(y, B)\, dp(x, y).$$

We may think of $q_{i_1, \ldots, i_k}(x, B)$ as the probability of the event that the process starting from $x \in A_m$ goes to $B \cap A_{i_k}$ after precisely k successive "jumps" to $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$.

For $k \in \boldsymbol{N}$, $m, n \in \boldsymbol{Z}$ we write

$$J_k(m, n) = \{(i_1, \ldots, i_k) \in \boldsymbol{Z}^k: i_1 \geqslant m+2,\ i_k = n,\ i_{j+1} - i_j \geqslant 2 \text{ for } j = 1, \ldots, k-1\}.$$

It is easy to notice that

$$J_{k+1}(m, n) = \{(i_1, \ldots, i_{k+1}) \in \boldsymbol{Z}^{k+1}: i_1 \geqslant m+2,\ (i_2, \ldots, i_{k+1}) \in J_k(i_1, n)\}.$$

LEMMA 3.1. *Let $k \in \boldsymbol{N}$, $m, n \in \boldsymbol{Z}$, $x \in A_m$, $B_n \subset A_n$ and $(i_1, \ldots, i_k) \notin J_k(m, n)$. Then we have*

$$q_{i_1, \ldots, i_k}(x, B_n) = 0.$$

Proof. The proof is by induction on k. For $k = 1$ we have $(i_1) \notin J_1(m, n)$ if and only if $i_1 \neq n$ or $i_1 < m+2$. When $i_1 \neq n$, we have $q_{i_1}(x, B_n) = p(x, B_n \cap A_{i_1}) = 0$. If $i_1 = n$ and $i_1 < m+2$, then $q_{i_1}(x, B_n) = P^x\big(X(\tau_{C_{m+1}}) \in B_n\big) = 0$ because $B_n \subset C_n \subset C_{m+1}$.

Assuming that lemma holds for k, we will prove it for $k+1$. Suppose that $(i_1, \ldots, i_{k+1}) \notin J_{k+1}(m, n)$. Then $i_1 < m+2$ or $(i_2, \ldots, i_{k+1}) \notin J_k(i_1, n)$. We have

$$q_{i_1, \ldots, i_{k+1}}(x, B) = \int_{A_{i_1}} q_{i_2, \ldots, i_{k+1}}(y, B)\, dp(x, y).$$

If $i_1 < m+2$, then $A_{i_1} \subset C_{m+1}$ and, consequently, $p(x, A_{i_1}) = P^x\big(X(\tau_{C_{m+1}}) \in A_{i_1}\big) = 0$. Thus the integral vanishes. When $i_1 \geqslant m+2$ and $(i_2, \ldots, i_{k+1}) \notin J_k(i_1, n)$, our assumption gives $q_{i_2, \ldots, i_{k+1}}(y, B) = 0$ for $y \in A_{i_1}$, which completes the proof.

LEMMA 3.2. *Let $k \in \boldsymbol{N}$, $m, n \in \boldsymbol{Z}$. If $n-m < 2k$, then $J_k(m, n)$ is empty. For $n-m \geqslant 2k$ the number of elements of $J_k(m, n)$ equals $\binom{n-m-k-1}{k-1}$.*

Proof. The first part of the lemma follows easily from the definition of $J_k(m, n)$. The second part will be proved by induction on k. If $k = 1$ and $n-m \geqslant 2$, the assertion is easy. Assume that lemma holds for k and all $n-m \geqslant 2k$. Let $n-m \geqslant 2(k+1)$. We have

$$J_{k+1}(m, n) = \bigcup_{j=m+2}^{n-2k} \{(i_1, \ldots, i_{k+1}) \in \mathbf{Z}^{k+1}: i_1 = j, (i_2, \ldots, i_{k+1}) \in J_k(j, n)\}.$$

Hence

$$\# J_{k+1}(m, n) = \sum_{j=m+2}^{n-2k} \# J_k(j, n) = \sum_{j=m+2}^{n-2k} \binom{n-j-k-1}{k-1} = \sum_{l=k-1}^{n-m-k-3} \binom{l}{k-1}.$$

The last sum equals $\binom{n-m-k-2}{k}$ and the lemma follows.

For $k \in N$ and $x \in V$ let us define

$$p_k(x, B) = \sum_{(i_1, \ldots, i_k) \in \mathbf{Z}^k} q_{i_1, \ldots, i_k}(x, B)$$

and $\sigma(x, B) = \sum_{k=1}^{\infty} p_k(x, B)$. We will write $d\sigma(x, y)$ for $\sigma(x, dy)$. Heuristically, $p_k(x, B)$ is the probability of the event that the process starting from $x \in V$ goes to $B \cap V$ after precisely k "jumps" of the considered type and $\sigma(x, B)$ is the probability that the process starting from $x \in V$ visits $B \cap V$ during these "jumps".

Now we are going to find a formula which expresses the harmonic measure $P^x(X(\tau_C) \in \cdot)$ in terms of $\sigma(x, \cdot)$.

LEMMA 3.3. *If* $x \in C_{-1}$ *and* $B_1 \subset C_1$, *then the following equality holds*:

$$\sigma(x, B_1) = p_1(x, B_1) + \int_{C_{-1}} \sigma(y, B_1)\, dp(x, y).$$

Proof. For $k \in N$ we have

$$p_{k+1}(x, B_1) = \sum_{(i_1, \ldots, i_{k+1}) \in \mathbf{Z}^{k+1}} q_{i_1, \ldots, i_{k+1}}(x, B_1)$$

$$= \sum_{(i_2, \ldots, i_{k+1}) \in \mathbf{Z}^k} \sum_{i_1 \in \mathbf{Z}} \int_{A_{i_1}} q_{i_2, \ldots, i_{k+1}}(y, B_1)\, dp(x, y).$$

Notice that $q_{i_2, \ldots, i_{k+1}}(y, B_1) = 0$ for $y \in V \setminus C_{-1}$. Hence $p_{k+1}(x, B_1)$ equals

$$\int_{C_{-1}} \sum_{(i_2, \ldots, i_{k+1}) \in \mathbf{Z}^k} q_{i_2, \ldots, i_{k+1}}(y, B_1)\, dp(x, y) = \int_{C_{-1}} p_k(y, B_1)\, dp(x, y).$$

Therefore

$$\sigma(x, B_1) - p_1(x, B_1) = \sum_{k=1}^{\infty} p_{k+1}(x, B_1) = \int_{C_{-1}} \sum_{k=1}^{\infty} p_k(x, B_1)\, dp(x, y)$$

$$= \int_{C_{-1}} \sigma(y, B_1)\, dp(x, y).$$

LEMMA 3.4. *Let $x \in C_{-1}$ and $B_1 \subset A_1$. Then we have the following formula:*

$$P^x(X(\tau_C) \in B_1) = \sigma(x, B_1) + \int_{A_0} P^y(X(\tau_C) \in B_1)\, d\sigma(x, y).$$

Proof. At first let $x \in A_{-1}$. By Lemma 3.2, $J_k(-1, 0)$ is empty for all $k \in \mathbf{N}$. Hence from Lemma 3.1 we get $p_k(x, A_0) = 0$ for $k \in \mathbf{N}$. Consequently, $\sigma(x, A_0) = 0$ and the integral in the assertion of the lemma vanishes. Let us recall that $C = C_0$. By Lemma 3.2, if $k \in \mathbf{N}$ and $k \geq 2$, then $J_k(1, 1)$ is empty, so $p_k(x, B_1) = 0$. Hence $\sigma(x, B_1) = p_1(x, B_1) = q_1(x, B_1) = P^x(X(\tau_{C_0}) \in B_1)$, which proves the lemma for $x \in A_{-1}$.

Now, let $m \in \mathbf{Z}$, $m \leq -1$, and assume that the lemma is true for $x \in A_{-1} \cup \ldots \cup A_m$. We will show that the lemma holds for $x \in A_{m-1}$. By the strong Markov property we obtain for $x \in A_{m-1}$

$$\begin{aligned} P^x(X(\tau_C) \in B_1) &= E^x\big(P^{X(\tau_{C_m})}(X(\tau_C) \in B_1)\big) \\ &= E^x\big(P^{X(\tau_{C_m})}(X(\tau_C) \in B_1);\ X(\tau_{C_m}) \in B_1\big) \\ &\quad + E^x\big(P^{X(\tau_{C_m})}(X(\tau_C) \in B_1);\ X(\tau_{C_m}) \in A_0\big) \\ &\quad + E^x\big(P^{X(\tau_{C_m})}(X(\tau_C) \in B_1);\ X(\tau_{C_m}) \in C_{-1} \setminus C_m\big) = \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

The first term in the last sum equals

$$\mathrm{I} = P^x(X(\tau_{C_m}) \in B_1) = q_1(x, B_1) = p_1(x, B_1).$$

The second one is equal to

$$\mathrm{II} = \int_{A_0} P^y(X(\tau_C) \in B_1)\, dp_1(x, y).$$

The third term equals

$$\mathrm{III} = \int_{C_{-1} \setminus C_m} P^y(X(\tau_C) \in B_1)\, dp(x, y).$$

Noticing that $C_{-1} \setminus C_m = \bigcup_{l=m+1}^{-1} A_l$, we infer by induction that

$$\mathrm{III} = \int_{C_{-1} \setminus C_m} \sigma(y, B_1)\, dp(x, y) + \int_{C_{-1} \setminus C_m} \int_{A_0} P^z(X(\tau_C) \in B_1)\, d\sigma(y, z)\, dp(x, y) = \mathrm{IV} + \mathrm{V}.$$

By Lemma 3.3 we get

$$\mathrm{IV} = \int_{C_{-1}} \sigma(y, B_1)\, dp(x, y) = \sigma(x, B_1) - p_1(x, B_1).$$

Also by Lemma 3.3 we have

$$\mathrm{V} = \int_{A_0} P^z(X(\tau_C) \in B_1)\, d\sigma(x, z) - \int_{A_0} P^z(X(\tau_C) \in B_1)\, dp_1(x, z).$$

Consequently,

$$\mathrm{II}+\mathrm{V} = \int_{A_0} P^z\big(X(\tau_C)\in B_1\big)\, d\sigma(x, z).$$

Since $\mathrm{I}+\mathrm{IV} = \sigma(x, B_1)$, the proof is complete.

Having Lemma 3.4 it turns out that in order to estimate $P^x\big(X(\tau_C)\in\cdot\big)$ it suffices to have an appropriate estimate for $\sigma(x, \cdot)$. We give at first some simple inequalities for the probability of the "single jump" $p(x, \cdot)$.

LEMMA 3.5. *Let $m, n\in \mathbf{Z}$, $x\in A_m$, $B_n\subset A_n$ and $n-m\geqslant 3$. Then we have*

$$p(x, B_n)\leqslant \frac{b_1}{2^{(n-m)\alpha}}\int_{B_n}\frac{dy}{|y|^d} \quad \textit{and} \quad p(x, A_n)\leqslant \frac{a_1\lambda^{d-1}}{2^{(n-m)\alpha}},$$

where

$$b_1 = \frac{c_\alpha^d\, 2^{2d+3\alpha}}{3^{d+\alpha/2}} \quad \textit{and} \quad a_1 = \frac{c_\alpha^d\,\omega_{d-1}\ln(2)\, 2^{2d+3\alpha}}{(d-1)\, 3^{d+\alpha/2}}.$$

Proof. Since $C_{m+1}\subset B(0, 2^{m+1}r)$, we easily get

$$\begin{aligned} p(x, B_n) &= P^x\big(X(\tau_{C_{m+1}})\in B_n\big)\leqslant P^x\big(X(\tau_{B(0,2^{m+1}r)})\in B_n\big)\\ &= c_\alpha^d\int_{B_n}\frac{(2^{2m+2}r^2-|x|^2)^{\alpha/2}\,dy}{(|y|^2-2^{2m+2}r^2)^{\alpha/2}\,|x-y|^d}.\end{aligned}$$

For $y\in B_n$ we have $|x-y|\geqslant 3|y|/4$ and $|y|^2-2^{2m+2}r^2\geqslant 2^{2n-2}r^2-2^{2m+2}r^2\geqslant 2^{2n-4}3r^2$. It follows that

$$p(x, B_n)\leqslant \frac{c_\alpha^d\, 2^{m\alpha+\alpha+2d}}{2^{n\alpha-2\alpha}\,3^{d+\alpha/2}}\int_{B_n}\frac{dy}{|y|^d},$$

which establishes the first inequality in the lemma. Consequently,

$$\begin{aligned} 2^{(n-m)\alpha} b_1^{-1} p(x, A_n) &\leqslant \int_{A_n} |y|^{-d}\,dy\\ &= \int_{2^{n-1}r}^{2^n r}\int_0^\lambda\int_0^\pi\cdots\int_0^\pi\int_0^{2\pi} \varrho^{-1}\sin^{d-2}(\varphi_1)\ldots\sin(\varphi_{d-2})\,d\varphi_{d-1}\ldots d\varphi_1\,d\varrho\\ &= \omega_{d-1}\int_{2^{n-1}r}^{2^n r}\varrho^{-1}\,d\varrho\int_0^\lambda\sin^{d-2}(\varphi_1)\,d\varphi_1\leqslant \omega_{d-1}\ln(2)\,\lambda^{d-1}(d-1)^{-1},\end{aligned}$$

which proves the lemma.

LEMMA 3.6. *Let $m, n\in\mathbf{Z}$, $x\in A_m$, $B_n\subset A_n$ and $n-m\geqslant 2$. Then we have*

$$p(x, B_n)\leqslant\frac{b_2\,\mu_n(B_n)}{2^{(n-m)\alpha}} \quad \textit{and} \quad p(x, A_n)\leqslant\frac{a_2\lambda^{d-1}}{2^{(n-m)\alpha}},$$

where

$$b_2 = c_\alpha^d 2^{d+3\alpha/2}, \quad a_2 = \frac{c_\alpha^d \omega_{d-1} 2^{d+3\alpha/2}}{(d-1)(1-\alpha/2)}, \quad \mu_n(B) = \int_{B\cap A_n} \frac{(2^{n-1}r)^{\alpha/2}\,dy}{(|y|-2^{n-1}r)^{\alpha/2}|y|^d}.$$

Proof. It is not difficult to check that $a_1 \leqslant a_2$, $b_1 \leqslant b_2$ and $\int_{B_n}|y|^{-d}\,dy \leqslant \mu_n(B_n)$. Hence for $n-m \geqslant 3$ we see that Lemma 3.6 follows from Lemma 3.5. What remains it is to consider the case $n-m = 2$.

As in the proof of Lemma 3.5 we get

$$p(x, B_n) \leqslant P^x\left(X(\tau_{B(0,2^{n-1}r)}) \in B_n\right) = c_\alpha^d \int_{B_n} \frac{(2^{2n-2}r^2-|x|^2)^{\alpha/2}\,dy}{(|y|^2-2^{2n-2}r^2)^{\alpha/2}|x-y|^d}.$$

For $y \in B_n$ we have $|x-y| \geqslant |y|/2$ and $|y|+2^{n-1}r \geqslant 2^n r$. Hence

$$p(x, B_n) \leqslant \frac{c_\alpha^d 2^{(n-1)\alpha/2+d}}{2^{n\alpha/2}} \int_{B_n} \frac{(2^{n-1}r)^{\alpha/2}\,dy}{(|y|-2^{n-1}r)^{\alpha/2}|y|^d} = \frac{c_\alpha^d 2^{d+3\alpha/2}}{2^{(n-m)\alpha}}\mu_n(B_n).$$

It follows that

$$\begin{aligned}
2^{(n-m)\alpha} b_2^{-1} p(x, A_n) &\leqslant \int_{A_n} \frac{(2^{n-1}r)^{\alpha/2}\,dy}{(|y|-2^{n-1}r)^{\alpha/2}|y|^d} \\
&= \omega_{d-1}(2^{n-1}r)^{\alpha/2} \int_{2^{n-1}r}^{2^n r} \frac{\varrho^{d-1}\,d\varrho}{(\varrho-2^{n-1}r)^{\alpha/2}\varrho^d} \int_0^\lambda \sin^{d-2}(\varphi_1)\,d\varphi_1 \\
&\leqslant \frac{\omega_{d-1}(2^{n-1}r)^{\alpha/2}\lambda^{d-1}}{2^{n-1}r(d-1)} \int_0^{2^{n-1}r} \frac{d\varrho}{\varrho^{\alpha/2}} \\
&= \frac{\omega_{d-1}\lambda^{d-1}}{(2^{n-1}r)^{1-\alpha/2}(d-1)} \frac{(2^{n-1}r)^{1-\alpha/2}}{1-\alpha/2} = \frac{\omega_{d-1}\lambda^{d-1}}{(d-1)(1-\alpha/2)},
\end{aligned}$$

which completes the proof.

LEMMA 3.7. *Let $m, n \in \mathbf{Z}$, $n-m \geqslant 2$, $x \in A_m$, $B_n \subset A_n$. Then we have*

$$\sigma(x, B_n) \leqslant c_1 |x|^{\alpha-\varepsilon} \int_{B_n} \frac{dy}{(|y|-2^{n-1}r)^{\alpha/2}|y|^{d+\alpha/2-\varepsilon}},$$

where $\varepsilon = \min(\alpha/2, u_1\lambda^{d-1})$, $u_1 = c_\alpha^d \omega_{d-1} 2^{d+3\alpha/2}\ln(2)(d-1)^{-1}(1-\alpha/2)^{-1}$ and $c_1 = c_1(d, \alpha)$.

Both the constants ε and u_1 are fixed in the sequel. The constant ε has basic significance in this paper.

Proof. Let $k \in \mathbf{N}$ and $(i_1, \ldots, i_k) \in J_k(m, n)$. At first we will prove that

$$q_{i_1,\ldots,i_k}(x, B_n) \leqslant b_2(a_2\lambda^{d-1})^{k-1} 2^{(m-n)\alpha}\mu_n(B_n), \tag{3.1}$$

where a_2, b_2 and μ_n are defined in Lemma 3.6.

Let $k = 1$. Since $(i_1) \in J_1(m, n)$, we have $i_1 = n$ and $q_{i_1}(x, B_n) = p(x, B_n)$. Hence the inequality (3.1) follows from Lemma 3.6. Now let $k \geqslant 2$. By Lemma 3.6 and by induction we get

$$q_{i_1,\ldots,i_k}(x, B_n) = \int_{A_{i_1}} q_{i_2,\ldots,i_k}(y, B_n)\, dp(x, y) \leqslant p(x, A_{i_1}) \sup_{y \in A_{i_1}} q_{i_2,\ldots,i_k}(y, B_n)$$
$$\leqslant a_2 \lambda^{d-1} 2^{(m-i_1)\alpha} b_2 (a_2 \lambda^{d-1})^{k-2} 2^{(i_1-n)\alpha} \mu_n(B_n)$$

and (3.1) follows.

Let us recall that for $2k > n-m$ the set $J_k(m, n)$ is empty, which yields $p_k(x, A_n) = 0$ for $2k > n-m$. Consequently, by Lemmas 3.1 and 3.2 and formula (3.1) we get

$$\sigma(x, B_n) = \sum_{\substack{2k \leqslant n-m \\ k \in N}} p_k(x, B_n) = \sum_{k=1}^{[(n-m)/2]} \sum_{(i_1,\ldots,i_k) \in J_k(m,n)} q_{i_1,\ldots,i_k}(x, B_n)$$
$$\leqslant b_2 \mu_n(B_n) 2^{(m-n)\alpha} \sum_{k=1}^{[(n-m)/2]} \binom{n-m-k-1}{k-1} (a_2 \lambda^{d-1})^{k-1}.$$

The last sum is less than

$$\sum_{k=1}^{n-m+1} \binom{n-m}{k-1} (a_2 \lambda^{d-1})^{k-1} = (1 + a_2 \lambda^{d-1})^{n-m} \leqslant 2^{(n-m)\ln(2) a_2 \lambda^{d-1}}.$$

The last expression equals $2^{(n-m) u_1 \lambda^{d-1}}$ since $u_1 = a_2 \ln(2)$.

Now, let us assume that $u_1 \lambda^{d-1} < \alpha/2$, i.e. $\varepsilon = u_1 \lambda^{d-1}$. It follows that $\sigma(x, B_n) \leqslant b_2 \mu_n(B_n) 2^{(n-m)(\varepsilon-\alpha)}$. For $y \in B_n$ we have $2^{n-m} \geqslant |y|/(2|x|)$, and therefore $2^{(n-m)(\varepsilon-\alpha)} \leqslant 2^\alpha |y|^{\varepsilon-\alpha} |x|^{\alpha-\varepsilon}$. For $y \in B_n$ we also have $2^{n-1} r \leqslant |y|$. Hence

$$\sigma(x, B_n) \leqslant b_2 2^\alpha |x|^{\alpha-\varepsilon} \int_{B_n} \frac{|y|^{\varepsilon-\alpha+\alpha/2}\, dy}{(|y| - 2^{n-1} r)^{\alpha/2} |y|^d},$$

which proves the lemma when $\varepsilon = u_1 \lambda^{d-1}$.

Now we will prove that the lemma holds when $\alpha/2 \leqslant u_1 \lambda^{d-1}$, i.e. $\varepsilon = \alpha/2$. This follows from estimates of the harmonic measure of open bounded sets with $C^{1,1}$ boundary and is independent of our preceding arguments. In fact, we will prove that the lemma is always true if we replace ε by $\alpha/2$ (recall that $\lambda \in (0, \pi/2)$).

Let $x_0 = (\varrho, \varphi_1, \ldots, \varphi_{d-1})$ be a point such that $\varrho = 1/4$, $\varphi_1 = \pi$ and consider $D = B(0, 1) \setminus \overline{B(x_0, 1/4)}$; D is an open bounded set with $C^{1,1}$ boundary satisfying $0 \in \partial D$ and $C_{n-1} \subset 2^{n-1} rD$. Write $t = 2^{n-1} r$ and let $y \in B_n \setminus \partial C_{n-1}$. By (2.4) and the scaling property of f_{tD}^x we obtain

$$f_{C_{n-1}}^x(y) \leqslant f_{tD}^x(y) = \frac{f_D^{x/t}(y/t)}{t^d} \leqslant \frac{b_3 |x/t|^{\alpha/2}}{t^d \delta_D^{\alpha/2}(y/t)\left(1 + \delta_D^{\alpha/2}(y/t)\right) |x/t - y/t|^d},$$

where $b_3 = b_3(D, \alpha)$. We have $\delta_D(y/t) = (|y|-2^{n-1}r)/t$ and $|y-x| \geqslant |y|/2$. Therefore

$$f_{C_{n-1}}^{x}(y) \leqslant \frac{2^d b_3 |x|^{\alpha/2}}{(|y|-2^{n-1}r)^{\alpha/2}|y|^d}.$$

We have

$$P^x(X(\tau_{C_{n-1}}) \in B_n) = \int_{B_n} f_{C_{n-1}}^{x}(y)\,dy \quad \text{and} \quad \sigma(x, B_n) \leqslant P^x(X(\tau_{C_{n-1}}) \in B_n).$$

It follows that the lemma is true for $\varepsilon = \alpha/2$.

Having Lemmas 3.4 and 3.7 we can now formulate estimates of the harmonic measure $P^x(X(\tau_C) \in \cdot)$.

Lemma 3.8. *Let $x \in C_{-1}$ and $B_1 \subset A_1$. Then the following inequality holds:*

$$P^x(X(\tau_C) \in B_1) \leqslant c_2 |x|^{\alpha-\varepsilon} r^{\varepsilon} \int_{B_1} \frac{dy}{(|y|-r)^{\alpha/2}|y|^{d+\alpha/2}}, \quad \text{where } c_2 = c_2(d, \alpha).$$

Proof. By the previous lemma we obtain

$$(3.2)\qquad \int_{A_0} P^y(X(\tau_C) \in B_1)\,d\sigma(x, y) \leqslant c_1 |x|^{\alpha-\varepsilon} r^{\varepsilon} \int_{A_0} \frac{P^y(X(\tau_C) \in B_1)\,dy}{(|y|-r/2)^{\alpha/2}|y|^{d+\alpha/2}}.$$

Since $P^y(X(\tau_C) \in B_1) \leqslant P^y(X(\tau_{B(0,r)}) \in B_1)$, the expression on the right-hand side of (3.2) is bounded from above by

$$(3.3)\qquad c_1 c_\alpha^d |x|^{\alpha-\varepsilon} r^{\varepsilon} \int_{A_0} \frac{(r^2-|y|^2)^{\alpha/2}}{(|y|-r/2)^{\alpha/2}|y|^{d+\alpha/2}} \int_{B_1} \frac{1}{(|z|^2-r^2)^{\alpha/2}|z-y|^d}\,dz\,dy.$$

Let us notice that for $y \in A_0$ and $z \in B_1$ we have $r-|y| = \operatorname{dist}(y, B^c(0, r)) \leqslant |y-z|$, $1/|y| \leqslant 4/|z|$ and $r+|y| \leqslant |z|+r$. Therefore (3.3) is less than or equal to

$$c_1 c_\alpha^d 4^{d+\alpha/2} |x|^{\alpha-\varepsilon} r^{\varepsilon} \int_{B_1} \int_{A_0} \frac{1}{(|y|-r/2)^{\alpha/2}|z-y|^{d-\alpha/2}}\,dy \frac{1}{(|z|-r)^{\alpha/2}|z|^{d+\alpha/2}}\,dz.$$

Now we will show that for all $z \in B_1$ the integral

$$\int_{A_0} \frac{dy}{(|y|-r/2)^{\alpha/2}|z-y|^{d-\alpha/2}}$$

is bounded by a constant which does not depend on z. To obtain this let us put $A_0' = \{y \in A_0 \colon |y| < 3r/4\}$ and $A_0'' = A_0 \setminus A_0'$. For $y \in A_0'$ and $z \in B_1$ we have $|y-z| \geqslant r/4$. Hence

$$\int_{A_0'} \frac{dy}{(|y|-r/2)^{\alpha/2}|z-y|^{d-\alpha/2}} \leqslant \frac{4^{d-\alpha/2}\omega_d}{r^{d-\alpha/2}} \int_{r/2}^{3r/4} \frac{\varrho^{d-1}}{(\varrho-r/2)^{\alpha/2}}\,d\varrho$$

$$\leqslant \frac{4^{d-\alpha/2}(3r)^{d-1}\omega_d}{r^{d-\alpha/2}4^{d-1}} \int_0^{r/4} \varrho^{-\alpha/2}\,d\varrho = \frac{3^{d-1}\omega_d}{1-\alpha/2}.$$

Let us notice that $A_0'' \subset B(0, r) \subset B(z, 3r)$ and $|y|-r/2 \geqslant r/4$ for $y \in A_0''$. Consequently,

$$\int_{A_0''} \frac{dy}{(|y|-r/2)^{\alpha/2}\,|z-y|^{d-\alpha/2}} \leqslant \frac{4^{\alpha/2}}{r^{\alpha/2}} \int_{B(z,3r)} \frac{dy}{|z-y|^{d-\alpha/2}} = \frac{12^{\alpha/2}\,\omega_d}{\alpha/2}.$$

Finally, from Lemmas 3.4 and 3.7 and the above estimates it follows that

$$\begin{aligned} P^x\big(X(\tau_C)\in B_1\big) &= \sigma(x, B_1) + \int_{A_0} P^y\big(X(\tau_C)\in B_1\big)\,d\sigma(x, y) \\ &\leqslant c_2\,|x|^{\alpha-\varepsilon} r^{\varepsilon} \int_{B_1} \frac{dy}{(|y|-r)^{\alpha/2}\,|y|^{d+\alpha/2}}. \end{aligned}$$

Using estimates of the harmonic measure proved in the previous lemma and the formula (2.2) we will get estimates of the expectation of the exit time τ_C.

PROPOSITION 3.9. *For all* $x \in C/2$ *we have*

$$E^x(\tau_C) \leqslant c_3\,|x|^{\alpha-\varepsilon} r^{\varepsilon}, \quad \text{where } c_3 = c_3(d, \alpha).$$

Proof. Using basic properties of Green operator (see Section 2) we obtain a well-known formula

$$E^x(\tau_C) = E^x \int_0^{\tau_C} 1_C(X_t)\,dt = \int_C G_C(x, y)\,dy.$$

Let us put $A_1' = \{z \in A_1: |z| > 3r/2\}$. For $z \in A_1'$ and $y \in C$ we have $|y-z| \leqslant 2|z|$ and $|y-z| \leqslant |z|+r \leqslant 5(|z|-r)$. By (2.2) it follows that

$$\begin{aligned} P^x\big(X(\tau_C)\in A_1'\big) &= \int_C G_C(x, y) \int_{A_1'} \frac{C_{d,\alpha}}{|y-z|^{d+\alpha}}\,dz\,dy \\ &\geqslant C_{d,\alpha}\,2^{-d-\alpha/2}\,5^{-\alpha/2}\,E^x(\tau_C) \int_{A_1'} \frac{dz}{(|z|-r)^{\alpha/2}\,|z|^{d+\alpha/2}}. \end{aligned}$$

On the other hand, by Lemma 3.8 we obtain

$$P^x\big(X(\tau_C)\in A_1'\big) \leqslant c_2\,|x|^{\alpha-\varepsilon} r^{\varepsilon} \int_{A_1'} \frac{dz}{(|z|-r)^{\alpha/2}\,|z|^{d+\alpha/2}}$$

and the proposition follows.

Using Lemma 3.8 and Proposition 3.9 with again formula (2.2) we can estimate the density of the harmonic measure $f_C^x(z)$ for all $z \in V\backslash C$ and $x \in C/2$. This gives an extension of Lemma 3.8.

PROPOSITION 3.10. *Let* $x \in C/2$ *and* $z \in V\backslash\bar{C}$. *The following inequality holds*:

$$f_C^x(z) \leqslant \frac{c_4\,|x|^{\alpha-\varepsilon} r^{\varepsilon}}{(|z|-r)^{\alpha/2}\,|z|^{d+\alpha/2}}, \quad \text{where } c_4 = c_4(d, \alpha).$$

Proof. For $z \in A_1$ the inequality follows directly from Lemma 3.8 with $c_4 = c_2$. Now let $B \subset V \backslash C_1$. By formula (2.2) we have

$$P^x\big(X(\tau_C) \in B\big) = \int_C G_C(x, y) \int_B \frac{C_{d,\alpha}}{|y-z|^{d+\alpha}} dz dy. \tag{3.4}$$

Let us notice that for $z \in V \backslash C_1$ and $y \in C$ we have $|y| \leqslant |z|/2$ and, consequently, $|y-z| \geqslant |z|/2$. Therefore (3.4) is bounded from above by

$$\int_C G_C(x, y) dy 2^{d+\alpha} C_{d,\alpha} \int_B |z|^{-d-\alpha} dz = 2^{d+\alpha} C_{d,\alpha} E^x(\tau_C) \int_B |z|^{-d-\alpha} dz.$$

Hence Proposition 3.10 follows from Proposition 3.9.

Until now we were concerned with estimates of the density $f_C^x(z)$ of the harmonic measure in terms of the distance $|x|$ from x to the edge 0 of the cone. Our next aim is to get estimates of $f_C^x(z)$ which depend also on $\delta_C(x)$ – the distance from x to the boundary of the cone.

Let l_1 be a fixed line through 0, perpendicular to the half-line $h_1 = \{x = (\varrho, \varphi_1, \ldots, \varphi_{d-1}): \varphi_1 = 0\}$. Fix a point x_0 from the line l_1 such that $|x_0| = 1/2$. Let D be a domain with $C^{1,1}$ boundary such that

$$D \subset B(0, 3/4) \cap \{x = (\varrho, \varphi_1, \ldots, \varphi_{d-1}): \varphi_1 < \pi/2\}$$

and the line segment $\overline{0x_0} \subset \partial D$. Then take $E = B(0, 1) \backslash D$; E is an open bounded set with $C^{1,1}$ boundary. The density of the harmonic measure for E satisfies the following inequality (see (2.4)):

$$f_E^x(z) \leqslant \frac{c_5\, \delta_E^{\alpha/2}(x)}{\delta_E^{\alpha/2}(z)\big(1+\delta_E^{\alpha/2}(z)\big)|x-z|^d}, \qquad x \in E,\ z \in \operatorname{int}(E^c).$$

The constant c_5 depends on E and α, but since E is fixed in each $\boldsymbol{R}^d$, we write $c_5 = c_5(d, \alpha)$.

Let $y \in \partial V$, $y \neq 0$, and denote by l_2 the line determined by 0 and y. Consider a line l_3 through y, perpendicular to l_2 and lying in the same plane as l_2 and h_1. The point y divides l_3 into two half-lines. One of them, which will be denoted by h_2, is contained in V^c. By h_3 denote a half-line beginning at 0, parallel to l_3 and with the same direction as h_2.

Let T_y be a fixed rotation around 0 mapping x_0 to $y/(2|y|)$ and h_1 to h_3. It follows that the line segment $\overline{0(y/2|y|)} \subset \partial(T_y E)$ and $T_y D \subset V^c$, so we have $V \cap B(0, 1) \subset T_y E$.

For $x \in T_y E$ and $z \in \operatorname{int}\big((T_y E)^c\big)$ we have

$$\begin{aligned} f_{T_y E}^x(z) = f_E^{T_y^{-1}x}(T_y^{-1} z) &\leqslant \frac{c_5\, \delta_E^{\alpha/2}(T_y^{-1} x)}{\delta_E^{\alpha/2}(T_y^{-1} z)\big(1+\delta_E^{\alpha/2}(T_y^{-1} z)\big)|T_y^{-1} x - T_y^{-1} z|^d} \\ &= \frac{c_5\, \delta_{T_y E}^{\alpha/2}(x)}{\delta_{T_y E}^{\alpha/2}(z)\big(1+\delta_{T_y E}^{\alpha/2}(z)\big)|x-z|^d}. \end{aligned} \tag{3.5}$$

Let us define a set $E(m, y) = 2^m r T_y E$ for $m \in \mathbf{Z}$ and $y \in \partial V$, $y \neq 0$. For abbreviation we put $t = 2^m r$. It is clear that $C_m = V \cap B(0, t) \subset E(m, y) \subset B(0, t)$, and also the line segment $\overline{0, ty/(2|y|)} \subset \partial E(m, y)$. Now, let $x \in E(m, y)$ and $z \in V \setminus \overline{B(0, t)}$. By the scaling property of the density of the harmonic measure and (3.5) we obtain

$$f_{E(m,y)}^{x}(z) = t^{-d} f_{T_y E}^{x/t}(z/t) \leqslant \frac{c_5\, \delta_{T_y E}^{\alpha/2}(x/t)}{t^d\, \delta_{T_y E}^{\alpha/2}(z/t)\left(1+\delta_{T_y E}^{\alpha/2}(z/t)\right)|x/t - z/t|^d}.$$

Notice that for all $B \subset \mathbf{R}^d$, $s > 0$, and $x \in \mathbf{R}^d$ we have $\delta_{sB}(sx) = s\delta_B(x)$. Thus $\delta_{T_y E}(x/t) = \delta_{E(m,y)}(x)/t$. For $z \in V \cap B^c(0, t)$ we also have $\delta_{T_y E}(z/t) = (|z|-t)/t$ and $1+\delta_{T_y E}^{\alpha/2}(z/t) \geqslant (2t)^{-\alpha/2}|z|^{\alpha/2}$. Consequently, we get

$$f_{E(m,y)}^{x}(z) \leqslant \frac{c_5\, \delta_{E(m,y)}^{\alpha/2}(x)\,(2^{m+1} r)^{\alpha/2}}{(|z|-2^m r)^{\alpha/2}\, |z|^{\alpha/2}\, |x-z|^d}, \tag{3.6}$$

where $x \in E(m, y)$ and $z \in V \setminus \overline{B(0, 2^m r)}$.

LEMMA 3.11. *Let $k, m \in \mathbf{Z}$, $m \leqslant -1$, and $x \in A_m$, $B \subset V \setminus C_{m+1}$, $k \geqslant m+2$. Then there exist constants $c_6 = c_6(d, \alpha)$ and $c_7 = c_7(d, \alpha)$ such that*

$$p(x, B) \leqslant \int_B \frac{c_6\, |x|^{\alpha/2}\, \delta_C^{\alpha/2}(x)}{(|z|-2^{m+1} r)^{\alpha/2}\, |z|^{d+\alpha/2}}\, dz \quad \text{and} \quad p(x, A_k) \leqslant \frac{c_7\, \lambda^{d-1}\, \delta_C^{\alpha/2}(x)}{2^{(k-m)\alpha}\, |x|^{\alpha/2}}.$$

Proof. Let x^* be such that $|x - x^*| = \delta_V(x)$. Of course, we have $0 < |x^*| < |x| < 2^m r$ and $\overline{0, 2^m r x^*/|x^*|} \subset \partial E(m+1, x^*)$. Therefore $\delta_{E(m+1,x^*)}(x) = |x - x^*| = \delta_C(x)$. From (3.6) it follows that

$$f_{C_{m+1}}^{x}(z) \leqslant f_{E(m+1,x^*)}^{x}(z) \leqslant \frac{c_5\, \delta_C^{\alpha/2}(x)\,(2^{m+2} r)^{\alpha/2}}{(|z|-2^{m+1} r)^{\alpha/2}\, |z|^{\alpha/2}\, |x-z|^d}, \tag{3.7}$$

where $z \in V \setminus \overline{C_{m+1}}$. The first inequality in the assertion of the lemma follows directly from (3.7). It suffices to notice that $p(x, B) = \int_B f_{C_{m+1}}^{x}(z)\, dz$, $2^{m-1} r \leqslant |x|$ and $|z| \leqslant 2|x-z|$ for $z \in B$. The second inequality follows from the first one. Indeed, we have

$$p(x, A_k) \leqslant \int_{A_k} \frac{c_6\, |x|^{\alpha/2}\, \delta_C^{\alpha/2}(x)}{(|z|-2^{m+1} r)^{\alpha/2}\, |z|^{d+\alpha/2}}\, dz \leqslant \frac{c_6\, |x|^{\alpha/2}\, \delta_C^{\alpha/2}(x)}{(2^{k-1} r)^{\alpha}} \int_{A_k} \frac{(2^{k-1} r)^{\alpha/2}}{(|z|-2^{k-1} r)^{\alpha/2}\, |z|^d}\, dz.$$

In the proof of Lemma 3.6 it was shown that the last integral is bounded from above by $\omega_{d-1}(d-1)^{-1}(1-\alpha/2)^{-1}\lambda^{d-1}$. We also have $|x|^{\alpha/2} \leqslant r^{\alpha}/(2^{-m\alpha}|x|^{\alpha/2})$, and the second inequality in the lemma follows.

Now we are able to prove our main upper bound estimate for the density of the harmonic measure of the bounded cone.

THEOREM 3.12. *Let $x \in C/2$ and $z \in V\setminus\bar{C}$. Then we have*

$$f_C^x(z) \leqslant c\frac{\delta_C^{\alpha/2}(x)\,|x|^{\alpha/2-\varepsilon}\,r^{\varepsilon}}{\delta_C^{\alpha/2}(z)\,|z|^{d+\alpha/2}}, \quad \textit{where } c = c(d, \alpha).$$

Proof. Let $m \in \mathbf{Z}$, $m \leqslant -1$, $x \in A_m$ and $B \subset V\setminus\bar{C}$. By the strong Markov property we get

(3.8)

$$P^x\big(X(\tau_C)\in B\big) = P^x\big(X(\tau_{C_{m+1}})\in B\big)+E^x\big(P^{X(\tau_{C_{m+1}})}(X(\tau_C)\in B);\ X(\tau_{C_{m+1}})\in A_0\big)$$
$$+\sum_{k=m+2}^{-1} E^x\big(P^{X(\tau_{C_{m+1}})}(X(\tau_C)\in B);\ X(\tau_{C_{m+1}})\in A_k\big) = \mathrm{I}+\mathrm{II}+\mathrm{III}.$$

We have $|x|^{\alpha/2} \leqslant |x|^{\alpha/2-\varepsilon} r^{\varepsilon}$ and $|z|-2^{m+1}r \geqslant \delta_C(z)$ for $z \in V\setminus C$. Therefore, by Lemma 3.11 we obtain

$$\mathrm{I} = p(x, B) \leqslant \int_B \frac{c_6\,|x|^{\alpha/2-\varepsilon}\,r^{\varepsilon}\,\delta_C^{\alpha/2}(x)}{\delta_C^{\alpha/2}(z)\,|z|^{d+\alpha/2}}\,dz. \tag{3.9}$$

Now we will estimate II. Noticing that if $m = -1$, then II vanishes, we assume that $m \leqslant -2$. In order to estimate II we will divide it into two parts. We have

$$\mathrm{II} = \int_{A_0}\int_{B\cap(V\setminus C_1)} f_C^y(z)\,dz\,dp(x, y)+\int_{A_0}\int_{B\cap A_1} f_C^y(z)\,dz\,dp(x, y) = \mathrm{IV}+\mathrm{V}.$$

For $z \in V\setminus C_1$ and $y \in A_0$ we obtain

$$f_C^y(z) \leqslant f_{B(0,r)}^y(z) = \frac{c_\alpha^d\,(r^2-|y|^2)^{\alpha/2}}{(|z|^2-r^2)^{\alpha/2}\,|z-y|^d} \leqslant \frac{c_\alpha^d\,2^d\,r^\alpha}{\delta_C^{\alpha/2}(z)\,|z|^{d+\alpha/2}}.$$

To get the last inequality it remains to notice that $|z-y| \geqslant |z|/2$. Consequently,

$$\mathrm{IV} \leqslant p(x, A_0)\int_{B\cap(V\setminus C_1)} \frac{c_\alpha^d\,2^d\,r^\alpha}{\delta_C^{\alpha/2}(z)\,|z|^{d+\alpha/2}}\,dz.$$

Noticing that $2^{m\alpha} \leqslant 2^\alpha |x|^{\alpha-\varepsilon} r^{\varepsilon-\alpha}$ we obtain by Lemma 3.11

$$\mathrm{IV} \leqslant \int_{B\cap(V\setminus C_1)} \frac{c_\alpha^d\,c_7\,2^{1+\alpha}\,\pi^{d-1}\,\delta_C^{\alpha/2}(x)\,|x|^{\alpha/2-\varepsilon}\,r^{\varepsilon}}{\delta_C^{\alpha/2}(z)\,|z|^{d+\alpha/2}}\,dz. \tag{3.10}$$

From Lemma 3.11 and the inequality $f_C^y(z) \leqslant f_{B(0,r)}^y(z)$ (for $z \in V\setminus\bar{C}$) we get

$$\mathrm{V} \leqslant \int_{A_0} \frac{c_6\,|x|^{\alpha/2}\,\delta_C^{\alpha/2}(x)}{(|y|-2^{m+1}r)^{\alpha/2}\,|y|^{d+\alpha/2}}\int_{B\cap A_1} \frac{c_\alpha^d\,(r^2-|y|^2)^{\alpha/2}}{(|z|^2-r^2)^{\alpha/2}\,|z-y|^d}\,dz\,dy. \tag{3.11}$$

For $y \in A_0$ and $z \in A_1$ we have $r+|y| \leqslant |z|+r$, $1/|y| \leqslant 4/|z|$ and $r-|y| = \operatorname{dist}(y, B^c(0, r)) \leqslant |y-z|$. For $m \leqslant -2$ we also have $|y|-2^{m+1}r \geqslant |y|-r/2$.

Therefore, the right-hand side of (3.11) is bounded from above by

$$\int_{B\cap A_1}\int_{A_0}\frac{1}{(|y|-r/2)^{\alpha/2}|y-z|^{d-\alpha/2}}dy\frac{c_\alpha^d c_6 4^{d+\alpha/2}|x|^{\alpha/2}\delta_C^{\alpha/2}(x)}{(|z|-r)^{\alpha/2}|z|^{d+\alpha/2}}dz.$$

In the proof of Lemma 3.8 it was shown that there exists a constant $c_8 = c_8(d, \alpha)$ such that

$$\int_{A_0}\frac{dy}{(|y|-r/2)^{\alpha/2}|y-z|^{d-\alpha/2}} \leqslant c_8.$$

Hence we obtain

$$\mathrm{V} \leqslant \int_{B\cap A_1}\frac{c_\alpha^d c_6 c_8 4^{d+\alpha/2}|x|^{\alpha/2}\delta_C^{\alpha/2}(x)}{(|z|-r)^{\alpha/2}|z|^{d+\alpha/2}}dz.$$

According to (3.10) and the last inequality there exists a constant $c_9 = c_9(d, \alpha)$ such that

$$\mathrm{II} = \mathrm{IV}+\mathrm{V} \leqslant c_9\,\delta_C^{\alpha/2}(x)\,|x|^{\alpha/2-\varepsilon}r^{\varepsilon}\int_B\frac{dz}{\delta_C^{\alpha/2}(z)\,|z|^{d+\alpha/2}}. \tag{3.12}$$

Our next aim is to estimate III. Let us notice that if $m \geqslant -2$, then III vanishes. Therefore we will assume that $m \leqslant -3$. We have

$$\mathrm{III} = \sum_{k=m+2}^{-1}\int_{A_k}\int_B f_C^y(z)\,dz\,dp(x, y).$$

For $y\in A_k$, $m+2\leqslant k\leqslant -1$ and $z\in V\setminus\bar{C}$ we get by Proposition 3.10

$$f_C^y(z) \leqslant \frac{c_4|y|^{\alpha-\varepsilon}r^{\varepsilon}}{\delta_C^{\alpha/2}(z)\,|z|^{d+\alpha/2}} \leqslant \frac{c_4 2^{k(\alpha-\varepsilon)}r^{\alpha}}{\delta_C^{\alpha/2}(z)\,|z|^{d+\alpha/2}}. \tag{3.13}$$

The last inequality follows from the fact that $|y| < 2^k r$ for $y\in A_k$. According to (3.13) and Lemma 3.11 we get

$$\begin{aligned}\mathrm{III} &\leqslant \sum_{k=m+2}^{-1}\int_{A_k}\int_B\frac{c_4 2^{k(\alpha-\varepsilon)}r^{\alpha}}{\delta_C^{\alpha/2}(z)\,|z|^{d+\alpha/2}}dz\,dp(x, y)\\ &= \int_B\frac{c_4 r^{\alpha}dz}{\delta_C^{\alpha/2}(z)\,|z|^{d+\alpha/2}}\sum_{k=m+2}^{-1}2^{k(\alpha-\varepsilon)}p(x, A_k)\\ &\leqslant \frac{c_4 c_7 r^{\alpha}\lambda^{d-1}\delta_C^{\alpha/2}(x)}{|x|^{\alpha/2}}\int_B\frac{dz}{\delta_C^{\alpha/2}(z)\,|z|^{d+\alpha/2}}\sum_{k=m+2}^{-1}2^{k(\alpha-\varepsilon)}2^{(m-k)\alpha}.\end{aligned}$$

The last sum equals

$$2^{m\alpha}\sum_{k=m+2}^{-1}2^{-k\varepsilon} = 2^{m\alpha}2^{(-m-2)\varepsilon}\frac{1-2^{-\varepsilon(-m-2)}}{1-2^{-\varepsilon}} \leqslant 2^{m(\alpha-\varepsilon)-2\varepsilon}\frac{2^{\varepsilon}}{2^{\varepsilon}-1}. \tag{3.14}$$

It is not difficult to see that $2^{\varepsilon}-1 \geqslant \min(2^{\alpha/2}-1, a_2\lambda^{d-1})$, where a_2 is defined in Lemma 3.6 and is such that $u_1 = a_2 \ln(2)$. Consequently, the right-hand side of (3.14) is bounded from above by

$$\frac{c_{10}2^{m(\alpha-\varepsilon)}}{\lambda^{d-1}} \leqslant \frac{c_{10}2^{\alpha}|x|^{\alpha-\varepsilon}}{r^{\alpha-\varepsilon}\lambda^{d-1}},$$

where $c_{10} = c_{10}(d, \alpha)$. It follows that

$$\mathrm{III} \leqslant c_4 c_7 c_{10} 2^{\alpha} \delta_C^{\alpha/2}(x)|x|^{\alpha/2-\varepsilon} r^{\varepsilon} \int_B \frac{dz}{\delta_C^{\alpha/2}(z)|z|^{d+\alpha/2}},$$

which with (3.8), (3.9) and (3.12) proves the theorem.

Having Theorem 3.12 we can strengthen the estimate of the expectation of τ_C which was given in Proposition 3.9. The proof is almost the same as the proof of Proposition 3.9, so it is omitted.

THEOREM 3.13. *There exists a constant $c = c(d, \alpha)$ such that for all $x \in C/2$*

$$E^x(\tau_C) \leqslant c\delta_C^{\alpha/2}(x)|x|^{\alpha/2-\varepsilon} r^{\varepsilon}.$$

Now our aim is to obtain lower bound estimates of $E^x(\tau_C)$ and $f_C^x(z)$. As in the proof of the upper bound estimate we will need some simple inequalities for a probability of a single "jump" $p(x, \cdot)$. For $m \in \mathbf{Z}$ let us put

$$\tilde{A}_n = \{x = (\varrho, \varphi_1, \ldots, \varphi_{d-1}) \colon \varrho \in [2^{n-1}r, 2^n r), \varphi_1 \in [0, \lambda/2)\}.$$

LEMMA 3.14. *Let $m, n \in \mathbf{Z}$, $n-m \geqslant 2$, $x \in A_m$ and $B_n \subset A_n$. Then we have*

$$p(x, B_n) \geqslant \frac{b_1 \delta_V^{\alpha/2}(x)}{2^{(n-m)\alpha}|x|^{\alpha/2}} \int_{B_n} \frac{dz}{|z|^d} \quad \text{and} \quad p(x, \tilde{A}_n) \geqslant \frac{a_1 \lambda^{d-1+\alpha}\delta_V^{\alpha/2}(x)}{2^{(n-m)\alpha}|x|^{\alpha/2}},$$

where $b_1 = c_\alpha^d 2^{d-\alpha} 3^{-d-\alpha} \sin^\alpha \lambda$ and $a_1 = c_\alpha^d \omega_{d-1} \ln(2) 2^{3d/2-2} 3^{-d-\alpha} \pi^{-d-\alpha+2} \times (d-1)^{-1}$. If $x \in \tilde{A}_m$, then we have

$$p(x, B_n) \geqslant \frac{b_1}{2^{(n-m)\alpha}} \int_{B_n} \frac{dz}{|z|^d} \quad \text{and} \quad p(x, \tilde{A}_n) \geqslant \frac{a_2 \lambda^{d-1+\alpha}}{2^{(n-m)\alpha}},$$

where $a_2 = c_\alpha^d \omega_{d-1} \ln(2) 2^{3d/2-2} 5^{-(d+\alpha)/2} \pi^{-d-\alpha+2} (d-1)^{-1}$.

Proof. Let $r_m = 2^{m-2} r \sin\lambda$. At first let us consider the case $\delta_V(x) < r_m$. Let x^* be the point on ∂V such that $|x-x^*| = \delta_V(x)$. It is easy to see that the half-line $l = \{z = (\varrho, \varphi_1, \ldots, \varphi_{d-1}) \colon \varphi_1 = 0\}$ is contained in the plane determined by points 0, x and x^*. Denote by $\tilde{x}$ the point of intersection of l and the line determined by x, x^*. Let x' be the point within the line segment $\overline{\tilde{x}x}$ such that $|x'-x^*| = r_m$. Since

$$|\tilde{x}-x^*| = |\tilde{x}|\sin\lambda \geqslant |x|\sin\lambda \geqslant 2^{m-1} r\sin\lambda > r_m,$$

such a point exists and is easy to notice that $\delta_V(x') = |x'-x^*|$. Consequently, $x \in B(x', r_m) \subset V$. If $y \in B(x', r_m)$, then

$$|y| \leqslant |x'| + r_m \leqslant |x| + 2r_m \leqslant 2^m r + 2^{m-1} r \sin\lambda < 2^{m+1} r,$$

so we also have $B(x', r_m) \subset C_{m+1}$. It follows that

$$p(x, B_n) \geqslant P^x\big(X(\tau_{B(x',r_m)}) \in B_n\big) = \int_{B_n} \frac{c_\alpha^d (r_m^2 - |x-x'|^2)^{\alpha/2}\, dz}{(|z-x'|^2 - r_m^2)^{\alpha/2}\, |z-x|^d}.$$

For $z \in A_n$ we have $|z-x| \leqslant |z|+|x| \leqslant 3|z|/2$, $|z-x'| \leqslant |z-x|+|x-x'| \leqslant 2|z|$ and $r_m - |x-x'| = \delta_V(x)$. Therefore

$$p(x, B_n) \geqslant c_\alpha^d r_m^{\alpha/2} 2^{d-\alpha} 3^{-d} \delta_V^{\alpha/2}(x) \int_{B_n} |z|^{-d-\alpha}\, dz.$$

Notice that for $z \in A_n$ we have $r_m^{\alpha/2} |z|^{-\alpha} \geqslant 2^{-3\alpha/2} |x|^{-\alpha/2} 2^{(m-n)\alpha} \sin^{\alpha/2}\lambda$. Hence

$$p(x, B_n) \geqslant c_\alpha^d 2^{d-5\alpha/2} 3^{-d} \delta_V^{\alpha/2}(x) |x|^{-\alpha/2} 2^{(m-n)\alpha} \sin^{\alpha/2}\lambda \int_{B_n} |z|^{-d}\, dz. \tag{3.15}$$

As in Lemma 3.5 we get

$$\int_{\tilde{A}_n} |z|^{-d}\, dz = \omega_{d-1} \int_{2^{n-1}r}^{2^n r} \varrho^{-1}\, d\varrho \int_0^{\lambda/2} \sin^{d-2}(\varphi_1)\, d\varphi_1.$$

Since $\sin\varphi \geqslant (2^{3/2}/\pi)\varphi$ for $\varphi \in [0, \pi/4]$, we obtain

$$\int_{\tilde{A}_n} |z|^{-d}\, dz \geqslant \omega_{d-1} \ln(2) \frac{2^{3d/2-3}}{\pi^{d-2}} \int_0^{\lambda/2} \varphi_1^{d-2}\, d\varphi_1 \geqslant \frac{\omega_{d-1} \ln(2)\, 2^{3d/2-3} \lambda^{d-1}}{2^{d-1} \pi^{d-2} (d-1)}.$$

Noticing that $\sin^{\alpha/2}\lambda \geqslant 2^\alpha \pi^{-\alpha} \lambda^\alpha$ we get from (3.15)

$$p(x, \tilde{A}_n) \geqslant \frac{c_\alpha^d \omega_{d-1} \ln(2)\, 2^{3d/2-3\alpha/2-2} \delta_V^{\alpha/2}(x) \lambda^{d-1+\alpha}}{3^d \pi^{d+\alpha-2} (d-1) |x|^{\alpha/2} 2^{(n-m)\alpha}}. \tag{3.16}$$

Now let us consider the case $\delta_V(x) \geqslant r_m$. When $y \in B(x, r_m)$, we have $|y| \leqslant |x| + r_m < 2^{m+1} r$ and, consequently, $B(x, r_m) \subset C_{m+1}$. Hence

$$p(x, B_n) \geqslant P^x\big(X(\tau_{B(x,r_m)}) \in B_n\big) = \int_{B_n} \frac{c_\alpha^d r_m^\alpha\, dz}{(|z-x|^2 - r_m^2)^{\alpha/2} |z-x|^d}. \tag{3.17}$$

Since for $z \in A_n$ we have $|z-x| \leqslant 3|z|/2$ and $r_m^\alpha |z|^{-\alpha} \geqslant 2^{(m-n)\alpha - 2\alpha} \sin^\alpha\lambda$, we get

$$\begin{aligned} p(x, B_n) &\geqslant c_\alpha^d r_m^\alpha 2^{d+\alpha} 3^{-d-\alpha} \int_{B_n} |z|^{-d-\alpha}\, dz \\ &\geqslant c_\alpha^d 2^{d-\alpha} 3^{-d-\alpha} \delta_V^{\alpha/2}(x) |x|^{-\alpha/2} 2^{(m-n)\alpha} \sin^\alpha\lambda \int_{B_n} |z|^{-d}\, dz. \end{aligned} \tag{3.18}$$

Hence, replacing $\sin^\alpha\lambda$ by $2^\alpha \pi^{-\alpha} \lambda^\alpha$ we obtain

$$p(x, \tilde{A}_n) \geqslant \frac{c_\alpha^d \omega_{d-1} \ln(2)\, 2^{3d/2-2} \delta_V^{\alpha/2}(x) \lambda^{d-1+\alpha}}{3^{d+\alpha} \pi^{d+\alpha-2} (d-1) |x|^{\alpha/2} 2^{(n-m)\alpha}}. \tag{3.19}$$

Consequently, (3.15), (3.16), (3.18) and (3.19) give the first two inequalities in the lemma.

Let us notice that if $x\in\tilde{A}_m$, then $\delta_V(x)\geqslant |x|\sin(\lambda/2)\geqslant 2^{m-1}r\sin(\lambda/2)\geqslant r_m$. Hence the third inequality in the lemma follows from (3.17) in a similar way as the inequality (3.18). To get the fourth inequality notice that for $x\in\tilde{A}_m$ and $z\in\tilde{A}_n$ we have

$$|z-x|^2=|z|^2+|x|^2-2\cos(\sphericalangle x0z)|z|\,|x|\leqslant |z|^2+|x|^2\leqslant 5|z|^2/4$$

and from (3.17) we finally get

$$p(x,\tilde{A}_n)\geqslant c_\alpha^d r_m^\alpha 2^{d+\alpha}5^{(-d-\alpha)/2}\int_{\tilde{A}_n}|z|^{-d-\alpha}\,dz\geqslant \frac{c_\alpha^d\omega_{d-1}\ln(2)\,2^{3d/2-2}\lambda^{d-1+\alpha}}{5^{(d+\alpha)/2}\pi^{d+\alpha-2}(d-1)\,2^{(n-m)\alpha}}.$$

LEMMA 3.15. *Let $m,n\in\mathbf{Z}$, $n-m\geqslant 2$, $x\in A_m$ and $B_n\subset A_n$. Then we have*

$$\sigma(x,B_n)\geqslant c_1\delta_V^{\alpha/2}(x)|x|^{\alpha/2-\varepsilon'}\int_{B_n}|z|^{-d-\alpha+\varepsilon'}\,dz,$$

where $\varepsilon'=u_2\lambda^{d-1+\alpha}$, $u_2=c_\alpha^d\omega_{d-1}\ln(2)\,2^{3d/2-5}\,5^{(2-d-\alpha)/2}\pi^{-d-\alpha+2}(d-1)^{-1}$ and $c_1=2^{-\alpha}a_1b_1/(12a_2)$ (a_1, a_2, b_1 are the same as is Lemma 3.14).

Similarly to ε and u_1 the constants ε' and u_2 are fixed in the whole paper. Let us notice that $u_2=5a_2/8$. Easy computations show that $c_\alpha^d\omega_{d-1}(d-1)^{-1}<\alpha/(2\sqrt{\pi})$, $u_2(\pi/2)^{d-1+\alpha}<\alpha/(4\sqrt{\pi})$, and hence $\varepsilon'<\alpha/2$.

Proof. At first we will prove that for $x\in\tilde{A}_m$, $k\in\mathbf{N}$ and $(i_1,\ldots,i_k)\in J_k(m,n)$ we have

$$q_{i_1,\ldots,i_k}(x,B_n)\geqslant\frac{b_1(a_2\lambda^{d-1+\alpha})^{k-1}}{2^{(n-m)\alpha}}\int_{B_n}\frac{dz}{|z|^d}. \tag{3.20}$$

For $k=1$ we have $i_1=n$ and $q_{i_1}(x,B_n)=p(x,B_n)$, so (3.20) follows from the third inequality in Lemma 3.14. Now let $k\geqslant 2$. Let us recall that if $(i_1,\ldots,i_k)\in J_k(m,n)$, then $n-i_1\geqslant 2$ and $(i_2,\ldots,i_k)\in J_{k-1}(i_1,n)$. By the forth inequality in Lemma 3.14 and by induction we get

$$q_{i_1,\ldots,i_k}(x,B_n)\geqslant\int_{\tilde{A}_{i_1}}q_{i_2,\ldots,i_k}(y,B_n)\,dp(x,y)$$
$$\geqslant p(x,\tilde{A}_{i_1})\inf_{y\in\tilde{A}_{i_1}}q_{i_2,\ldots,i_k}(y,B_n)\geqslant\frac{a_2\lambda^{d-1+\alpha}b_1(a_2\lambda^{d-1+\alpha})^{k-2}}{2^{(i_1-m)\alpha}2^{(n-i_1)\alpha}}\int_{B_n}\frac{dz}{|z|^d},$$

which proves (3.20).

Now we are going to prove that for $x\in A_m$, $k\in\mathbf{N}$ and $(i_1,\ldots,i_k)\in J_k(m,n)$ we have

$$q_{i_1,\ldots,i_k}(x,B_n)\geqslant\frac{a_1b_1\delta_V^{\alpha/2}(x)(a_2\lambda^{d-1+\alpha})^{k-1}}{a_2|x|^{\alpha/2}2^{(n-m)\alpha}}\int_{B_n}\frac{dz}{|z|^d}. \tag{3.21}$$

For $k=1$ we have $q_{i_1}(x, B_n) = p(x, B_n)$ and (3.21) follows from the first inequality in Lemma 3.14 and an easy inequality $a_1 < a_2$. Let $k \geqslant 2$. By the second inequality in Lemma 3.14 and by (3.20) we obtain

$$\begin{aligned} q_{i_1,\dots,i_k}(x, B_n) &\geqslant \int_{\tilde{A}_{i_1}} q_{i_2,\dots,i_k}(y, B_n)\, dp(x, y) \\ &\geqslant p(x, \tilde{A}_{i_1}) \inf_{y \in \tilde{A}_{i_1}} q_{i_2,\dots,i_k}(y, B_n) \\ &\geqslant \frac{a_1 \lambda^{d-1+\alpha}\, \delta_V^{\alpha/2}(x)\, b_1 (a_2 \lambda^{d-1+\alpha})^{k-2}}{|x|^{\alpha/2}\, 2^{(i_1-m)\alpha}\, 2^{(n-i_1)\alpha}} \int_{B_n} \frac{dz}{|z|^d}, \end{aligned}$$

and (3.21) follows. Consequently, for $x \in A_m$

$$\begin{aligned} (3.22) \qquad \sigma(x, B_n) &= \sum_{k=1}^{[(n-m)/2]} \sum_{(i_1,\dots,i_k) \in J_k(m,n)} q_{i_1,\dots,i_k}(x, B_n) \\ &\geqslant \frac{a_1 b_1 \delta_V^{\alpha/2}(x)}{a_2 |x|^{\alpha/2}\, 2^{(n-m)\alpha}} \int_{B_n} \frac{dz}{|z|^d} \sum_{k=1}^{[(n-m)/2]} \binom{n-m-k-1}{k-1} (a_2 \lambda^{d-1+\alpha})^{k-1}. \end{aligned}$$

Let us put $c = a_2 \lambda^{d-1+\alpha}$ and $w = [(n-m-2)/4]$. We have $[(n-m)/2] \geqslant 2w+1$ and $n-m \geqslant 4w+2$. Hence the last sum in (3.22) is bounded from below by

$$\sum_{k=1}^{2w+1} \binom{4w-(k-1)}{k-1} c^{k-1} = \sum_{l=0}^{2w} \binom{4w-l}{l} c^l \geqslant \sum_{l=0}^{w} \binom{4w-l}{l} c^l.$$

It is not difficult to show that

$$\binom{4w-l}{l} \geqslant \left(\frac{3}{2}\right)^l \binom{2w}{l}$$

for $l \leqslant w$ and $l, w \in N \cup \{0\}$. According to the remark before the proof we have $c = \lambda^{d-1+\alpha} 8u_2/5 < 1/2$. Hence

$$\sum_{l=0}^{w} \binom{4w-l}{l} c^l \geqslant \sum_{l=0}^{w} \left(\frac{3}{2}c\right)^l \binom{2w}{l} \geqslant \frac{1}{2}\left(1+\frac{3}{2}c\right)^{2w}.$$

Since $c < 1/2$, it is easy to notice that $1+3c/2 \geqslant (1+5c/8)^2$ and $(1+5c/8) \geqslant 2^{5c/8}$. We also have $\varepsilon' = 5c/8$, $4w \geqslant n-m-6$ and $(1+5/16)^6 < 6$. Consequently, the last sum in (3.22) is bounded from below by

$$(3.23) \qquad \frac{1}{2}\left(1+\frac{3}{2}c\right)^{2w} \geqslant \frac{1}{2}\left(1+\frac{5}{8}c\right)^{n-m-6} \geqslant \frac{1}{12} 2^{(n-m)\varepsilon'}.$$

It is clear that $2^{n-m} \leqslant 2|z|/|x|$ for $z \in A_n$, $x \in A_m$, which yields

$$2^{(n-m)(\varepsilon'-\alpha)} \geqslant 2^{-\alpha} |x|^{\alpha-\varepsilon'} |z|^{\varepsilon'-\alpha}.$$

Thus the lemma follows from (3.22) and (3.23).

Let us notice that apart from its vertex the bounded cone C has other "singularities" at $\partial V \cap \partial B(0, r)$ (the points for which $\varrho = r$ and $\varphi_1 = \lambda$). It was not the aim of the paper to consider this kind of singularities. Therefore, in the following estimate of $P^x(X(\tau_C) \in B_1)$ (where $B_1 \subset A_1$) we consider only such sets B_1 that are "far" from $\partial V \cap \partial B(0, r)$.

LEMMA 3.16. *Let* $x \in C_{-1}$. *Then we have*

$$P^x(X(\tau_C) \in B_1) \geqslant c_2 \delta_C^{\alpha/2}(x) |x|^{\alpha/2-\varepsilon'} r^{\varepsilon'} \int_{B_1} \frac{dz}{\delta_C^{\alpha/2}(z) |z|^{d+\alpha/2}}$$

for $B_1 \subset A_1 \setminus S$, *where*

$$S = \{z = (\varrho, \varphi_1, \ldots, \varphi_{d-1}) \colon \varrho \in [r, r + (r \sin \lambda)/8), \varphi_1 \in [\lambda/2, \lambda)\},$$

$$c_2 = \min(c_1, c_3) c_\alpha^d \omega_d 2^{-d-5\alpha} 3^{-d} d^{-1} \sin^{\alpha/2} \lambda, \qquad c_3 = c_\alpha^d 2^{-d-4\alpha} \sin^{\alpha/2} \lambda,$$

and c_1 *is the same as in Lemma* 3.15.

Proof. Set $s = (r \sin \lambda)/8$. At first assume $B_1 \subset \{z \in A_1 \colon |z| \geqslant r + s\}$. By Lemmas 3.4 and 3.15 we get

$$P^x(X(\tau_C) \in B_1) \geqslant \sigma(x, B_1) \geqslant c_1 \delta_C^{\alpha/2}(x) |x|^{\alpha/2-\varepsilon'} r^{\varepsilon'} \int_{B_1} \frac{dz}{|z|^{d+\alpha}}.$$

We have $|z|^{-\alpha/2} \geqslant 2^{-2\alpha} \delta_C^{-\alpha/2}(z) \sin^{\alpha/2} \lambda$ for $z \in B_1$. Let us also notice that $c_\alpha^d \omega_d < 1$. Hence the inequality in the lemma holds for all sets B_1 which are contained in $\{z \in A_1 \colon |z| \geqslant r + s\}$.

Now our task is to prove the lemma for $B_1 \subset \{z \in A_1 \setminus S \colon |z| < r + s\}$. Let us assume that $x \in C_{-2}$. Then choose $z_0 \in A_1 \setminus S$ such that $|z_0| \in (r, r+s)$. Let y_0 be the point belonging to the line segment $\overline{0 z_0}$ such that $|y_0| = r - s$. Notice that $\delta_V(y_0) \geqslant \sin(\lambda/2) r/2 \geqslant s$, and so $B(y_0, s) \subset A_0$. Put $w = (|z_0| - r)/2 \leqslant s/2$. Since $\delta_V(z_0) \geqslant r \sin(\lambda/2) \geqslant s$, it is easy to check that $B(z_0, w) \subset A_1$. From Lemma 3.4 we obtain

$$P^x(X(\tau_C) \in B(z_0, w)) \geqslant \int_{A_0} P^y(X(\tau_C) \in B(z_0, w)) \, d\sigma(x, y). \tag{3.24}$$

For $y \in B(y_0, s/2)$ we have $P^y(X(\tau_C) \in B(z_0, w)) \geqslant P^y(X(\tau_{B(y_0, s)}) \in B(z_0, w))$. From this and Lemma 3.15 we infer that the right-hand side of (3.24) is bounded from below by

$$\int_{B(y_0, s/2)} \int_{B(z_0, w)} \frac{c_\alpha^d (s^2 - |y - y_0|^2)^{\alpha/2}}{(|z - y_0|^2 - s^2)^{\alpha/2} |y - z|^d} dz \frac{c_1 |x|^{\alpha/2-\varepsilon'} \delta_C^{\alpha/2}(x)}{|y|^{d+\alpha-\varepsilon'}} dy. \tag{3.25}$$

For $z \in B(z_0, w)$ we have $|z - y_0| - s = \delta_{B(y_0, s)}(z) \leqslant \delta_{B(y_0, s)}(z_0) + |z - z_0|$. The last sum is smaller than $3w \leqslant 3\delta_C(z)$, so $|z - y_0| - s < 4\delta_C(z)$. For $z \in B(z_0, w)$ and $y \in B(y_0, s/2)$ we also have $|y - z| \leqslant 3s$, $|z - y_0| + s \leqslant 4s$ and $s^2 - |y - y_0|^2 > s^2/4$.

Hence (3.25) is bounded from below by

$$\frac{c_1 c_\alpha^d 2^{-\alpha} s^\alpha |x|^{\alpha/2-\varepsilon'} \delta_C^{\alpha/2}(x) r^{\varepsilon'-\alpha/2}}{2^{2\alpha} s^{\alpha/2} 3^d s^d} \int_{B(y_0,s/2)} dy \int_{B(z_0,w)} \frac{dz}{\delta_C^{\alpha/2}(z)|z|^{d+\alpha/2}}.$$

Since $m(B(y_0, s/2)) = s^d 2^{-d} \omega_d d^{-1}$ and $s^{\alpha/2} r^{-\alpha/2} \geqslant 2^{-2\alpha} \sin^{\alpha/2} \lambda$, the lemma is true for $B_1 = B(z_0, w)$ if $x \in C_{-2}$. Consequently, the lemma holds for all sets $B_1 \subset \{z \in A_1 \backslash S: |z| < r+s\}$ when $x \in C_{-2}$.

It remains to consider the case $x \in A_{-1}$. This is left so far aside since we could not apply (3.24) because $\sigma(x, A_0) = 0$ for $x \in A_{-1}$. We only give main ideas of the proof in this case. Choose s, z_0, y_0 and w as in the case $x \in C_{-2}$. Let us adopt the notation from the proof of Lemma 3.14. In that notation $r_{-1} = 2^{-3} r \sin\lambda$. As in Lemma 3.14 at first assume $\delta_V(x) < r_{-1}$. Consider the ball $B(x', r_{-1}) \subset C$ and notice that for $z \in B(x', r_{-1})$ we have $|z| \leqslant |x| + 2r_{-1} < 3r/4$, so $B(x', r_{-1}) \cap B(y_0, s/2)$ is empty. Consequently, by the strong Markov property we have

$$(3.26)\quad P^x(X(\tau_C) \in B(z_0, w)) = E^x\big(P^{X(\tau_{B(x',r_{-1})})}(X(\tau_C) \in B(z_0, w))\big)$$
$$\geqslant \int_{B(y_0,s/2)} P^y(X(\tau_C) \in B(z_0, w)) P_{r_{-1}}(x-x', y-x')\, dy.$$

For $y \in B(y_0, s/2)$ we have $|y-x'| \leqslant 2|y|$, so

$$(3.27)\quad P_{r_{-1}}(x-x', y-x') \geqslant \frac{c_\alpha^d \delta_C^{\alpha/2}(x) r_{-1}^{\alpha/2}}{2^{d+\alpha}|y|^{d+\alpha}} \geqslant \frac{c_\alpha^d \delta_C^{\alpha/2}(x)|x|^{\alpha/2-\varepsilon'} \sin^{\alpha/2}\lambda}{2^{d+3\alpha}|y|^{d+\alpha-\varepsilon'}}.$$

Now assume $\delta_V(x) \geqslant r_{-1}$ and consider the ball $B(x, r_{-1}) \subset C$. As before, $B(x, r_{-1}) \cap B(y_0, s/2)$ is empty and we get

$$(3.28)\quad P^x(X(\tau_C) \in B(z_0, w)) \geqslant \int_{B(y_0,s/2)} P^y(X(\tau_C) \in B(z_0, w)) P_{r_{-1}}(0, y-x)\, dy.$$

For $y \in B(y_0, s/2)$ we have

$$(3.29)\quad P_{r_{-1}}(0, y-x) \geqslant \frac{c_\alpha^d r_{-1}^\alpha}{2^{d+\alpha}|y|^{d+\alpha}} \geqslant \frac{c_\alpha^d \delta_C^{\alpha/2}(x)|x|^{\alpha/2-\varepsilon'} \sin^{\alpha/2}\lambda}{2^{d+4\alpha}|y|^{d+\alpha-\varepsilon'}}.$$

From inequalities (3.26)–(3.29) it follows that $P^x(X(\tau_C) \in B(z_0, w))$ is bounded from below by (3.25) with c_1 replaced by c_3. This proves the lemma.

We can now formulate a lower bound estimate of the expectation of the exit time from the bounded cone C.

THEOREM 3.17. *There exist a constant $c = c(d, \alpha, \lambda)$ such that for all $x \in C/2$*

$$E^x(\tau_C) \geqslant c\delta_C^{\alpha/2}(x)|x|^{\alpha/2-\varepsilon'} r^{\varepsilon'}.$$

As a constant c one can take $c = c_1 C_{d,\alpha}^{-1} 4^{-d-\alpha}$, where c_1 is from Lemma 3.15.

Proof. The proof is almost the same as the proof of Proposition 3.9. Let us put $A'_1 = \{z \in A_1 \colon |z| > 3r/2\}$. For $z \in A'_1$ and $y \in C$ we have $|y-z| \geqslant |z|/4$. From (2.2) it follows that

$$E^x(\tau_C)\, C_{d,\alpha} 4^{d+\alpha} \int_{A'_1} \frac{dz}{|z|^{d+\alpha}} \geqslant \int_C G_C(x, y) \int_{A'_1} \frac{C_{d,\alpha}}{|y-z|^{d+\alpha}}\, dz\, dy = P^x\big(X(\tau_C) \in A'_1\big).$$

On the other hand, by Lemmas 3.4 and 3.15 we get

$$P^x\big(X(\tau_C) \in A'_1\big) \geqslant \sigma(x, A'_1) \geqslant c_1\, \delta_C^{\alpha/2}(x)\, |x|^{\alpha/2-\varepsilon'} r^{\varepsilon'} \int_{A'_1} |z|^{-d-\alpha}\, dz$$

and the theorem follows.

The following theorem is our main lower bound estimate of the density of the harmonic measure of C.

THEOREM 3.18. *We have*

$$f_C^x(z) \geqslant \frac{c'\, \delta_C^{\alpha/2}(x)\, |x|^{\alpha/2-\varepsilon'} r^{\varepsilon'}}{\delta_C^{\alpha/2}(z)\, |z|^{d+\alpha/2}}$$

for all $x \in C/2$ *and* $z \in V \backslash (\bar{C} \cup S)$, *where* $c' = c'(d, \alpha, \lambda)$ *and*

$$S = \big\{z = (\varrho, \varphi_1, \ldots, \varphi_{d-1}) \colon \varrho \in [r, r + (r \sin\lambda)/8),\ \varphi_1 \in [\lambda/2, \lambda)\big\}.$$

Moreover, the inequality is true for all $z \in V \backslash \bar{C}$ *when* $\delta_C^{\alpha/2}(z)$ *is replaced by* $|z|^{\alpha/2}$.

As a constant c' one can take $c' = \min(c_2, c C_{d,\alpha} 2^{-d-\alpha})$, where c_2 is from Lemma 3.16 and c is from Theorem 3.17.

Proof. For $z \in A_1 \backslash S$ the inequality follows directly from Lemma 3.16. Now let $B \subset V \backslash C_1$. By formula (2.2) we have

$$P^x\big(X(\tau_C) \in B\big) = \int_C G_C(x, y) \int_B \frac{C_{d,\alpha}}{|y-z|^{d+\alpha}}\, dz\, dy. \tag{3.30}$$

For $y \in C$ and $z \in V \backslash C_1$ we have $|y-z| \leqslant 3|z|/2$ and $|y-z| \leqslant 3\delta_C(z)$. Therefore

$$P^x\big(X(\tau_C) \in B\big) \geqslant C_{d,\alpha} 2^{d+\alpha/2} 3^{-d-\alpha} E^x(\tau_C) \int_B \frac{dz}{\delta_C^{\alpha/2}(z)\, |z|^{d+\alpha/2}}.$$

Applying Theorem 3.17 we see that the inequality in the theorem is true for $z \in V \backslash C_1$. To get the last statement in the theorem assume that $B \subset V \backslash \bar{C}$. Then again (3.30) holds and for $y \in C$ and $z \in V \backslash \bar{C}$ we have $|y-z| \leqslant 2|z|$. Hence

$$P^x\big(X(\tau_C) \in B\big) \geqslant C_{d,\alpha} 2^{-d-\alpha} E^x(\tau_C) \int_B |z|^{-d-\alpha}\, dz$$

and the theorem follows by Theorem 3.17.

Now, at the end of this section, let us give some comments on the results which are obtained. One of the aims of this paper was to investigate how estimates of the harmonic measure, the Green function and the expectation of the exit time for the bounded cone differ from those for open bounded sets with $C^{1,1}$ boundary. To simplify the discussion we focus on the estimates of the expectation of the exit time. If D is an open bounded set with $C^{1,1}$ boundary, the inequality (2.5) shows that $E^x(\tau_D)$ behaves like $\delta_D^{\alpha/2}(x)$. Unfortunately, we do not have sharp estimates for the bounded cone C, since the methods we use do not give optimal constants ε and ε' (in particular, $\varepsilon' < \varepsilon$). On the other hand, both ε and ε' tend to 0 when λ tends to 0. Therefore, for fixed d and α there exists a sufficiently narrow opening λ_0 of the cone C such that for all $\lambda \in (0, \lambda_0)$ we have $\varepsilon < \alpha/2$. Theorem 3.13 then shows that for such "narrow" cones the lower bound inequality in (2.5) does not hold. To see what are the values of ε and ε' and how "small" λ must be to have $\varepsilon < \alpha/2$ we made a numerical calculation (based on explicit expressions for the constants). For $d = 3$, $\alpha = 1$ and $\lambda = \pi/12$ we have $\varepsilon \simeq 0.34$, $\varepsilon' \simeq 0.000028$ and for $\lambda = \pi/180$ we have $\varepsilon \simeq 0.0015$, $\varepsilon' \simeq 8.4 \cdot 10^{-9}$.

4. Green function. In this section we obtain estimates of Green function of the unbounded and bounded cone. This is done by applying the results from the previous section.

At first we need an auxiliary estimate for Green function for a ball. The following lemma is a direct consequence of estimates obtained in [15].

LEMMA 4.1. *Let $u \in \mathbf{R}^d$ and $s > 0$. There exists a constant $c_1 = c_1(d, \alpha)$ such that for any $v \in B(u, 2s)$ we have*

$$\int_{B(u,s)} G_{B(u,2s)}(v, z)\, dz \geq c_1 s^{\alpha/2} \delta_{B(u,2s)}^{\alpha/2}(v).$$

Proof. From Theorem 3.4 in [15] we have

$$G_{B(u,2s)}(v, z) \geq c \min\left(\frac{1}{|v-z|^{d-\alpha}}, \frac{\delta_{B(u,2s)}^{\alpha/2}(v)\, \delta_{B(u,2s)}^{\alpha/2}(z)}{|v-z|^d}\right) \tag{4.1}$$

for $v, z \in B(u, 2s)$, where $c = c(d, \alpha)$. For $v \in B(u, 2s)$ and $z \in B(u, s)$ we obtain

$$\frac{1}{|v-z|^{d-\alpha}} \geq \frac{1}{(4s)^{d-\alpha}} \geq \frac{\delta_{B(u,2s)}^{\alpha/2}(v)}{4^d s^{d-\alpha/2}} \quad \text{and} \quad \frac{\delta_{B(u,2s)}^{\alpha/2}(v)\, \delta_{B(u,2s)}^{\alpha/2}(z)}{|v-z|^d} \geq \frac{\delta_{B(u,2s)}^{\alpha/2}(v)\, s^{\alpha/2}}{4^d s^d}.$$

Hence (4.1) yields

$$\int_{B(u,s)} G_{B(u,2s)}(v, z)\, dz \geq c 4^{-d} s^{\alpha/2-d} \delta_{B(u,2s)}^{\alpha/2}(v)\, \omega_d d^{-1} s^d,$$

which gives our claim.

Now we prove the lower bound estimate of Green function for the bounded cone.

PROPOSITION 4.2. *There exists a constant $c_2 = c_2(d, \alpha, \lambda)$ such that for all $x, y \in C/2$ we have*

$$G_C(x, y) \geqslant c_2 \min\left(\frac{1}{|x-y|^{d-\alpha}}, \frac{\delta_C^{\alpha/2}(x)\,\delta_C^{\alpha/2}(y)}{|x-y|^d}\left(\frac{\min(|x|, |y|)}{\max(|x|, |y|)}\right)^{\alpha/2-\varepsilon'}\right).$$

Proof. Let $x, y \in C/2$. Since $G_C(x, y) = G_C(y, x)$, we may and do assume that $|y| \geqslant |x|$. We will consider two cases: $4|x| < |y|$ and $4|x| \geqslant |y| \geqslant |x|$.

At first let us assume that $4|x| < |y|$. We define $U = V \cap B(0, |y|/2)$. Let us notice that $x \in U/2$. By formula (2.1) and Theorem 3.18 (applied for U instead of C) we obtain

$$(4.2)\quad G_C(x, y) \int_{U^c} G_C(z, y) f_U^x(z)\,dz$$
$$\geqslant c'\,\delta_C^{\alpha/2}(x)\,|x|^{\alpha/2-\varepsilon'}\,|y|^{\varepsilon'}\,2^{-\varepsilon'} \int_{U^c} G_C(z, y)\,|z|^{-d-\alpha}\,dz.$$

We will now proceed similarly to the proof of Lemma 3.14. Set $s = (|y| \sin \lambda)/8$. We begin by considering the case $\delta_V(y) < 2s$. Let y^* be the point on ∂V such that $|y-y^*| = \delta_V(y)$. It is easy to see that the half-line $l = \{z = (\varrho, \varphi_1, \ldots, \varphi_{d-1})\colon \varphi_1 = 0\}$ is contained in the plane determined by points 0, y and y^*. Denote by $\tilde{y}$ the point of intersection of l and the line determined by y, y^*. Let y' be the point within the line segment $\overline{\tilde{y}y}$ such that $|y'-y^*| = 2s$. Since $|\tilde{y}-y^*| = |\tilde{y}| \sin \lambda \geqslant |y| \sin \lambda > 2s$, such a point exists and it is easy to notice that $\delta_V(y') = |y'-y^*|$. Consequently, $y \in B(y', 2s) \subset V$. If $z \in B(y', 2s)$, then $|z| \geqslant |y| - |y-z| \geqslant |y| - 4s > |y|/2$, so we have $B(y', 2s) \subset U^c$. On the other hand, for $z \in B(y', 2s)$ we have $|z| \leqslant |y| + 4s < 2|y| < r$, so $B(y', 2s) \subset C$. Hence

$$(4.3)\quad \int_{U^c} G_C(z, y)\,|z|^{-d-\alpha}\,dz \geqslant \int_{B(y',s)} G_{B(y',2s)}(z, y)\,|z|^{-d-\alpha}\,dz.$$

From Lemma 4.1 and the inequality $|z| < 2|y|$ (for $z \in B(y', s)$) we obtain

$$(4.4)\quad \int_{B(y',s)} G_{B(y',2s)}(z, y)\,|z|^{-d-\alpha}\,dz \geqslant 2^{-d-\alpha} c_1\,|y|^{-d-\alpha}\,s^{\alpha/2}\,\delta_{B(y',2s)}^{\alpha/2}(y).$$

We have $\delta_{B(y',2s)}(y) = \delta_C(y)$ and $|y| < 2|x-y|$ (because we have assumed that $4|x| < |y|$). Therefore (4.2)–(4.4) yield

$$(4.5)\quad G_C(x, y) \geqslant c' c_1\, 2^{-2d-3\alpha} (\sin^{\alpha/2} \lambda)\, \delta_C^{\alpha/2}(x)\, \delta_C^{\alpha/2}(y)\, |x|^{\alpha/2-\varepsilon'}\, |y|^{\varepsilon'-\alpha/2}\, |x-y|^{-d}.$$

Now let us consider the case $\delta_V(y) \geqslant 2s$. If $z \in B(y, 2s)$, then $|z| \geqslant |y| - 2s > |y|/2$ and $|z| \leqslant |y| + 2s < 2|y| < r$, so $B(y, 2s) \subset C \cap U^c$. From this and Lemma 4.1 we obtain

$$(4.6)\quad \int_{U^c} G_C(z, y)\,|z|^{-d-\alpha}\,dz$$
$$\geqslant \int_{B(y,s)} G_{B(y,2s)}(z, y)\,|z|^{-d-\alpha}\,dz \geqslant 2^{-d-\alpha} c_1\,|y|^{-d-\alpha}\,s^{\alpha/2}\,\delta_{B(y,2s)}^{\alpha/2}(y).$$

We have $\delta_{B(y,2s)}(y) = 2s = (|y| \sin\lambda)/4 \geqslant \delta_C(y)/4$ and as before $|y| < 2|x-y|$. Hence from (4.2) and (4.6) it follows that

$$G_C(x, y) \geqslant c'c_1 2^{-2d-4\alpha} (\sin^{\alpha/2}\lambda)\, \delta_C^{\alpha/2}(x)\, \delta_C^{\alpha/2}(y)\, |x|^{\alpha/2-\varepsilon'} |y|^{\varepsilon'-\alpha/2} |x-y|^{-d}.$$

When $4|x| < |y|$, the inequality above and (4.5) give the assertion of the proposition.

Now we assume $4|x| \geqslant |y| \geqslant |x|$. In this case the proposition follows from estimates of Green function of open bounded sets with $C^{1,1}$ boundary. Let us put

$$U_1 = \{z = (\varrho, \varphi_1, \ldots, \varphi_{d-1}):\ 1/8 < \varrho < 5/4,\ 0 < \varphi_1 < \lambda\}$$

and

$$U_2 = \{z = (\varrho, \varphi_1, \ldots, \varphi_{d-1}):\ 1/16 < \varrho < 3/2,\ 0 < \varphi_1 < \lambda\}.$$

Let D be a fixed domain with a $C^{1,1}$ boundary such that $U_1 \subset D \subset U_2$. Let us consider the domain $|y|D$. We have $|y|D \subset |y|U_2 \subset B(0, 3|y|/2)$, so $|y|D \subset C$. It is also clear that $x, y \in |y|D$. By the scaling property of Green function and the inequality (2.3) we obtain

$$\begin{aligned} (4.7)\qquad G_C(x, y) &\geqslant G_{|y|D}(x, y) = |y|^{\alpha-d} G_D(x/|y|, y/|y|) \\ &\geqslant c|y|^{\alpha-d} \min\left(\frac{1}{\big|x/|y| - y/|y|\big|^{d-\alpha}}, \frac{\delta_D^{\alpha/2}(x/|y|)\, \delta_D^{\alpha/2}(y/|y|)}{\big|x/|y| - y/|y|\big|^{d}}\right) \\ &= c \min\left(\frac{1}{|x-y|^{d-\alpha}}, \frac{\delta_{|y|D}^{\alpha/2}(x)\, \delta_{|y|D}^{\alpha/2}(y)}{|x-y|^{d}}\right), \end{aligned}$$

where $c = c(D, \alpha)$. Let us notice that $\delta_{|y|D}(y) = \min(\delta_V(y), \delta_{(V\setminus|y|D)}(y))$. We have $\delta_V(y) \leqslant |y|$ and $\delta_{(V\setminus|y|D)}(y) \geqslant |y|/4$. Hence $\delta_{|y|D}(y) \geqslant \min(\delta_V(y), |y|/4) \geqslant \delta_V(y)/4$. But $\delta_V(y) = \delta_C(y)$, so finally $\delta_{|y|D}(y) \geqslant \delta_C(y)/4$. Similarly, $\delta_{|y|D}(x) \geqslant \delta_C(x)/8$. Therefore, when $4|x| \geqslant |y| \geqslant |x|$, the proposition follows from (4.7).

We will need the following auxiliary fact:

LEMMA 4.3. *Let D be an open nonempty bounded set with $C^{1,1}$ boundary. For $R \in (0, \infty)$ let us define $W = \{z \in D^c\colon \operatorname{dist}(z, D) < R\}$. Then we have*

$$\int_W \frac{dz}{\delta_D^{\alpha/2}(z)} < c_3, \qquad \text{where } c_3 = c_3(D, R, \alpha).$$

Proof. It is possible to obtain the above result directly, but it will be convenient to use the estimate (2.4) instead. Fix $x_0 \in D$. From (2.4) we have for any $z \in W$

$$f_D^{x_0}(z) \geqslant \frac{c\delta_D^{\alpha/2}(x_0)}{\delta_D^{\alpha/2}(z)\big(1+\delta_D^{\alpha/2}(z)\big)|x_0 - z|^d} \geqslant \frac{c\delta_D^{\alpha/2}(x_0)}{\delta_D^{\alpha/2}(z)(1+R^{\alpha/2})\big(\operatorname{diam}(D)+R\big)^d},$$

where $c = c(D, \alpha)$. It follows that

$$1 \geqslant \int_W f_D^{x_0}(z)\,dz \geqslant \frac{c\delta_D^{\alpha/2}(x_0)}{(1+R^{\alpha/2})\big(\operatorname{diam}(D)+R\big)^d} \int_W \frac{dz}{\delta_D^{\alpha/2}(z)},$$

which completes the proof.

The following proposition may be treated as the local version of the upper bound estimate in the inequality (2.3).

PROPOSITION 4.4. *Let B be an open set, $B \neq \boldsymbol{R}^d$, and $D \subset B$ be an open bounded set with $C^{1,1}$ boundary. For $s > 0$ define $U = \{z \in D\colon \operatorname{dist}(z, B \backslash D) > s\}$. Then there exists a constant $c_4 = c_4(B, D, s, \alpha)$ such that for all $x, y \in U$ we have*

$$G_B(x, y) \leqslant c_4 \frac{\delta_B^{\alpha/2}(x)\,\delta_B^{\alpha/2}(y)}{|x-y|^d}.$$

This estimate is especially interesting for such $x, y \in U$ which are "near" $\partial D \cap \partial B$. Of course, in general, $\partial D \cap \partial B$ may be empty.

Proof. Let $x, y \in U$. For $x = y$ the proposition holds trivially, so we may and do assume that $x \neq y$. By the definition of Green function we get

$$G_B(x, y) - G_D(x, y) = \int_{D^c} u(z, y)\,d\omega_D^x(z) - \int_{B^c} u(z, y)\,d\omega_B^x(z). \tag{4.8}$$

We have $\int_{B^c} u(z, y)\,d\omega_B^x(z) = E^x u\big(X(\tau_B), y\big)$ adopting the convention (see [5]) that $u\big(X(\tau_B), y\big) = 0$ for $\tau_B = \infty$. According to Lemma 2.4 in [15] we have

$$E^x u\big(X(\tau_B), y\big) = E^x\big(E^{X(\tau_D)} u\big(X(\tau_B), y\big)\big) = \int_{D^c}\int_{B^c} u(w, y)\,d\omega_B^z(w)\,d\omega_D^x(z).$$

Hence (4.8) yields

$$G_B(x, y) - G_D(x, y) = \int_{D^c} \Big(u(z, y) - \int_{B^c} u(w, y)\,d\omega_B^z(w)\Big) f_D^x(z)\,dz. \tag{4.9}$$

We next claim that

$$u(z, y) - \int_{B^c} u(w, y)\,d\omega_B^z(w) = G_B(z, y) \tag{4.10}$$

for $z \in D^c$ except possibly on a set of Lebesgue measure zero. For $z \in B$, (4.10) is exactly the definition of $G_B(z, y)$. For $z \in B^c$ we have $G_B(z, y) = 0$, so we want to show that also the left-hand side of (4.10) equals zero. To do this we need to introduce the definition of regular points.

For $A \in \mathscr{B}(\boldsymbol{R}^d)$ let us put $T_A = \inf\{t > 0\colon X_t \in A\}$. For each $x \in \boldsymbol{R}^d$, $P^x(T_A = 0)$ is either zero or one according to the Blumenthal zero-one law. A point $x \in \boldsymbol{R}^d$ is called *regular* for $A \in \mathscr{B}(\boldsymbol{R}^d)$ if $P^x(T_A = 0) = 1$, and x is called *irregular* for A if $P^x(T_A = 0) = 0$. We denote by A^r the set of all points which are regular for the set A. It is known [5] that the set $A \backslash A^r$ is of Lebesgue measure zero.

For $z \in (B^c)^r$ we have $\int_{B^c} u(w, y)\, d\omega_B^z(w) = u(z, y)$. Since $B^c \backslash (B^c)^r$ is of Lebesgue measure zero, (4.10) holds. By (4.9) and (4.10) we obtain

$$G_B(x, y) = G_D(x, y) + \int_{D^c} G_B(z, y)\, f_D^x(z)\, dz. \tag{4.11}$$

From (2.3) we have

$$G_D(x, y) \leqslant c\, \frac{\delta_D^{\alpha/2}(x)\, \delta_D^{\alpha/2}(y)}{|x-y|^d} \leqslant c\, \frac{\delta_B^{\alpha/2}(x)\, \delta_B^{\alpha/2}(y)}{|x-y|^d}, \quad \text{where } c = c(D, \alpha).$$

So, having (4.11), it is clear that to prove the proposition it suffices to show

$$\int_{B\backslash D} G_B(z, y)\, f_D^x(z)\, dz \leqslant a_1 \frac{\delta_B^{\alpha/2}(x)\, \delta_B^{\alpha/2}(y)}{|x-y|^d}, \tag{4.12}$$

where $a_1 = a_1(B, D, s, \alpha)$. To obtain (4.12) we first show that there exists a constant $a_2 = a_2(B, D, s, \alpha)$ such that for all $z \in B\backslash D$ we have

$$G_B(z, y) \leqslant a_2\, \delta_B^{\alpha/2}(y). \tag{4.13}$$

When $\delta_B(y) \geqslant s/2$, (4.13) is easy. Since $z \in B\backslash D$ and $y \in U$, we have $|z-y| \geqslant s$. Consequently,

$$G_B(z, y) \leqslant \frac{A_{d,\alpha}}{|z-y|^d} \leqslant \frac{A_{d,\alpha}\, \delta_B^{\alpha/2}(y)}{(s/2)^{\alpha/2}\, |z-y|^d} \leqslant \frac{2^{\alpha/2} A_{d,\alpha}\, \delta_B^{\alpha/2}(y)}{s^{d+\alpha/2}}.$$

We now turn to the case $\delta_B(y) < s/2$. Let y^* be the point on ∂B such that $|y-y^*| = \delta_B(y)$. We have $y^* \in B(y, s)$. Since $y \in U$, the set $B(y, s) \cap (B\backslash D)$ is empty, which yields $y^* \in \partial D$.

Now we recall one of the geometric properties of the open bounded set with $C^{1,1}$ boundary. Namely, it is well known (see [18]) that there exists a constant $r_0 = r_0(D)$ such that for any $w \in \partial D$ and $r_1 \in (0, r_0]$ there exists a ball $B(w', r_1)$ (with w' depending on w and r_1) such that $B(w', r_1) \subset \boldsymbol{R}^d \backslash \bar{D}$ and $w \in \partial B(w', r_1)$.

Set $r_1 = \min(r_0, s/4)$. Then there exists y' such that $B(y', r_1) \subset \boldsymbol{R}^d \backslash \bar{D}$ and $y^* \in \partial B(y', r_1)$. If $z \in B(y', r_1)$, then $|z-y| \leqslant |z-y^*| + |y^*-y| < s$, so $B(y', r_1) \subset B(y, s) \subset (B\backslash D)^c$. Since $B(y', r_1) \subset \boldsymbol{R}^d \backslash \bar{D}$, we obtain $B(y', r_1) \subset B^c$.

We also need an estimate of Green function of a complement of a ball. Let $w \in \boldsymbol{R}^d$ and $t > 0$. According to Lemma 2.5 in [10] there exists a constant $a_3 = a_3(d, \alpha, t)$ such that

$$G_{B^c(w,t)}(u, v) \leqslant a_3\, |u-w|^{\alpha/2} \frac{\delta_{B(w,t)}^{\alpha/2}(v)}{|u-v|^{d-\alpha/2}}, \quad u, v \in B^c(w, t).$$

Applying this estimate to $G_{B^c(y', r_1)}(z, y)$ we obtain

$$G_{B^c(y', r_1)}(z, y) \leqslant a_3\, |z-y'|^{\alpha/2} \frac{\delta_{B(y', r_1)}^{\alpha/2}(y)}{|z-y|^{d-\alpha/2}}, \tag{4.14}$$

where $a_3 = a_3(d, \alpha, r_1)$. Since $B(y', r_1) \subset B^c$, we have $G_B(z, y) \leqslant G_{B^c(y',r_1)}(z, y)$. We also have $\delta_{B(y',r_1)}(y) = |y-y^*| = \delta_B(y)$. Recall that $|z-y| \geqslant s$. Notice that $|y-y'| \leqslant |y-y^*|+|y^*-y'| < 3s/4 < |z-y|$. This implies $|z-y'| \leqslant |z-y|+ +|y-y'| \leqslant 2|z-y|$. Hence (4.14) yields $G_B(z, y) \leqslant a_3 2^{\alpha/2} s^{\alpha-d} \delta_B^{\alpha/2}(y)$, which gives (4.13).

Our next aim is to prove (4.12). From (2.4) we have

$$f_D^x(z) \leqslant \frac{a_4 \delta_B^{\alpha/2}(x)}{\delta_D^{\alpha/2}(z)(1+\delta_D^{\alpha/2}(z))|x-z|^d}, \qquad z \in D^c, \tag{4.15}$$

where $a_4 = a_4(D, \alpha)$. We divide the set $B \backslash D$ into two sets $W_1 = \{z \in B\backslash D\colon \operatorname{dist}(z, D) \leqslant \operatorname{diam}(D)\}$ and $W_2 = \{z \in B\backslash D\colon \operatorname{dist}(z, D) > \operatorname{diam}(D)\}$. By (4.13), (4.15) and Lemma 4.3 we obtain

$$\begin{aligned} \int_{W_1} G_B(z, y) f_D^x(z)\, dz &\leqslant a_2 a_4 \delta_B^{\alpha/2}(y) \delta_D^{\alpha/2}(x) s^{-d} \int_{W_1} \delta_D^{-\alpha/2}(z)\, dz \\ &\leqslant \frac{a_2 a_4 c_3 (\operatorname{diam}(D))^d}{s^d} \frac{\delta_B^{\alpha/2}(x) \delta_B^{\alpha/2}(y)}{|x-y|^d}. \end{aligned} \tag{4.16}$$

For $z \in W_2$ we have $|z-x| \leqslant \delta_D(z)+\operatorname{diam}(D) \leqslant 2\delta_D(z)$. From this and inequalities (4.13), (4.15) we get

$$\int_{W_2} G_B(z, y) f_D^x(z)\, dz \leqslant a_2 a_4 2^\alpha (\operatorname{diam}(D))^d \frac{\delta_B^{\alpha/2}(x) \delta_B^{\alpha/2}(y)}{|x-y|^d} \int_{W_2} \frac{dz}{|z-x|^{d+\alpha}}. \tag{4.17}$$

The last integral is bounded from above by $\omega_d \alpha^{-1} (\operatorname{diam}(D))^{-\alpha}$, so (4.16) and (4.17) give (4.12) and the proposition is proved.

Now we need the following simple estimate of Green function of a half-space:

LEMMA 4.5. *Let $H = \{x = (x_1, \ldots, x_d) \in \mathbf{R}^d\colon x_1 > 0\}$. There exists a constant $c_5 = c_5(d, \alpha)$ such that for any $x, y \in H$ we have*

$$G_H(x, y) \leqslant c_5 \frac{\delta_H^{\alpha/2}(x)}{|x-y|^{d-\alpha/2}}.$$

Proof. From [10], Lemma 2.4, we know that there exists a constant $c = c(d, \alpha)$ such that for any ball $B(z, r)$, $z \in \mathbf{R}^d$, $r > 0$, and all $x, y \in B(z, r)$ we have

$$G_{B(z,r)}(x, y) \leqslant c \frac{\delta_{B(z,r)}^{\alpha/2}(x)}{|x-y|^{d-\alpha/2}}. \tag{4.18}$$

For $n \in N$ set $z_n = (n, 0, \ldots, 0) \in \mathbf{R}^d$ and $r_n = n$. Clearly, we have $B(z_n, r_n) \subset H$ and $\tau_{B(z_n,r_n)} \leqslant \tau_H$. From (4.18) we obtain

$$G_{B(z_n,r_n)}(x, y) \leqslant c \frac{\delta_{B(z_n,r_n)}^{\alpha/2}(x)}{|x-y|^{d-\alpha/2}}, \qquad x, y \in B(z_n, r_n). \tag{4.19}$$

By well-known results for the distribution of the exit time from the half-line of the one-dimensional symmetric stable process [4] we infer that for each $x \in H$ we have $P^x(\tau_H < \infty) = 1$. Therefore by the quasi-left continuity of our process X_t we get $\lim_{n\to\infty} X(\tau_{B(z_n,r_n)}) = X(\tau_H)$ a.s. P^x for all $x \in H$. Hence for all $x, y \in H$ we have $\lim_{n\to\infty} G_{B(z_n,r_n)}(x, y) = G_H(x, y)$ and the lemma follows from (4.19).

We can now state the main result of this section.

THEOREM 4.6. *There exist constants* $c_6 = c_6(d, \alpha, \lambda)$ *and* $c_7 = c_7(d, \alpha, \lambda)$ *such that for all* $x, y \in V$ *we have*

$$G_V(x, y) \geqslant c_6 \min\left(\frac{A_{d,\alpha}}{|x-y|^{d-\alpha}}, \frac{\delta_V^{\alpha/2}(x)\,\delta_V^{\alpha/2}(y)}{|x-y|^d}\left(\frac{\min(|x|, |y|)}{\max(|x|, |y|)}\right)^{\alpha/2-\varepsilon'}\right),$$

$$G_V(x, y) \leqslant \min\left(\frac{A_{d,\alpha}}{|x-y|^{d-\alpha}}, c_7\frac{\delta_V^{\alpha/2}(x)\,\delta_V^{\alpha/2}(y)}{|x-y|^d}\left(\frac{\min(|x|, |y|)}{\max(|x|, |y|)}\right)^{\alpha/2-\varepsilon}\right).$$

Proof. The lower bound estimate follows directly from Proposition 4.2 and the fact that $G_V(x, y) \geqslant G_C(x, y)$, so we only have to prove the upper bound estimate.

The inequality $G_V(x, y) \leqslant A_{d,\alpha}|x-y|^{\alpha-d}$ is obvious; hence it remains to show the inequality with the second term under the minimum. Since $G_V(x, y) = G_V(y, x)$, we may and do assume that $|y| \geqslant |x|$. As in the proof of Proposition 4.2 we will consider two cases: $4|x| < |y|$ and $4|x| \geqslant |y| \geqslant |x|$.

Let us first assume that $4|x| < |y|$. We begin with the observation that

$$G_V(z, y) \leqslant a_1 \frac{\delta_V^{\alpha/2}(y)}{|z-y|^{d-\alpha/2}}, \qquad z, y \in V, \tag{4.20}$$

where $a_1 = a_1(d, \alpha)$. Indeed, let $y^* \in \partial V$ be such that $|y-y^*| = \delta_V(y)$. It is clear that there exists a half-space H such that $V \subset H$ and the line determined by $0, y^*$ is contained in ∂H. Therefore the inequality (4.20) follows from Lemma 4.5.

Define $U = V \cap B(0, |y|/2)$ and notice that $x \in U/2$. By formula (2.1), inequality (4.20) and Theorem 3.12 we obtain

$$(4.21)\quad G_V(x, y) = \int_{U^c} G_V(z, y)\, d\omega_U^x(z)$$

$$\leqslant c\delta_V^{\alpha/2}(x)|x|^{\alpha/2-\varepsilon}2^{-\varepsilon}|y|^{\varepsilon} a_1 \delta_V^{\alpha/2}(y) \int_{V\setminus\bar{U}} \frac{dz}{\delta_U^{\alpha/2}(z)|z|^{d+\alpha/2}|z-y|^{d-\alpha/2}}.$$

Consequently, to get the upper bound estimate in the theorem (in case $4|x| < |y|$) it is sufficient to show that the last integral in (4.21) is bounded from above by $c|y|^{-d-\alpha/2}$, where $c = c(d, \alpha)$, and to notice that $|y| \geqslant (4/5)|x-y|$.

In order to estimate the last integral in (4.21) we divide $V\setminus\bar{U}$ into three sets

$$W_1 = \{z \in V\colon\ |z| \in (|y|/2, 3|y|/4]\},$$

$$W_2 = \{z \in V\colon\ |z| \in (3|y|/4, 2|y|)\} \quad \text{and} \quad W_3 = \{z \in V\colon\ |z| \geqslant 2|y|\}.$$

For abbreviation let us put

$$I_i = \int_{W_i} \frac{dz}{\delta_U^{\alpha/2}(z)\,|z|^{d+\alpha/2}\,|z-y|^{d-\alpha/2}}, \qquad i = 1, 2, 3.$$

For $z \in W_1$ we have $|z-y| \geqslant |y|/4$, so

$$I_1 \leqslant \frac{2^{d+\alpha/2}\,4^{d-\alpha/2}}{|y|^{d+\alpha/2}\,|y|^{d-\alpha/2}} \int_{W_1} \frac{dz}{\delta_U^{\alpha/2}(z)} \leqslant \frac{2^{3d-\alpha/2}\,\omega_d}{|y|^{2d}} \int_{|y|/2}^{3|y|/4} \frac{\varrho^{d-1}}{(\varrho-|y|/2)^{\alpha/2}}\,d\varrho$$

$$\leqslant \frac{2^{3d-\alpha/2}\,\omega_d}{|y|^{d+1}} \int_0^{|y|/4} \frac{d\varrho}{\varrho^{\alpha/2}} = \frac{2^{3d+\alpha/2-2}\,\omega_d}{(1-\alpha/2)\,|y|^{d+\alpha/2}}.$$

Let us notice that $W_2 \subset B(y, 3|y|)$ and $\delta_U(z) \geqslant |y|/4$ for $z \in W_2$. Hence we obtain

$$I_2 \leqslant \frac{4^{\alpha/2}\,4^{d+\alpha/2}}{|y|^{\alpha/2}\,3^{d+\alpha/2}\,|y|^{d+\alpha/2}} \int_{B(y,3|y|)} \frac{dz}{|z-y|^{d-\alpha/2}} = \frac{4^{d+\alpha}\,\omega_d}{3^d\,(\alpha/2)\,|y|^{d+\alpha/2}}.$$

For $z \in W_3$ we have $|z-y| \geqslant |z|/2$ and $\delta_U(z) \geqslant |z|/2$. Therefore

$$I_3 \leqslant \int_{B^c(0,2|y|)} \frac{2^d\,dz}{|z|^{2d+\alpha/2}} = \frac{1}{2^{\alpha/2}\,(d+\alpha/2)\,|y|^{d+\alpha/2}}.$$

This proves the upper bound estimate of $G_V(x, y)$ in the case $4|x| < |y|$.

Now let us assume that $4|x| \geqslant |y| \geqslant |x|$. In this case the upper bound estimate of $G_V(x, y)$ follows from Proposition 4.4. As in the proof of Proposition 4.2 we put

$$U_1 = \{z = (\varrho, \varphi_1, \ldots, \varphi_{d-1})\colon\ 1/8 < \varrho < 5/4,\ 0 < \varphi_1 < \lambda\}$$

and

$$U_2 = \{z = (\varrho, \varphi_1, \ldots, \varphi_{d-1})\colon\ 1/16 < \varrho < 3/2,\ 0 < \varphi_1 < \lambda\}.$$

Let D be a fixed domain with a $C^{1,1}$ boundary such that $U_1 \subset D \subset U_2$. Let us also put $U = \{z \in D\colon\ \operatorname{dist}(z, V\backslash D) > 1/16\}$. It is easy to check that $x/|y| \in U$ and $y/|y| \in U$. By Proposition 4.4 and the scaling property of Green function we obtain

$$G_V(x, y) = |y|^{\alpha-d}\,G_V(x/|y|, y/|y|)$$

$$\leqslant c_4\,|y|^{\alpha-d}\,\frac{\delta_V^{\alpha/2}(x/|y|)\,\delta_V^{\alpha/2}(y/|y|)}{\big|x/|y|-y/|y|\big|^d} = c_4\,\frac{\delta_V^{\alpha/2}(x)\,\delta_V^{\alpha/2}(y)}{|x-y|^d}.$$

This completes the proof of the theorem.

As a simple corollary to the results proved in this section we can formulate lower and upper bound estimates of the Green function of the bounded cone C. Since apart from its vertex the bounded cone C has other "singularities" at $\partial V \cap \partial B(0, r)$, we state our estimates of $G_C(x, y)$ only for $x, y \in C/2$.

THEOREM 4.7. *There exist constants* $c_8 = c_8(d, \alpha, \lambda)$ *and* $c_9 = c_9(d, \alpha, \lambda)$ *such that for all* $x, y \in C/2$ *we have*

$$G_C(x, y) \geqslant c_8 \min\left(\frac{A_{d,\alpha}}{|x-y|^{d-\alpha}}, \frac{\delta_C^{\alpha/2}(x)\,\delta_C^{\alpha/2}(y)}{|x-y|^d}\left(\frac{\min(|x|, |y|)}{\max(|x|, |y|)}\right)^{\alpha/2-\varepsilon'}\right),$$

$$G_C(x, y) \leqslant \min\left(\frac{A_{d,\alpha}}{|x-y|^{d-\alpha}}, c_9\frac{\delta_C^{\alpha/2}(x)\,\delta_C^{\alpha/2}(y)}{|x-y|^d}\left(\frac{\min(|x|, |y|)}{\max(|x|, |y|)}\right)^{\alpha/2-\varepsilon}\right).$$

Moreover, the upper bound remains true for all $x, y \in C$ *if we replace* $\delta_C^{\alpha/2}(x)\,\delta_C^{\alpha/2}(y)$ *by* $\delta_V^{\alpha/2}(x)\,\delta_V^{\alpha/2}(y)$.

The lower bound estimate follows from Proposition 4.2 and the upper bound estimate follows from Theorem 4.6 and the inequality $G_C(x, y) \leqslant G_V(x, y)$. Using formula (2.2) and Theorem 4.7 one can obtain estimates of the density of the harmonic measure of the bounded and unbounded cones. For example, one can obtain some estimates of the growth of $f_C^x(z)$ when $x \in C/2$ is fixed and $z \in V^c$ tends to the vertex of the cone.

5. Exit time. It is easy to check that $E^x(\tau_V) = \infty$ for $x \in V$. This follows for example from Theorem 3.17. The aim of this section is to investigate for which $p \in (0, 1)$, $E^x(\tau_V^p)$ is finite for $x \in V$. Theorem 3.2 in [3] gives an analytic condition for the finiteness of $E^x(\tau_V^p)$. This theorem, which is an analogue of the classical result of Burkholder [9], states that for any region $D \subset \mathbf{R}^d$, $x \in D$ and $p \in (0, 1)$ we have $E^x(\tau_D^p) < \infty$ if and only if there is a function u which is α-harmonic on D and $u(x) \geqslant |x|^{p\alpha}$ for all x. We found it difficult to check this last condition. Instead we give in this section direct estimates of the critical value $p_0 = p_0(d, \alpha, \lambda)$ such that for all $x \in V$ we have $E^x(\tau_V^p) < \infty$ for $0 \leqslant p < p_0$ and $E^x(\tau_V^p) = \infty$ for $p > p_0$. This is done by applying our previous results.

Throughout the whole section we assume that $r = 1$ in the definition of the bounded cone C, i.e. $C = V \cap B(0, 1)$. As in Section 3 we put $C_k = 2^k C = V \cap B(0, 2^k)$ for $k = 0, 1, 2, \ldots$ Before formulating our main result of this section we will prove two auxiliary lemmas.

LEMMA 5.1. *There exists a constant* $c = c(d, \alpha)$ *such that for all* $x \in C/2$ *and* $k \in N$ *we have*

$$P^x\big(X(\tau_{C_{k-1}}) \in V\big) \leqslant c\,2^{k(\varepsilon-\alpha)} \quad \text{and} \quad E^x(\tau_{C_k}) \leqslant c\,2^{k\varepsilon}.$$

Proof. By Theorem 3.12 and the scaling property of the harmonic measure we obtain

$$P^x\big(X(\tau_{C_{k-1}}) \in V\big) = P^{x/(2^{k-1})}\big(X(\tau_C) \in V\big)$$

$$\leqslant c\left|\frac{x}{2^{k-1}}\right|^{\alpha-\varepsilon} \int\limits_{V\setminus C} \frac{dz}{\delta_C^{\alpha/2}(z)\,|z|^{d+\alpha/2}} \leqslant c\,2^{(k-1)(\varepsilon-\alpha)} \int\limits_{V\setminus C} \frac{dz}{\delta_C^{\alpha/2}(z)\,|z|^{d+\alpha/2}},$$

where $c = c(d, \alpha)$. To get the first inequality in the lemma it suffices to show that the last integral is finite. To obtain this we divide this integral into two: one over $(2C)\backslash C$ and the other over $V\backslash(2C)$. We have

$$\int_{(2C)\backslash C} \frac{dz}{\delta_C^{\alpha/2}(z)\,|z|^{d+\alpha/2}} \leqslant \omega_d \int_1^2 \frac{\varrho^{d-1}}{(\varrho-1)^{\alpha/2}\,\varrho^{d+\alpha/2}}\,d\varrho \leqslant \frac{\omega_d}{1-\alpha/2}$$

and

$$\int_{V\backslash(2C)} \frac{dz}{\delta_C^{\alpha/2}(z)\,|z|^{d+\alpha/2}} \leqslant \int_{V\backslash(2C)} \frac{2^{\alpha/2}\,dz}{|z|^{d+\alpha}} = 2^{-\alpha/2}\,\alpha^{-1}\,\omega_d.$$

The second inequality in the lemma follows directly from Theorem 3.13. Indeed, from Theorem 3.13 and the scaling property of the exit time we obtain

$$E^x(\tau_{C_k}) = 2^{k\alpha}\,E^{x/2^k}(\tau_C) \leqslant c2^{k\alpha}\,|x|^{\alpha-\varepsilon}\,2^{k(\varepsilon-\alpha)} \leqslant c\,2^{k\varepsilon}, \quad \text{where } c = c(d, \alpha).$$

Observe that for $D \in \mathscr{B}(\mathbf{R}^d)$ we have

$$E^x(\tau_D) = \int_0^\infty P^x(\tau_D > s)\,ds \geqslant \int_0^1 P^x(\tau_D > s)\,ds, \qquad x \in \mathbf{R}^d.$$

We also have the following reverse inequality.

LEMMA 5.2. *Let D be an open bounded set. There exists a constant $c_1 = c_1(d, \alpha, m(D))$ such that for all $x \in D$ we have*

$$\int_0^1 P^x(\tau_D > s)\,ds \geqslant c_1\,E^x(\tau_D).$$

Proof. Let us recall (see Section 2) that $p_D(t, x, y)$ is the transition density for P_t^D, the semigroup generated by the process killed on exiting D. Obviously, we have $p_D(t, x, y) \leqslant p(t, x, y)$. According to [19] there exists a constant $c_2 = c_2(d, \alpha)$ such that $p(t, x, y) \leqslant c_2\,t^{-d/\alpha}$ for all $x, y \in \mathbf{R}^d$ and $t > 0$.

For $x, y \in D$ and $t > 0$ we have

$$\begin{aligned}\int_t^\infty p_D(s, x, y)\,ds &= \int_t^\infty \int_D p_D(s-t, x, z)\,p_D(t, z, y)\,dz\,ds\\ &= \int_D p_D(t, z, y) \int_t^\infty p_D(s-t, x, z)\,ds dz = \int_D p_D(t, z, y) \int_0^\infty p_D(s, x, z)\,ds\,dz\\ &= \int_D p_D(t, z, y)\,G_D(x, z)\,dz \leqslant c_2\,t^{-d/\alpha} \int_D G_D(x, z)\,dz = c_2\,t^{-d/\alpha}\,E^x(\tau_D).\end{aligned}$$

It follows that

$$\begin{aligned}\int_t^\infty P^x(\tau_D > s)\,ds &= \int_t^\infty \int_D p_D(s, x, y)\,dy ds\\ &= \int_D \int_t^\infty p_D(s, x, y)\,ds dy \leqslant c_2\,m(D)\,t^{-d/\alpha}\,E^x(\tau_D).\end{aligned}$$

Hence

$$\int_0^t P^x(\tau_D > s)\,ds = \int_0^\infty P^x(\tau_D > s)\,ds - \int_t^\infty P^x(\tau_D > s)\,ds \geqslant \left(1 - c_2\, m(D)\, t^{-d/\alpha}\right) E^x(\tau_D).$$

Let us take $t \geqslant 1$ such that $c_2\, m(D)\, t^{-d/\alpha} < 1/2$. Since $P^x(\tau_D > s)$ is not increasing, we obtain

$$\int_0^1 P^x(\tau_D > s)\,ds \geqslant \frac{1}{t}\int_0^t P^x(\tau_D > s)\,ds \geqslant \frac{1}{2t} E^x(\tau_D),$$

which completes the proof.

We can now formulate the main result of this section.

THEOREM 5.3. *There exists a constant* $p_0 = p_0(d, \alpha, \lambda) \in [1/2, 1)$ *such that for all* $x \in V$ *we have* $E^x(\tau_V^p) < \infty$ *for* $0 \leqslant p < p_0$ *and* $E^x(\tau_V^p) = \infty$ *for* $p > p_0$. *The constant* p_0 *satisfies the following inequality:*

$$\frac{\alpha - \varepsilon}{\alpha} \leqslant p_0 \leqslant \frac{\alpha - \varepsilon'}{\alpha}.$$

Let us note that an immediate conclusion from this theorem is that p_0 tends to 1 when λ tends to 0.

Proof. The fact that if $p > 0$ is fixed, then $E^x(\tau_V^p)$ is either finite for all $x \in V$ or infinite for all $x \in V$ is not difficult and well known. Indeed, this holds not only for the cone V but for an arbitrary open set. The statement that this is true for an arbitrary open region and $p \in (0, 1)$ follows from the above-mentioned Theorem 3.2 in [3].

At first we will prove that $p_0 \geqslant (\alpha - \varepsilon)/\alpha$. Let $x \in C/2$ and $p \in (0, 1)$. We have

$$E^x(\tau_V^p) = \sum_{k=1}^\infty E^x(\tau_V^p;\ \tau_V = \tau_{C_k} > \tau_{C_{k-1}}) + E^x(\tau_V^p;\ \tau_V = \tau_{C_0})$$

$$\leqslant \sum_{k=1}^\infty E^x(\tau_{C_k}^p;\ \tau_{C_k} > \tau_{C_{k-1}}) + E^x(\tau_C^p).$$

By the Hölder inequality we obtain for $k \in N$

$$E^x(\tau_{C_k}^p;\ \tau_{C_k} > \tau_{C_{k-1}}) = E^x(\tau_{C_k}^p 1_{\{\tau_{C_k} > \tau_{C_{k-1}}\}})$$

$$\leqslant \left(E^x\left((\tau_{C_k}^p)^{1/p}\right)\right)^p \left(E^x(1_{\{\tau_{C_k} > \tau_{C_{k-1}}\}}^{1/(1-p)})\right)^{1-p} \leqslant \left(E^x(\tau_{C_k})\right)^p \left(P^x\left(X(\tau_{C_{k-1}}) \in V\right)\right)^{1-p}.$$

From Lemma 5.1 we get

$$\left(E^x(\tau_{C_k})\right)^p \left(P^x\left(X(\tau_{C_{k-1}}) \in V\right)\right)^{1-p} \leqslant c\, 2^{k\varepsilon p}\, 2^{k(\varepsilon - \alpha)(1-p)} = c\, 2^{k(\varepsilon - \alpha + p\alpha)},$$

where $c = c(d, \alpha)$.

It follows that

$$E^x(\tau_V^p) \leqslant c \sum_{k=1}^{\infty} 2^{k(\varepsilon-\alpha+p\alpha)} + E^x(\tau_C^p).$$

Of course, $E^x(\tau_C^p)$ is finite. Hence if $\varepsilon-\alpha+p\alpha < 0$ (i.e. $p < (\alpha-\varepsilon)/\alpha$), then we have $E^x(\tau_V^p) < \infty$. This proves that $p_0 \geqslant (\alpha-\varepsilon)/\alpha$.

Now we are going to show that $p_0 \leqslant (\alpha-\varepsilon')/\alpha$. We will need the following equality:

$$EY^p = p \int_0^{\infty} t^{p-1} P(Y > t)\, dt,$$

which holds for an arbitrary nonnegative random variable Y and all $p > 0$. Let $x \in C/2$, $p \in (0, 1]$ and $n \in N$. We have

$$E^x(\tau_V^p) \geqslant E^x(\tau_{nC}^p) = n^{p\alpha} E^{x/n}(\tau_C^p) = n^{p\alpha} p \int_0^{\infty} t^{p-1} P^{x/n}(\tau_C > t)\, dt$$

$$\geqslant n^{p\alpha} p \int_0^{1} P^{x/n}(\tau_C > t)\, dt \geqslant c_1 p n^{p\alpha} E^{x/n}(\tau_C),$$

where $c_1 = c_1(d, \alpha, \lambda)$. The last inequality follows from Lemma 5.2. By Theorem 3.17 we obtain

$$E^{x/n}(\tau_C) \geqslant c\delta_C^{\alpha/2}\left(\frac{x}{n}\right)\left|\frac{x}{n}\right|^{\alpha/2-\varepsilon'} = c\delta_C^{\alpha/2}(x)\,|x|^{\alpha/2-\varepsilon'} n^{\varepsilon'-\alpha}, \qquad \text{where } c = c(d, \alpha, \lambda).$$

It follows that

$$E^x(\tau_V^p) \geqslant cc_1 p\delta_C^{\alpha/2}(x)\,|x|^{\alpha/2-\varepsilon'} n^{\varepsilon'-\alpha+p\alpha}.$$

This inequality holds for an arbitrary $n \in N$, so if $\varepsilon'-\alpha+p\alpha > 0$ (i.e. $p > (\alpha-\varepsilon')/\alpha$), then we have $E^x(\tau_V^p) = \infty$. This proves that $p_0 \leqslant (\alpha-\varepsilon')/\alpha$.

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