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POINT REGULARITY OF p-STABLE DENSITY IN $\mathscr{R}^d$ AND FISHER INFORMATION

BY

MACIEJ LEWANDOWSKI* (WROCŁAW)

Abstract. In the paper we prove that the n-th directional derivative of a p-stable density $f(x)$ in the direction a can be estimated by

$$|D_a^n f(x)| \leqslant \frac{C(u)}{1+|x|}[f(x)]^{(1-u)[p/(1+p)]},$$

where $0 < u < 1$, and C depends also on geometrical properties of the Lévy measure. This inequality helps us to calculate the Fisher information of stable measures.

Introduction. In this paper, $|\cdot|$ denotes Euclidean norm and $\langle \cdot, \cdot \rangle$ is a scalar product in $\mathscr{R}^d$. We say that $f(y)$ is a *density of a p-stable vector* in $\mathscr{R}^d$ $(0 < p < 2)$ if the Fourier transform of $f(x)$ is of the form

$$\int_{\mathscr{R}^d} e^{i\langle x,y\rangle} f(y)\,dy = \exp\big(-\int_{S^{d-1}} |\langle x, s\rangle|^p \sigma(ds)\big)$$

for a certain symmetric Borel measure σ. Moreover, σ is finite, positive, concentrated on S^{d-1} and $\operatorname{lin}(\operatorname{supp}\sigma) = \mathscr{R}^d$. The measure σ (called *spectral*) is unique. Let us write

$$|x|_\sigma^p = \int_{S^{d-1}} |\langle x, s\rangle|^p \sigma(ds), \qquad \tau(\sigma) = \inf_{|x|=1} |x|_\sigma^p.$$

Notice that if $\tau(\sigma) > 0$, then the Fourier transform of $f(x)$ can be approximated by $\exp\big(-\tau(\sigma)|x|^p\big)$, so that $f(x) \in C^\infty$.

It was proved in Głowacki [3] that

$$|D_{e_i}^n f(x)| \leqslant \frac{C}{1+|x|^{n+p}}$$

for some constant C. Unfortunately, this result does not allow us to answer the question how to estimate the derivatives using the density. Inspiration for this question comes from the theory of admissible translates (Kakutani Theorem)

* Institute of Mathematics, University of Technology, Wrocław.

for products of measures. In order to answer the question whether $\underline{a} = (a_1, a_2, \ldots)$ is an admissible translate of $\underline{\mu} = (f_1\,dx)\times(f_2\,dx)\times \ldots$, where f_i are densities on $\mathscr{R}^d$, one has to check if

$$\sum a_i^2 \int \frac{|\nabla f|^2}{f_i}\,dx < \infty.$$

Our main result allows us to characterize admissible translates of infinite product of 2-dimensional stable measures; this situation occurs in the case of harmonizable p-stable process.

Preliminaries. For the rest of the paper, σ will denote the spectral measure of p-stable measure $\mu(dx) = f(x)\,dx$. We will consider only finite positive Borel measures σ.

For $\tau(\sigma)$ we list some simple facts.

FACT 1. *Let M be the set of all symmetric probability measures concentrated on S^{d-1}. The function $\tau(\sigma)$: $M \to \mathscr{R}$ is continuous in the weak topology.*

Proof. If σ is fixed, the function $x \to \int_{S^{d-1}} |\langle x, s\rangle|^p \sigma(ds)$ is continuous; hence there exists $x(\sigma)\in S^{d-1}$ such that

$$\tau(\sigma) = \int |\langle x(\sigma), s\rangle|^p \sigma(ds) = \int\limits_{S^{d-1}\times S^{d-1}} |\langle y, s\rangle|^p\, d(\delta_{x(\sigma)}\times\sigma)(y, s),$$

and we have

$$\tau(\sigma) \leqslant \int |x|^p |s|^p \sigma(ds) = \sigma(S^{d-1}) = 1,$$

where $\delta_{x(\sigma)}$ denotes the unit measure concentrated at the point $x(\sigma)$.

Let $\sigma_n \xrightarrow{w} \sigma_0$ (w means "weakly") and $\tau_0 = \lim_{k\to\infty}\tau(\sigma_{n_k})$ for a subsequence n_k. We can also assume that $\lim_{k\to\infty} x(\sigma_{n_k}) = x_0$ for any $x_0\in S^{d-1}$. Thus

$$\delta_{x(\sigma_{n_k})}\times\sigma_{n_k} \xrightarrow{w} \delta_{x_0}\times\sigma_0 \quad\text{ and }\quad \int |\langle y, s\rangle|^p\, d(\delta_{x(\sigma_{n_k})}\times\sigma_{n_k}) \leqslant \int |\langle x, s\rangle|^p \sigma_{n_k}(ds)$$

for every $x\in S^{d-1}$. By the inequality

$$\lim_{k\to\infty} \int |\langle y, s\rangle|^p\, d(\delta_{x(\sigma_{n_k})}\times\sigma_{n_k}) = \int |\langle x_0, s\rangle|^p \sigma_0(ds) \leqslant \int |\langle x, s\rangle|^p \sigma_0(ds),$$

we set $\tau(\sigma_0) = \lim\tau(\sigma_{n_k})$, which completes the proof.

The next result is needed only for technical reasons.

FACT 2. *Let $Z_1, Z_2, \ldots, Z_d$ denote independent random variables with identical distributions equal to σ (σ is the normalized spectral measure). Then*:

$$\tau(\sigma) > 0 \Leftrightarrow \exists 0 < q \leqslant 1\ \exists\alpha > 0\ \ P\big(|\det(Z_1, \ldots, Z_d)| > \alpha\big) = q.$$

Proof. ($\Leftarrow$) If $\tau(\sigma) = 0$, then there exists $x_0\in S^{d-1}$ such that

$$\int |\langle x_0, s\rangle|^p \sigma(ds) = 0,$$

so that $\operatorname{supp}\sigma \subset \{x_0\}^{\perp}$. Then $P(Z_1 \in \{x_0\}^{\perp}, \ldots, Z_d \in \{x_0\}^{\perp}) = 1$, and hence $P(\det(Z_1, \ldots, Z_d) = 0) = 1$, a contradiction.

($\Rightarrow$) Let $\tau(\sigma) > 0$. We will find $x_1, x_2, \ldots, x_d$ such that $x_i \in \operatorname{supp}\sigma$ and $x_1, x_2, \ldots, x_d$ are linearly independent. Recall that

$$\operatorname{supp}\sigma = \{x\colon \forall \varepsilon > 0,\ \sigma(B(x, \varepsilon)) > 0\},$$

where $B(x, \varepsilon)$ is a ball with center x and radius ε.

Take arbitrary $x_1 \in \operatorname{supp}\sigma$, $x_1 \neq 0$. Next take $w \in S^{d-1}$ such that $\langle w, x_1 \rangle = 0$. Since also $\int |\langle w, s\rangle|^p \sigma(ds) > 0$, there exists $x_2 \in \operatorname{supp}\sigma - \operatorname{lin}\{x_1\}$, so we have found x_1 and x_2.

If we have chosen $x_1, x_2, \ldots, x_r$, then take $w \in \operatorname{lin}\{x_1, x_2, \ldots, x_r\}^{\perp}$, $w \in S^{d-1}$. Since $\int |\langle w, s\rangle|^p \sigma(ds) > 0$, there exists

$$x_{r+1} \in \operatorname{supp}\sigma - \operatorname{lin}\{x_1, x_2, \ldots, x_r\}$$

and we have found x_{r+1}. This process ends when we reach $r = d$. Of course,

$$(x_1, x_2, \ldots, x_d) \in A = \{(y_1, y_2, \ldots, y_d)\colon \det[y_1, y_2, \ldots, y_d] \neq 0\},$$

and A is open in $(\mathscr{R}^d)^d$, so there exist $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d$ such that $B(x_1, \varepsilon_1) \times \ldots \times B(x_d, \varepsilon_d) \in A$. Then

$$\begin{aligned} P(|\det(Z_1, \ldots, Z_d)| > 0) &\geqslant P(Z_1 \in B(x_1, \varepsilon_1), \ldots, Z_d \in B(x_i, \varepsilon_i)) \\ &= P(Z_1 \in B(x_1, \varepsilon_1)) \cdot \ldots \cdot P(Z_d \in B(x_d, \varepsilon_d)) > 0. \end{aligned}$$

This completes the proof.

Now we present the main tool of our proof (see [5] and [6]). Assume that $X_1, X_2, \ldots$ are i.i.d., $P(X_i \geqslant x) = \exp(-x)$, $x \geqslant 0$; $\Gamma_n = X_1 + X_2 + \ldots + X_n$; $g_1, g_2, \ldots$ are i.i.d., $g_i \sim N(0, 1)$; $Z_1, Z_2, \ldots$ are i.i.d., $Z_i \stackrel{d}{=} \sigma$; moreover, (X_i), (g_i), and (Z_i) are mutually independent. In this case, $\sum_{i=1}^{\infty} (\Gamma_i)^{-1/p} Z_i g_i$ converges a.s. and

$$P(\eta \in A) = \mathrm{E}\, P\Big(C_p \sum_{i=1}^{\infty} (\Gamma_i)^{-1/p} g_i Z_i \in A \mid \Gamma, Z\Big), \tag{1}$$

where $\eta(dx) = f(x)\,dx$ is a p-stable measure with spectral measure σ and

$$1/C_p = \int_0^{\infty} x^{-p} \sin x\, dx.$$

Since for fixed (Γ, Z) the series $\sum_{i=1}^{\infty} (\Gamma_i)^{-1/p} Z_i g_i$ is a Gaussian vector, we will need some simple facts about Gaussian distributions.

A symmetric measure μ is *Gaussian* (*stable with parameter* $p = 2$) if

$$\int \exp(i\langle x^*, x\rangle)\, \mu(dx) = \exp\big(-\tfrac{1}{2} \int \langle x^*, x\rangle^2 \mu(dx)\big), \qquad x^* \in \mathscr{R}^d.$$

The measure μ generates quadratic form on $\mathscr{R}^d$: for $\bar{x}, \bar{y} \in \mathscr{R}^d$

$$\langle \bar{x}, \bar{y} \rangle_1 = \int \langle \bar{x}, x \rangle \langle \bar{y}, x \rangle \mu(dx).$$

Then for a matrix $A = [a_{ij}]_{d \times d}$ with entries $a_{ij} = \langle e_i, e_j \rangle_1$, we have

$$Ax = \sum_{i=1}^{d} v_i \langle x, v_i \rangle \lambda_i^2$$

for some orthonormal system $\{v_1, \ldots, v_d\}$ in $\mathscr{R}^d$ and

$$\mu \stackrel{d}{=} v_1 g_1 \lambda_1 + \ldots + v_n g_d \lambda_d,$$

where g_1 are i.i.d., and $g_i \sim N(0, 1)$.

Observe that if

$$\mu \stackrel{d}{=} \sum_{i=1}^{d} g_i Z_i,$$

where Z_i are arbitrary vectors from $\mathscr{R}^d$ with $\operatorname{lin}(Z_1, \ldots, Z_d) = \mathscr{R}^d$, then $A = [Z] \cdot [Z^T]$, where $[Z]$ is a matrix with columns $Z_1, Z_2, \ldots, Z_d$.

Let us put the eigenvalues in the following order: $\lambda_1^2 \geqslant \lambda_2^2 \geqslant \ldots \geqslant \lambda_d^2$.

PROPOSITION 1 (see Bierezin [1]). *If*

$$\mu = \mathscr{L}\Big(\sum_{i=1}^{d} v_i \lambda_i g_i\Big) = \mathscr{L}\Big(\sum_{i=1}^{d} Z_i g_i\Big),$$

and $g_i \sim N(0, 1)$, $\{v_1, v_2, \ldots, v_d\}$ *is an orthonormal system*, $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_d > 0$, *then*

(a) $$\lambda_1^2 = \sup_{|x|=1} \int \langle x, n \rangle^2 \mu(dn);$$

(b) $$\lambda_d^2 = \inf_{|x|=1} \int \langle x, n \rangle^2 \mu(dn);$$

(c) $$\lambda_d^2 \geqslant (\det[Z])^2 \cdot \left(\frac{d-1}{\sum_{i=1}^{d} |Z_i|^2}\right)^{d-1}.$$

Proof. (a) and (b). We have

$$\int \langle x, n \rangle^2 \mu(dn) = \mathrm{E}\left\langle x, \sum v_i \lambda_i g_i \right\rangle^2 = \sum \langle x, v_i \rangle^2 \lambda_i^2.$$

Hence for $|x| = 1$ we obtain $\lambda_d^2 \leqslant \int \langle x, n \rangle^2 \mu(dn) \leqslant \lambda_1^2$, but

$$\int \langle v_1, n \rangle^2 \mu(dn) = \lambda_1^2 \quad \text{and} \quad \int \langle v_d, n \rangle^2 \mu(dn) = \lambda_d^2.$$

(c) Let $A = [Z] \cdot [Z]^T$, the covariance matrix of μ,

$$A = \begin{bmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_d^2 \end{bmatrix}$$

in the basis $\{v_1, \ldots, v_d\}$. Then $\operatorname{Tr} A = \lambda_1^2 + \ldots + \lambda_d^2$ and, by the inequality between arithmetic and geometric means,

$$\frac{\operatorname{Tr} A - \lambda_d^2}{d-1} = \frac{\lambda_1^2 + \ldots + \lambda_{d-1}^2}{d-1} \geqslant \sqrt[d-1]{\frac{\lambda_1^2 \cdot \ldots \cdot \lambda_d^2}{\lambda_d^2}} = \frac{\sqrt[d-1]{\det A}}{\sqrt[d-1]{\lambda_d^2}}.$$

Hence

$$\lambda_d^2 \geqslant (\det A) \cdot \left(\frac{d-1}{\operatorname{Tr} A}\right)^{d-1}$$

and

$$\operatorname{Tr} A = \sum_{i=1}^{d} a_{ii} = \sum_{i=1}^{d} \mathrm{E} \langle e_i, \sum_j Z_j g_j \rangle^2$$
$$= \sum_{i=1}^{d} \left(\sum_j \langle e_i, Z_j \rangle^2\right) = \sum_i \sum_j \langle e_i, Z_j \rangle^2 = \sum_i |Z_i|^2,$$

which proves (c).

The main result. Now we can formulate our result.

THEOREM 1. *There exists a function $C(\cdot, \cdot, \cdot)$: $\mathscr{R}^3 \to \mathscr{R}_+$ such that, for every p-stable density $f(x)$ on $\mathscr{R}^d$ with spectral measure σ, we have*

$$\forall x \in \mathscr{R}^d \;\; \forall 0 < \alpha < 1 \;\; \forall a \in S^{d-1} \;\; \forall n \in N$$
$$|D_a^n f(x)| \leqslant \frac{C\left(\tau(\sigma), \alpha, n\right)}{1+|x|} [f(x)]^{(1-\alpha)[p/(1+p)]},$$

where D_a^n denotes the n-th directional derivative in the direction a.

The idea of proof is to look at μ as a mixture of Gaussian measures. Let

$$\mu = \mathrm{E}\, \mu_{(\Gamma, Z)}, \quad \text{where } \mu_{(\Gamma, Z)} \stackrel{d}{=} C_p \sum_{i=1}^{\infty} (\Gamma_i)^{-1/p} Z_i g_i$$

and (Γ, Z) are fixed. Let $\lambda_d(\Gamma, Z) \leqslant \ldots \leqslant \lambda_1(\Gamma, Z)$ be eigenvalues and $\left(v_k(\Gamma, Z)\right)_{k=1}^{d}$ be an orthonormal basis generated by $\mu_{(\Gamma, Z)}$.

For $\lambda_1(\Gamma, Z)$ and $\lambda_d(\Gamma, Z)$ we have two lemmas.

LEMMA 1. $\mathrm{E}\, \lambda_1^{\beta}(\Gamma, Z) \leqslant C_p^2\, \mathrm{E}\left(\sum_{i=1}^{\infty} \Gamma^{-2/p}\right)^{\beta/2} < \infty$ *for* $0 \leqslant \beta < p$.

Proof. We have

$$X_{(\Gamma, Z)} = C_p \sum_{i=1}^{\infty} (\Gamma_i)^{-1/p} Z_i g_i$$

and, by Proposition 1 (a),

$$\lambda_1^2(\Gamma, Z) = \sup_{|x|=1} \mathrm{E} \langle x, X_{(\Gamma, Z)} \rangle^2 = C_p^2 \sup_{|x|=1} \sum_{i=1}^{\infty} (\Gamma_i)^{-2/p} \langle x, Z_i \rangle^2 \leqslant C_p^2 \sum_{i=1}^{\infty} (\Gamma)^{-2/p}.$$

A series $\sum_{i=1}^{\infty}(\Gamma_i)^{-2/p}$ is a $(p/2)$-stable variable (Linde [6]), so it has moments less than $p/2$. This completes the proof.

LEMMA 2. $\mathrm{E}\,\lambda_d^{-\beta}(\Gamma, Z) < \infty$ *for* $\beta \geqslant 0$. *Moreover,*

$$\mathrm{E}\,\lambda_d^{-\beta}(\Gamma, Z) \leqslant C_d \frac{1}{\alpha} w_\beta(1-q),$$

where

$$w_\beta(x) = 1 + \sum_{r=1}^{\infty} \left(d(r+1) + \frac{2\beta}{p}\right)^{(2\beta/p)+1} x^r$$

for α *and* q *as described in Fact* 2.

Proof. We have

$$X_{(\Gamma,Z)} = C_p \sum (\Gamma_i)^{-1/p} g_i Z_i = \sum_{k=0}^{\infty} X_k(\Gamma, Z),$$

where

$$X_k(\Gamma, Z) = C_p \sum_{r=0}^{d-1} (\Gamma_{dk+r})^{-1/p} Z_{dk+r}\, g_{dk+r}.$$

Then

$$\lambda_d^2(\Gamma, Z) = \inf_{|x|=1} \mathrm{E}_{(\Gamma,Z)} \langle x, X_{(\Gamma,Z)}\rangle^2 = \inf_{|x|=1} \sum_{k=0}^{\infty} \langle x, X_k\rangle^2 \geqslant \sum_{k=0}^{\infty} \lambda_{d,k}^2,$$

where

$$\begin{aligned}\lambda_{d,k}^2 &= \inf_{|x|=1} \mathrm{E}_{(\Gamma,Z)} \langle x, X_k\rangle^2 = \inf_{|x|=1} C_p^2 \sum_{r=0}^{d-1} \mathrm{E}_{(\Gamma,Z)} (\Gamma_{dk+r})^{-2/p} \langle x, Z_{dk+r}\rangle^2 \\ &\geqslant C_p^2 \inf_{|x|=1} \mathrm{E}_{(\Gamma,Z)} \Gamma_{d(k+1)}^{-2/p} \Big(\sum_{r=0}^{d-1} \langle x, Z_{dk+r}\rangle^2\Big) \\ &\geqslant C_p^2 \Gamma_{d(k+1)}^{-2/p} \cdot (\det[Z_{dk}, \ldots, Z_{dk+d-1}])^2 \left(\frac{d-1}{d}\right)^{d-1}.\end{aligned}$$

In the above we have used Proposition 1 (c) and the fact that Γ_k is increasing.

Now we follow the paper of Pap [8]. Since $\operatorname{lin}(\operatorname{supp}\sigma) = \mathscr{R}^d$, by Fact 2 there exist $\alpha > 0$ and $0 < q \leqslant 1$ such that

$$P(|\det[Z_1, \ldots, Z_d]|^2 > \alpha) = q.$$

Consequently,

$$\mathrm{E}\,\lambda_d^{-2\beta} \leqslant C_d \mathrm{E}\Big[\sum_{k=0}^{\infty} \Gamma_{d(k+1)}^{-2/p} \cdot (\det[Z_{dk}, \ldots, Z_{dk+d-1}])^2\Big]^{-\beta}.$$

Let us put $t = \inf_{k \geqslant 1} (\det[Z_{dk}, \ldots, Z_{dk+d-1}])^2 \geqslant \alpha$; then

$$\mathrm{E}\Big[\sum_{k=0}^{\infty} \Gamma_{d(k+1)}^{-2/p} (\det[Z_{dk}, \ldots, Z_{dk+d-1}])^2\Big]^{-\beta}$$

$$= \sum_{r=1}^{\infty} \mathrm{E}\Big[\sum_{k=0}^{\infty} \Gamma_{d(k+1)}^{-2/p} (\det[Z_{dk}, \ldots, Z_{dk+d-1}])^2\Big]^{-\beta} \cdot \mathbb{1}(t=r)$$

$$\leqslant \sum_{r=1}^{\infty} \mathrm{E}\,\Gamma_{d(r+1)}^{2\beta/p} \frac{1}{\alpha^\beta} \cdot \mathbb{1}(t=r) = \alpha^{-\beta} \sum_{r=1}^{\infty} \Gamma_{d(r+1)}^{2\beta/p} \cdot q(1-q)^{r-1}.$$

But $\mathrm{E}\,\Gamma_k^r = k(k+1)\ldots(k+r-1) \leqslant (k+r-1)^r$ for $k, r \in N$ (see Fisz [2]). Thus

$$\mathrm{E}\,\Gamma_{d(r+1)}^{2\beta/p} \leqslant \mathrm{E}\,\Gamma_{d(r+1)}^{[2\beta/p]} + 1 \leqslant \left(d(r+1) + \frac{2\beta}{p}\right)^{(2\beta/p)+1} + 1$$

and

$$\sum_{r=1}^{\infty} \mathrm{E}(\Gamma_{d(r+1)}^{2\beta/p})\, q(1-q)^{r-1} \leqslant 1 + \sum_{r=1}^{\infty} \left(d(r+1) + \frac{2\beta}{p}\right)^{(2\beta/p)+1} \cdot q(1-q)^r,$$

whence

$$\mathrm{E}\,\lambda_d^{-2\beta} \leqslant C_d \alpha^{-\beta} \cdot W_\beta(1-q),$$

where

$$W_\beta(x) = 1 + \sum_{r=1}^{\infty} \left(d(r+1) + \frac{2\beta}{p}\right)^{(2\beta/p)+1} \cdot x^r, \qquad |x| < 1.$$

Now we have to prove the following

FACT 3. *Let σ be a spectral measure such that $\tau(\sigma) > 0$. Define*

$$U(\sigma) = \{(\alpha, q)\colon P(|\det[Z_1, \ldots, Z_d]| > \alpha) = q\}$$

and

$$M_\varepsilon = \{\sigma\colon \tau(\sigma) \geqslant \varepsilon,\ \operatorname{supp}\sigma = S^{d-1},\ \sigma(S^{d-1}) = 1\}.$$

Then

$$\sup_{\sigma \in M_\varepsilon} \inf_{(\alpha,q)\in U(\sigma)} \frac{1}{\alpha^\beta} W_\beta(1-q) < \infty.$$

Proof. In Fact 2 we have proved that, for $\sigma \in M_\varepsilon$,

$$\inf_{(\alpha,q)\in U(\sigma)} \frac{1}{\alpha^\beta} W_\beta(1-q) < \infty.$$

Let $\sigma_n \in M_\varepsilon$ be such that

$$\inf_{(\alpha,q)\in U(\sigma_n)} \frac{1}{\alpha^\beta} W_\beta(1-q) \to \infty \qquad \text{as } n \to \infty.$$

By Fact 1 we can choose a subsequence $\sigma_{n_k} \xrightarrow{w} \sigma_0$, $\sigma_0 \in M(\varepsilon)$. From Fact 2 we deduce that there exists α_0 such that

$$P_{\sigma_0}(|\det[Z_1, \ldots, Z_d]|^2 > \alpha_0) > 0 \quad \text{and} \quad P_{\sigma_0}(|\det[Z_1, \ldots, Z_d]|^2 = \alpha_0) = 0,$$

P_{σ_0} denotes such a measure that $\mathscr{L}(Z_i) = \sigma_0$. Since $\sigma_{n_k} \to \sigma_0$, $\det[x_1, x_2, \ldots, x_d]$ is continuous, we have

$$\lim_{n\to\infty} P_{\sigma_{n_k}}(|\det[Z_1, \ldots, Z_d]|^2 > \alpha_0) = P_{\sigma_0}(|\det[Z_1, \ldots, Z_d]|^2 > \alpha_0),$$

and thus

$$\limsup \frac{1}{\alpha_0^\beta} W_\beta\big(1 - P_{\sigma_n}(|\det[Z_1, \ldots, Z_d]|^2 > \alpha_0)\big) < \infty.$$

This completes the proof.

Recall that $\mu(\cdot) = \mathrm{E}\mu_{(\Gamma,Z)}(\cdot)$. Denote by $f_{(\Gamma,Z)}(x)$ the density of a Gaussian measure $\mu_{(\Gamma,Z)}$. If $f(x)$ is the density of μ, then we have $f(x) = \mathrm{E}f_{(\Gamma,Z)}(x)$.

FACT 4. *For $n \in N$, $a \in S^{d-1}$,*

(a)
$$D_a^n f(x) = \mathrm{E}\, D_a^n f_{(\Gamma,Z)}(x),$$

and for every $0 < \varepsilon < 1$, $x \neq 0$,

(b)
$$|D_a^n f_{(\Gamma,Z)}(x)| \leqslant \frac{C_{n,\varepsilon}}{|x|} \frac{\lambda_1}{\lambda_d^n} f_{(\Gamma,Z)}(\sqrt{1-\varepsilon}\, x).$$

Proof. (a) We have

$$f_{(\Gamma,Z)}(x) = \frac{1}{(2\pi)^{d/2} \lambda_1 \cdot \ldots \cdot \lambda_d} \exp\Big(-\frac{1}{2} \sum \frac{\langle v_i, x\rangle^2}{\lambda_i^2}\Big).$$

First of all we prove that

$$(*) \quad \sup_{|t|\leqslant 1} |[e^{h(t)}]^{(n)}| \leqslant \Big(n + \frac{1}{\lambda_d^2}(d|x|+1)\Big)^n, \qquad \text{where } h(t) = -\frac{1}{2}\sum \frac{\langle v_i, x+ta\rangle^2}{\lambda_i^2}.$$

Put $G_n(t)$ such that $e^{h(t)} \cdot G_n(t) = [e^{h(t)}]^{(n)}$. Therefore, $G_{n+1} = h'(t) \cdot G_n(t) + G_n'(t)$. The function $G_n(t)$ is a polynomial of degree n, $G_n(t) = a_0 + a_1 t + \ldots + a_n t^n$; putting $|G_n| = |a_0| + |a_1| + \ldots + |a_n|$, we get

$$|G_{n+1}| \leqslant |h'| \cdot |G_n| + n|G_n| = |G_n|(|h'| + n) \quad \text{ and } \quad |G_n| \leqslant (n + |h'|)^n.$$

Now

$$h'(t) = \Big(-\frac{1}{2}\sum \frac{\langle v_i, x\rangle^2 + 2t\langle v_i, x\rangle\langle v_i, a\rangle + t^2\langle v_i, a\rangle}{\lambda_i^2}\Big)'$$

$$= -\sum \Big[\frac{\langle v_i, x\rangle\langle v_i, a\rangle}{\lambda_i^2} + t\frac{\langle v_i, a\rangle^2}{\lambda_i^2}\Big],$$

$$|h'| \leqslant \frac{1}{\lambda_d^2}\sum |\langle v_i, x\rangle\langle v_i, a\rangle| + \langle v_i, a\rangle^2 \leqslant \frac{1}{\lambda_d^2}(|x|d+1)$$

because $\sum \langle v_i, a\rangle^2 = |a^2| = 1$ and $|\langle v_i, x\rangle \langle v_i, a\rangle| \leqslant |x|$. Thus

$$\sup_{|t|\leqslant 1} |[e^{h(t)}]^{(n)}| \leqslant \sup_{|t|\leqslant 1} |G_n(t)| \cdot e^{h(t)} \leqslant |G_n| \leqslant \left(n + \frac{1}{\lambda_d^2}(d|x|+1)\right)^n,$$

which proves (*).

Now we are ready to prove (a); the proof goes by an easy induction. For $n = 1$

$$\left|\frac{f_{(\Gamma,Z)}(x+ta) - f_{(\Gamma,Z)}(x)}{t}\right| = \left|\frac{d}{dt} f_{(\Gamma,Z)}(x+ta)|_{t=\lambda}\right| \leqslant \frac{1}{(2\pi)^{d/2}\lambda_d^d}\left(1 + \frac{1}{\lambda_d^2}(d|x|+1)\right).$$

Using the Lebesgue Dominated Convergence Theorem we get

$$\lim_{t\to 0} \frac{f(x+ta) - f(x)}{t} = \lim_{t\to 0} \mathrm{E} \frac{f_{(\Gamma,Z)}(x+ta) - f_{(\Gamma,Z)}(x)}{t} = \mathrm{E} D_a^{(1)} f_{(\Gamma,Z)}(x)$$

because, by Lemma 2, $1/\lambda_d$ has all positive moments.

Similar arguments show that

$$\left|\frac{D_a^n f_{(\Gamma,Z)}(x+ta) - D_a^n f_{(\Gamma,Z)}(x)}{t}\right|$$

$$= |[f_{(\Gamma,Z)}(x+ta)]^{(n+1)}| \leqslant \left(n+1 + \frac{1}{\lambda_d^2}(d|x|+1)\right)^{n+1} \frac{1}{\lambda_d^d} \frac{1}{(2\pi)^{d/2}}.$$

By the Lebesgue Theorem and Lemma 2 we get the desired result.

(b) For a moment denote a scalar product by

$$\langle x, y\rangle_1 = \sum_{i=1}^{d} \frac{\langle v_i, x\rangle \langle v_i, y\rangle}{\lambda_i^2},$$

put $g_{(\Gamma,Z)}(x) = \exp(-\frac{1}{2}|x|_1^2)$ and

$g_{(\Gamma,Z)}(x+ta) = \exp(-\frac{1}{2}\langle x+ta, x+ta\rangle_1) = \exp(-\frac{1}{2}|x|_1^2 - t\langle x, a\rangle_1 - \frac{1}{2}t^2\langle a, a\rangle_1)$.

Let us compute

$$\frac{d^n}{dt^n} g_{(\Gamma,Z)}(x+ta)|_{t=0} = \exp(-\tfrac{1}{2}|x|_1^2) \cdot [\exp(-t\langle x, a\rangle_1) \cdot \exp(-\tfrac{1}{2}t^2\langle a, a\rangle_1)]^{(n)}$$

$$= \exp(-\tfrac{1}{2}|x|_1^2)\left(\sum_{\substack{k=0\\ n-k \text{ even}}}^{n} \binom{n}{k} (\exp(-t\langle x, a\rangle_1))^{(k)} \cdot (\exp(-\tfrac{1}{2}t^2\langle a, a\rangle_1))_{t=0}^{(n-k)}\right).$$

Hence

$$|(g_{(\Gamma,Z)}(x+ta))_{t=0}^{(n)}| \leqslant \exp(-\tfrac{1}{2}|x|_1^2) \cdot C_n\Big(\sum_{\substack{k=0\\ n-k \text{ even}}}^{n} |\langle x, a\rangle_1|^k (\langle a, a\rangle_1^{(n-k)/2})\Big)$$

$$\leqslant g_{(\Gamma,Z)}(x)\cdot C_n\Big(\sum_{k=0}^{n}|x|_1^k\,|a|_1^n\Big)\leqslant C_n\,g_{(\Gamma,Z)}(x)\,|a|_1^n\Big(\sum_{k=0}^{n}|x|_1^k\Big)$$

$$\leqslant C_{n,\varepsilon}\frac{1}{\lambda_d^n}\frac{1}{|x|_1}\exp\big(-\tfrac{1}{2}(1-\varepsilon)|x|_1^2\big)\leqslant\frac{C_{n,\varepsilon}}{|x|}\frac{\lambda_1}{\lambda_d^n}\exp\big(-\tfrac{1}{2}(1-\varepsilon)|x|_1^2\big)$$

and, finally,

$$|D_a^{(n)}f_{(\Gamma,Z)}(x)|\leqslant\frac{C_{n,\varepsilon}}{|x|}\frac{\lambda_1}{\lambda_d^n}f_{(\Gamma,Z)}(\sqrt{1-\varepsilon}\,x).$$

Now we are able to prove Theorem 1. From Fact 4 we infer that for every $0<\varepsilon<1$

$$|D_a^{(n)}f(x)|\leqslant \mathrm{E}\frac{C_{n,\varepsilon}}{|x|}\frac{\lambda_1}{\lambda_d^n}\cdot\frac{\exp\big(-\tfrac{1}{2}(1-\varepsilon)|x|_1^2\big)}{(2\pi)^{d/2}\lambda_1\cdot\ldots\cdot\lambda_d}.$$

Using the Hölder inequality to the above inequality with $p=1/(1-\varepsilon)$ and $q=1/\varepsilon$, we get

$$|D_a^{(n)}f(x)|\leqslant\frac{C_{n,\varepsilon}}{|x|}\left[\mathrm{E}\left(\frac{\lambda_1}{\lambda_d^n}\right)^{1/\varepsilon}\frac{1}{\lambda_1\cdot\ldots\cdot\lambda_d}\right]^{\varepsilon}\left[\mathrm{E}\frac{\exp\big(-\tfrac{1}{2}|x|_1^2\big)}{(2\pi)^{d/2}\lambda_1\cdot\ldots\cdot\lambda_d}\right]^{1-\varepsilon}$$

$$\leqslant\frac{C'_{n,\varepsilon}}{|x|}\big[\mathrm{E}\,\lambda_1^{1/\varepsilon-1}\cdot\lambda_d^{-n/\varepsilon-(d-1)}\big]^{\varepsilon}\big[f(x)\big]^{1-\varepsilon},$$

where we have used the fact that

$$f(x)=\mathrm{E}\,f_{(\Gamma,Z)}(x)\quad\text{and}\quad f_{(\Gamma,Z)}=\frac{\exp\big(-\tfrac{1}{2}|x|_1^2\big)}{(2\pi)^{d/2}\lambda_1\cdot\ldots\cdot\lambda_d}.$$

Further:

$$\mathrm{E}\lambda_1^{1/\varepsilon-1}\cdot\lambda_d^{-[n/\varepsilon+d-1]}\leqslant(\mathrm{E}\lambda_1^{u(1/\varepsilon-1)})^{1/u}(\mathrm{E}\lambda_d^{-(n/\varepsilon+d-1)u^*})^{1/u^*}$$

if $u(1/\varepsilon-1)<p$ (Lemma 1) and $u>1$ with $1/u+1/u^*=1$.

This implies that $1<u<\varepsilon(1-\varepsilon)^{-1}p$, so that we can find such u if $\varepsilon(1-\varepsilon)^{-1}p>1$. Hence

$$0<1-\varepsilon<1-\frac{1}{1+p}=\frac{p}{1+p}.$$

To complete our proof we have to show that

(1) $\inf_{\sigma\in M_\varepsilon}\inf_{|x|\leqslant 1}f_\sigma(x)>0$,

(2) $\sup_{\sigma\in M_\varepsilon}\sup_{|x|\leqslant 1}\sup_{a\in S^{d-1}}|D_a^{(n)}f_\sigma(x)|<\infty$, where $M_\varepsilon=\{\sigma\colon\tau(\sigma)\geqslant\varepsilon\}$ and $f_\sigma(x)$ is the density of p-stable measure with spectral measure σ.

The function $(\sigma,x)\to f_\sigma(x)$ is continuous because its Fourier transform is less than $\exp(-\varepsilon|x|^p)$. Since M_ε is compact, $\inf_{\sigma\in M_\varepsilon}\inf_{|x|\leqslant 1}f_\sigma(x)=f_{\sigma_0}(x_0)$ for certain $\sigma_0\in M_\varepsilon$ and $|x_0|\leqslant 1$. But p-stable densities are always positive for every $x\in\mathscr{R}^d$.

By similar arguments we obtain

$$\sup_{\sigma\in M_\varepsilon}\sup_{|x|\leqslant 1}\sup_{a\in S^{d-1}}|D_a^{(m)}f_\sigma(x)| = |D_{a_0}^{(m)}f_{\sigma_0}(x_0)| < \infty$$

for certain $\sigma_0 \in M_\varepsilon$, $|x_0| \leqslant 1$, and $a_0 \in S^{d-1}$.

Applications. We will use our inequality to a problem of admissible translates. Recall that two probability measures τ, ν are *equivalent* ($\tau \sim \nu$) if, for every measurable set A, $\mu(A) = 0$ if and only if $\nu(A) = 0$. We write $\mu \perp \nu$ if there exists A such that $\mu(A) = 1$ and $\nu(A) = 0$. A vector a is an *admissible translate* of μ if $\mu(\cdot - a) \sim \mu(\cdot)$.

Let us consider the series $\sum (X_n \cos 2\pi nt + Y_n \sin 2\pi nt)$ in $L_2([0, 1], dx)$, where $(X_n, Y_n)_{n=1}^{\infty}$ are independent 2-dimensional variables. Since the functions $X_n \cos 2\pi nt + Y_n \sin 2\pi nt$ belong to mutually orthogonal subspaces of L_2, we consider

$$\mu = \prod_{n=1}^{\infty} \mu_n, \quad \text{where } \mu_n = \mathscr{L}\left(\frac{X_n}{\sigma_n^{1/p}(S)}, \frac{Y_n}{\sigma_n^{1/p}(S)}\right)$$

on $(\mathscr{R}^2)^N$, and σ_n is the spectral measure of $\mathscr{L}(X_n, Y_n)$.

The problem is the following: find all vectors $\underline{a} = (a_1, a_2, \ldots)$ such that the measures $\mu(\cdot - \underline{a})$ and $\mu(\cdot)$ are equivalent. The question was investigated in [7] and [10], but only in the case when μ_n are 1-dimensional.

For the sake of completeness we recall below some basic facts and theorems (see Shepp [9]).

If μ, v are any probability measures, then there exists a measure m such that $m \gg \mu$ and $m \gg v$ (for example, $m = \frac{1}{2}(\mu + v)$). In this case $\varphi(x) = d\mu/dm$ and $\psi(x) = dv/dm$. Let us put

$$H(\mu, v) = \int \sqrt{\varphi(x)\cdot\psi(x)}\, dm.$$

Observe that $H(\mu, v)$ does not depend on the choice of measure m and, by the Schwarz inequality, $0 \leqslant H(\mu, v) \leqslant 1$.

THEOREM (Kakutani [4]). *Put $\mu = \prod_{n=i}^{\infty} \mu_n$ and $v = \prod_{n=1}^{\infty} v_n$, where μ_n and v_n are probability measures. Then*

$$H(\mu, v) = \prod_{n=1}^{\infty} H(\mu_n, v_n).$$

Moreover, $\mu \perp v$ if and only if $H(\mu, v) = 0$. If $\mu_n \sim v_n$, $n = 1, 2, \ldots$, then $\mu \sim v$ if and only if $H(\mu, v) > 0$.

The Kakutani Theorem implies that if $\mu_n \sim v_n$, then $\mu \perp v$ or $\mu \sim v$.

Now we repeat some observations of Shepp [9]. Let μ be a measure on $\mathscr{R}^2$ with density $\varphi(x)$, and $\varphi(x) > 0$ for almost all x with respect to Lebesgue measure. Since

$$H(\mu, \mu(\cdot - a)) = \int \sqrt{\varphi(x)\cdot\varphi(x-a)}\, dx = \int \sqrt{\varphi(x)}\cdot\sqrt{\varphi(x-a)}\, dx,$$

$\sqrt{\varphi} \in L_2(dx)$, we have by Parseval's identity

$$\int \sqrt{\varphi(x)} \cdot \sqrt{\varphi(x-a)}\, dx = (2\pi)^{-2} \int (\sqrt{\varphi})\hat{}\, \overline{(\sqrt{\varphi(x-a)})\hat{}}\, dx$$
$$= (2\pi)^{-2} \int_{\mathscr{R}^2} \cos\langle x, a\rangle |(\varphi^{1/2})\hat{}|^2\, dx.$$

But

$$1 - H(\mu, \mu(\cdot - a)) = (2\pi)^{-2} \int_{\mathscr{R}^2} (1 - \int_{\mathscr{R}^2} (1 - \cos\langle a, x\rangle) |(\varphi^{1/2})\hat{}|^2\, dx)$$
$$\leqslant C \int_{\mathscr{R}^2} \langle a, x\rangle^2 |(\varphi^{1/2})\hat{}|^2\, dx$$
$$= C \int_{\mathscr{R}^2} \big(i\langle x, a\rangle (\varphi^{1/2})\hat{}\big)\overline{\big(i\langle x, a\rangle (\varphi^{1/2})\hat{}\big)}\, dx$$
$$= C \int_{\mathscr{R}^2} |D_a \varphi^{1/2}|^2\, dx = C \int_{\mathscr{R}^2} \left[\frac{D_a \varphi}{\varphi^{1/2}}\right]^2 dx = C \int_{\mathscr{R}^2} \frac{(D_a \varphi)^2}{\varphi}\, dx$$

with

$$C = \frac{1}{8\pi^2} \quad \text{and} \quad D_a \varphi = \lim_{t \to 0} \frac{\varphi(x+ta) - \varphi(x)}{t}.$$

Thus, if $\mu = \prod_{n=1}^{\infty} \mu_n$, $\varphi_n = d\mu_n/dx$, $\underline{a} = (a_1, a_2, \ldots)$, and

$$(*) \qquad \sum \int_{\mathscr{R}^2} \frac{(D_{a_i} \varphi_i)^2}{\varphi_i^2}\, dx < \infty,$$

then the measures $\mu(\cdot - a)$ and μ are equivalent. The quantity $\int ((D_a \varphi)^2/\varphi)\, dx$ is called the *Fisher information.*

Now we are ready to prove the following

THEOREM 2. *Let μ_n be 2-dimensional symmetric p-stable measures with spectral measures σ_n, $1 < p < 2$. Additionally, we assume that $\sigma_n(S) = 1$ and $\inf_n \tau(\sigma_n) > 0$. Then $\underline{a} = (a_1, a_2, \ldots)$ is an admissible translate of μ if and only if $\sum_i |a_i|^2 < \infty$.*

Proof. First we prove that if $\underline{a}$ is an admissible translate, then $\underline{a} \in l_2(N)$.

Let $\underline{\mu} = \sum_{n=1}^{\infty} \mu_n$ on $(\mathscr{R}^2)^N$. Take any sequence of non-zero vectors in $\mathscr{R}^2$, say (y_i), define T: $(\mathscr{R}^2)^N \to \mathscr{R}^N$, where for $\underline{x} = (\underline{x}_1, \underline{x}_2, \ldots) \in (\mathscr{R}^2)^N$

$$T(\underline{x}) = \left(\left\langle \frac{y_1}{|y_1|\sigma_1}, x_1\right\rangle, \left\langle \frac{y_2}{|y_2|\sigma_2}, x_2\right\rangle, \ldots\right),$$

and $\langle \cdot, \cdot \rangle$ denotes scalar product; $|y_i|\sigma_i > 0$ because $\inf_n \tau(\sigma_n) > 0$. Since T is measurable and linear, we can define

$$\underline{\mu}_T(A) = \underline{\mu}(T^{-1}(A)) = \prod_{n=1}^{\infty} \mu_{y_i}$$

and

$$\underline{\mu}_T^{\underline{a}}(A) = \mu\big(T^{-1}(A) - \underline{a}\big) = \prod_{n=1}^{\infty} \mu_{y_i}\big(\cdot - T(\underline{a})_i\big),$$

where

$$\mu_{y_i}(A) = \mu_i\left(\left\langle \frac{y_i}{|y_i|_{\sigma_i}}, \cdot \right\rangle^{-1}(A)\right) \quad \text{and} \quad T(\underline{a}) = \big(T(\underline{a})_1, \ldots\big).$$

Measures μ_{y_i} are 1-dimensional p-stable with unit spectral measure (equivalently, for every n, $\int e^{itx}\mu_{y_n}(dx) = \exp(-|t|^p)$). Further, measures $\underline{\mu}_T^{\underline{a}}$ and $\underline{\mu}_T$ on $\mathscr{R}^N$ are also equivalent (as images of equivalent measures). Hence $\big(T(\underline{a})_1, T(\underline{a})_2, \ldots\big)$ is an admissible translate, and using Proposition 8 from Zinn [10] we infer that

$$\sum_{i=1}^{\infty} \frac{\langle y_i, a_i\rangle^2}{|y_i|_{\sigma_i}^2} < \infty$$

for every sequence $(y_1, y_2, \ldots)$. Putting $y_i = a_i$ we get

$$\sum_{i=1}^{\infty} \frac{(a_i, a_i)^2}{|a_i|_{\sigma_i}^2} < \infty.$$

Since $|a_i|_{\sigma_i}^2 \leqslant |a_i|^2$, we obtain also $\sum_{i=1}^{\infty}(a_i, a_i) < \infty$.

Now we prove that if $\sum_{i=1}^{\infty}(a_i, a_i) < \infty$, then $\underline{a} = (a_1, a_2, \ldots)$ is an admissible translate.

By (*) we have to calculate

$$\int \frac{(D_{a_i}\varphi_i)^2}{\varphi_i}\,dx = |a_i|^2 \int \frac{(D_{a_i/|a_i|}\varphi_i)^2}{\varphi_i}\,dx.$$

We will prove that

$$\sup_i \sup_{\underline{a}\in S} \int \frac{(D_{\underline{a}}\varphi_i)^2}{\varphi_i}\,dx < \infty \quad \text{if only} \quad \inf_i \tau(\sigma_i) > 0.$$

From our inequality (Theorem 1), if we take $0 < \alpha < (p-1)/2p$, we get

$$\int \frac{(D_a\varphi)^2}{\varphi}\,dx \leqslant \int \frac{C(\tau, \alpha)}{(1+|x|)^2}[\varphi(x)]^{2(1-\alpha)[p/(1+p)]-1}\,dx = C(\tau, u)\int \frac{1}{(1+|x|)^2}[\varphi(x)]^u\,dx,$$

where $0 < u < (p-1)/(p+1)$.

Let k be any number such that $k > (1+p)/(p-1)$ and $u = 1/k$. Then $k^* > 1$, where $1/k^* + 1/k = 1$. Using the Hölder inequality with k and k^*, we get

$$\int \frac{1}{(1+|x|)^2}[\varphi(x)]^u\,dx \leqslant \left(\int \frac{1}{(1+|x|)^{2k^*}}\,dx\right)^{1/k^*} \left(\int [\varphi(x)]^{uk}\,dx\right)^{1/k} < \infty$$

because

$$\int \frac{1}{(1+|x|)^{2k^*}} dx < \infty \quad \text{and} \quad \int \varphi(x) = 1.$$

Finally,

$$\sup_{\underline{a} \in S} \int \frac{(D_{\underline{a}}, \varphi_i)^2}{\varphi_i} dx \leqslant \left(\int \frac{1}{(1+|x|)^{2k^*}} dx \right)^{1/k^*} C\big(\tau(\sigma_i), u\big)$$

and (Fact 3)

$$\sup_{\tau(\sigma_i) \geqslant \inf_n \tau(\sigma_n)} C\big(\tau(\sigma_i), u\big) < \infty.$$

This completes the proof.

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Institute of Mathematics, University of Technology
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

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