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THE SECOND ORDER OPTIMALITY OF TESTS AND ESTIMATORS FOR MINIMUM CONTRAST FUNCTIONALS. II

BY

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This paper is a continuation of part I (see [7]). It presumes that the reader is familiar with the concepts and notation introduced there. Part II contains lemmas and proofs of the results given in part I.

9. Some auxiliary results. First we derive some asymptotic expansions which are needed in the proofs.

Let $P_* \in \mathfrak{P}$ and $\Delta > 0$. Let $P_n \in \mathfrak{P}$, $n \in N$, be a sequence fulfilling

$$\varkappa_0(P_n) = \varkappa_0(P_*) - n^{-1/2}\Delta \tag{9.1}$$

and admitting a P_*-density

$$p_n := 1 - n^{-1/2}\Delta\sigma_{00}^{-1}(P_*) f_0(\cdot, P_*) + n^{-1}\bar{r}_n \tag{9.2}$$

such that

$$P_*(\bar{r}_n^2) = o(n). \tag{9.3}$$

Assume that the following regularity conditions are fulfilled:

(9.4) $\quad M_4\big(P_* * f^\alpha(\cdot, \varkappa(P_*))\big)$ for $|\alpha| = 1, 2$,

$\quad\quad M_2\big(P_* * f^\alpha(\cdot, \varkappa(P_*))\big)$ for $|\alpha| = 3$;

(9.5) $\quad L_4(\varkappa(P_*), P_*)$ for $f^\alpha\colon X \times T \to \boldsymbol{R}$ if $|\alpha| = 2$,

$\quad\quad L_2(\varkappa(P_*), P_*)$ for $f^\alpha\colon X \times T \to \boldsymbol{R}$ if $|\alpha| = 3$.

If a fixed p-measure P_* is given, we omit the argument P_* in expressions depending on P_*, if this is convenient.

We first derive an asymptotic expansion for $\varkappa(P_n)$, $n \in N$. By a Taylor expansion of $t \to f^{(i)}(x, t)$ about $t = \varkappa(P_*)$, we infer from (9.2)-(9.5) that

$$P_n\big(f^{(i)}(\cdot, \varkappa(P_*) + n^{-1/2}\Delta a)\big) = o(n^{-1/2}) \tag{9.6}$$

for

(9.7) $$a_l := -\sigma_{00}^{-1} A_{li} A_{0j} F_{i,j}, \quad l = 0, \ldots, p.$$

Let g_n, $n \in N$, be defined by

$$g_n(t) := P_n(f^{\cdot}(\cdot, t)).$$

By condition (9.5), g_n is differentiable in some neighborhood $V(\varkappa(P_*))$ of $\varkappa(P_*)$, and the order of differentiation and integration may be interchanged. As $P_n \to P_*$, $n \in N$, in the strong topology, (8.4) implies the existence of a constant $\lambda_0 > 0$ and of an ε-neighborhood $V_\varepsilon(\varkappa(P_*)) \subset V(\varkappa(P_*))$ of $\varkappa(P_*)$ such that for all sufficiently large $n \in N$

(9.8) $$\|g_n(t) - g_n(t')\| \geq \lambda_0 \|t - t'\| \quad \text{for } t, t' \in V_\varepsilon(\varkappa(P_*)).$$

By (8.5), $\varkappa(P_n) \in V_{\varepsilon/2}(\varkappa(P_*))$ for all sufficiently large $n \in N$. Since $V_{\varepsilon/2}(\varkappa(P_n)) \subset V_\varepsilon(\varkappa(P_*))$ and $g_n(\varkappa(P_n)) = 0$, (9.8) implies the existence of a δ-neighborhood $V_\delta(0)$ such that g_n^{-1} exists on $V_\delta(0)$ for all sufficiently large $n \in N$, and

(9.9) $$\|g_n^{-1}(v) - g_n^{-1}(v')\| \leq \frac{1}{\lambda_0} \|v - v'\| \quad \text{for } v, v' \in V_\delta(0).$$

As $g_n(\varkappa(P_*) + n^{-1/2} \Delta a)$ is in $V_\delta(0)$ for sufficiently large $n \in N$ by (9.6), it follows from (9.6) and (9.9) that

(9.10) $$\varkappa(P_n) = \varkappa(P_*) + n^{-1/2} \Delta a + n^{-1/2} R_n,$$

where, by (9.1),

(9.11) $$R_{n,l} = o(n^0) \text{ for } l = 1, \ldots, p, \quad R_{n,0} = 0.$$

(Notice that $a_0 = -1$.)

By a Taylor expansion and (9.10),

$$F_{ij}(P_n) = F_{ij} + n^{-1/2} \sigma_{00}^{-1} (A_{0k} F_{ij,k} - A_{kl} A_{0p} F_{l,p} F_{ijk}) + o(n^{-1/2}),$$

and therefore

(9.12) $$A_{0i}(P_n) = A_{0i} + n^{-1/2} \Delta e_i + o(n^{-1/2}),$$

where

(9.13) $$e_i := \sigma_{00}^{-1} A_{it} A_{0r} A_{0s} (A_{pq} F_{q,r} F_{stp} - F_{st,r}), \quad i = 0, \ldots, p.$$

Furthermore,

(9.14) $$F_{i,j}(P_n) = F_{i,j} + n^{-1/2} \Delta \sigma_{00}^{-1} A_{0k} \big(A_{lp} F_{k,p} (F_{i,jl} + F_{j,il}) + F_{i,j,k}\big) + o(n^{-1/2}).$$

By (9.12) and (9.14),

(9.15) $$\sigma_{00}(P_n) = \sigma_{00} + n^{-1/2} \Delta c + o(n^{-1/2}),$$

where

$$(9.16)\qquad c := A_{0i}A_{0j}A_{0v}\big(A_{kq}F_{v,q}(4F_{ik,j}-F_{i,l}A_{ls}F_{sjk})-2F_{i,j,v}\big).$$

From (9.15), by a Taylor expansion of $x \to x^{1/2}$ about $x = \sigma_{00}$, we get

$$(9.17)\qquad \sigma_0(P_n) = \sigma_0+\tfrac{1}{2}n^{-1/2}\Delta\sigma_0^{-1}c+o(n^{-1/2}).$$

If in (9.2) we take $\bar{r}_n = \Delta^2 h+n^{-1/2}r_n$ with

$$(9.18)\qquad M_2(P_* * h)$$

and

$$(9.19)\qquad P_*(r_n^2) = o(n),$$

similarly as in (9.6)-(9.11) we obtain

$$(9.20)\qquad \varkappa(P_n) = \varkappa(P_*)+n^{-1/2}\Delta a+n^{-1}\Delta^2 b+o(n^{-1}),$$

where a_l $(l = 0,\ldots,p)$ are given by (9.7), and

$$(9.21)\qquad b_l = -A_{li}\big((\sigma_{00}^{-1}A_{0k}F_{k,ij}a_j+\tfrac{1}{2}a_j a_k F_{ijk})+P_*(hf^{(i)})\big),\qquad l = 0,\ldots,p.$$

Moreover, $b_0 = 0$ by (9.1).

The essential point of the following lemma is that the power function of the sequence of c.r. $\{F_n(\cdot,\varkappa_0(P_*)-n^{-1/2}\Delta) > 0\}$ does not depend on the polynomial M occurring in the stochastic expansion of $F_n(\cdot,t_0)$.

(9.22) LEMMA. *Let $P_n\in\mathfrak{P}$, $n\in N$, be a sequence fulfilling* (9.1)-(9.3). *Let F_n, $n\in N$, be a sequence of test functions for $\varkappa_0$ of type S which is asymptotically similar of level $\alpha+o(n^{-1/2})$ for $U_{n,\delta}(P_*)$ for every $\delta\in(0,1)$.*

Then

$$P_*^n\{F_n(\cdot,\varkappa_0(P_*)-n^{-1/2}\Delta) > 0\} = \pi_n(\Delta,\alpha)+o(n^{-1/2}),$$

where $\pi_n(\Delta,\alpha)$ is given by (5.7).

This holds true under conditions (9.4) *and* (9.5).

Proof. We first note that, by Lemma (9.35), $P_n\in U_{n,\delta}(P_*)$ for all sufficiently large $n\in N$ if

$$\delta > 2\big(1-\Phi(\tfrac{1}{2}\sigma_0^{-1}\Delta)\big).$$

Furthermore, we may assume without loss of generality that $U_{n,\delta}(P_*) \subset U_*$ for all $n\in N$.

By a Taylor expansion of $t\to f^{\alpha}(\cdot,t)$ about $t = \varkappa(P_*)$ for $|\alpha| = 1$, we infer from (9.3) and (9.4) that $M_3^*(\{P_* * f_0(\cdot,P_n)\colon n\in N\})$ is fulfilled.

Let

$$f_{0,n} := f_0(\cdot,P_n)-P_*(f_0(\cdot,P_n)),\qquad g_n := g(\cdot,P_n)-P_*(g(\cdot,P_n)).$$

Since $f_0(\cdot,P_*)$ and $g_i(\cdot,P_*)$ are P_*-uncorrelated, by (4.11), (9.3) and (4.14) we have

(9.23) $P_n(g_i(\cdot, P_n))$
$$= P_*(g_i(\cdot, P_n)) - n^{-1/2}\Delta\sigma_{00}^{-1} P_*(f_0(\cdot, P_*) g_i(\cdot, P_n)) + o(n^{-1/2})$$
$$= P_*(g_i(\cdot, P_n)) + o(n^{-1/2}).$$

Hence $P_n(g_i(\cdot, P_n)) = 0$ implies

(9.24) $$P_*(g_i(\cdot, P_n)) = o(n^{-1/2}).$$

Therefore, for $\tilde{\tilde{g}}_{i,n}(x) := n^{-1/2} \sum_{\nu=1}^{n} g_{i,n}(x_\nu)$ we have

(9.25) $$\tilde{\tilde{g}}_{i,n} = \tilde{g}_i(\cdot, P_n) + o(n^0).$$

Moreover, by a Taylor expansion of $t \to f^{(i)}(\cdot, t)$ about $\varkappa_0(P_*)$, (9.10) and (9.12),

(9.26) $$P_*(f_0(\cdot, P_n)) = n^{-1/2}\Delta - n^{-1}\Delta^2(\tfrac{1}{2} a_j a_k A_{0i} F_{ijk} + e_i a_j F_{ij}) + o(n^{-1}).$$

Thus, for $\tilde{\tilde{f}}_{0,n}(x) := n^{-1/2} \sum_{\nu=1}^{n} f_{0,n}(x_\nu)$ we get

(9.27) $$\tilde{\tilde{f}}_{0,n} = \tilde{f}_0(\cdot, P_n) - \Delta + n^{-1/2}\Delta^2(\tfrac{1}{2} a_j a_k A_{0i} F_{ijk} + e_i a_j F_{ij}) + o(n^{-1/2}).$$

Using (9.17), (9.25) and (9.27) and the fact that $F_n(\cdot, \varkappa_0(P_*) - n^{-1/2}\Delta)$ is asymptotically similar of level $\alpha + o(n^{-1/2})$ for P_n, from (4.8) we obtain

(9.28) $F_n(\cdot, \varkappa_0(P_*) - n^{-1/2}\Delta)$
$$= \tilde{\tilde{f}}_{0,n} + N_\alpha \sigma_0 + \Delta - n^{-1/2}\big(\Delta^2(\tfrac{1}{2} a_j a_k A_{0i} F_{ijk} + e_i a_j F_{ij}) +$$
$$+ \tfrac{1}{2}\Delta N_\alpha \sigma_0^{-1} c + M(\tilde{\tilde{f}}_{0,n} + \Delta, \tilde{\tilde{g}}_n, P_n)\big) +$$
$$+ n^{-1/2} o_n(\tfrac{1}{2}) \quad \text{with respect to } P_*.$$

Let $\sigma_n := P_*(f_{0,n}^2)^{1/2}$. By a Taylor expansion, from (9.10) and (9.12) we obtain

(9.29) $$\sigma_n^2 = \sigma_{00} + n^{-1/2}\Delta(A_{0i} A_{0j} a_k F_{ik,j} + A_{0j} e_i F_{i,j}) + o(n^{-1/2}).$$

Thus, by a Taylor expansion of $x \to x^{1/2}$ about $x = \sigma_{00}$,

(9.30) $$\sigma_n = \sigma_0 + \tfrac{1}{2} n^{-1/2} \sigma_0^{-1}\Delta(A_{0i} A_{0j} a_k F_{ik,j} + A_{0j} e_i F_{i,j}) + o(n^{-1/2}).$$

In virtue of conditions (4.10)-(4.15), Lemma (9.63), Lemma 5.25 in [8], p. 20, and (9.28) we get

(9.31) $P_*^n\{F_n(\cdot, \varkappa_0(P_*) - n^{-1/2}\Delta) > 0\}$
$$= \Phi((N_\alpha \sigma_0 + \Delta)\sigma_n^{-1}) - n^{-1/2}\sigma_0^{-1}\varphi(N_\alpha + \Delta\sigma_0^{-1})\big(k(-N_\alpha\sigma_0 - \Delta) -$$
$$- \int dv\, \varphi_{\Sigma_0}(v) M(-N_\alpha \sigma_0, v, P_*) +$$
$$+ \Delta(\Delta(\tfrac{1}{2} a_j a_k A_{0i} F_{ijk} + e_i a_k F_{ik}) - \tfrac{1}{2} c\sigma_0^{-1} N_\alpha)\big) + o(n^{-1/2}),$$

where $k(t) := \frac{1}{6}\sigma_{00}^{-1} P_*(f_0^3)(1-\sigma_{00}^{-1} t^2)$ and Σ_0 is the covariance matrix of $P_* * g(\cdot, P_*)$.

Using a Taylor expansion, from (9.30) we obtain

(9.32) $$\Phi\left((N_\alpha \sigma_0 + \Delta)\sigma_n^{-1}\right) = \Phi(N_\alpha + \Delta\sigma_0^{-1}) - $$
$$- n^{-1/2}\sigma_0^{-3}\varphi(N_\alpha + \Delta\sigma_0^{-1})(N_\alpha + \Delta\sigma_0^{-1})\Delta(A_{0i}A_{0j}a_k F_{ik,j} + F_{i,j}A_{0j}e_i) + o(n^{-1/2}).$$

For $\Delta = 0$ and $P_n = P_*$, making use of (9.31) and the fact that $P_*^n\{F_n(\cdot, \varkappa(P_*)) > 0\} = \alpha + o(n^{-1/2})$ we get

(9.33) $$\int dv\, \varphi_{\Sigma_0}(v) M(-N_\alpha \sigma_0, v, P_*) = k(-N_\alpha \sigma_0).$$

The assertion of the lemma now follows easily from (9.31)-(9.33).

(9.34) Remark. A result corresponding to (9.22) can be obtained for $\Delta < 0$.

(9.35) Lemma. *Let $P_n \in \mathfrak{P}$, $n \in N$, be a sequence admitting a P_*-density*

(9.36) $$p_n = 1 + n^{-1/2}\Delta g + n^{-1}\bar{r}_n.$$

Assume that $\bar{r}_n = \Delta^2 h + n^{-1/2} r_n$ with

(9.37) $$M_2^*(P_* * h),$$

(9.38) $$M_{3/2}^*(\{P_* * r_n \colon n \in N\}).$$

If φ_n, $n \in N$, is asymptotically of level $\alpha + o(n^{-1/2})$ for P_n, then

$$P_*^n(\varphi_n) \leqslant \Phi(N_\alpha + \Delta\sigma) +$$
$$+ n^{-1/2}\varphi(N_\alpha + \Delta\sigma)\sigma^{-1}\Delta\left(\Delta\left(P_*(gh) - \tfrac{1}{6}P_*(g^3)\right) + \tfrac{1}{6}P_*(g^3)N_\alpha\sigma^{-1}\right) +$$
$$+ o(n^{-1/2}),$$

where $\sigma := P_*(g^2)^{1/2}$.

This holds true under the following regularity conditions:

(9.39) $$M_{9/2}^*(P_* * g),$$

(9.40) $$C(P_* * g).$$

If (9.38), $P_*(r_n^2) = o(n)$, $g = -\sigma_{00}^{-1} f_0$, and (9.1) are fulfilled, we obtain

(9.41) $$P_*^n(\varphi_n) \leqslant \pi_n(\Delta, \alpha) + o(n^{-1/2}),$$

where $\pi_n(\Delta, \alpha)$ is given by (5.7).

Proof. For $r \in \boldsymbol{R}$ let

$$D_n(r) := \left\{x \in X^n \colon \prod_{\nu=1}^{n} P_n(x_\nu) \leqslant r\right\}, \qquad r_{n,\alpha} := \inf\left\{r \in \boldsymbol{R} \colon P_n^n(D_n(r)) \geqslant \alpha\right\}.$$

We have $P_n^n(D_n(r_{n,\alpha})) \geqslant \alpha$.

Let now φ_n, $n \in N$, be asymptotically of level $\alpha + o(n^{-1/2})$ for P_n. Let $\alpha_n := \max\{\alpha, P_n^n(\varphi_n)\}$. We have

$$\alpha \leqslant \alpha_n \leqslant \alpha + o(n^{-1/2}).$$

Since $\alpha\alpha_n^{-1}\varphi_n$ is of level α, by the Neyman-Pearson lemma we obtain

$$P_*^n(\alpha\alpha_n^{-1}\varphi_n) \leqslant P_*^n(D_n(r_{n,\alpha})).$$

Therefore

$$P_*^n(\varphi_n) \leqslant P_*^n(D_n(r_{n,\alpha})) + o(n^{-1/2}). \tag{9.42}$$

Let

$$A_n := \{\Delta|g| \leqslant \tfrac{1}{4}n^{1/2} \text{ and } |\bar{r}_n| \leqslant \tfrac{1}{4}n\}, \tag{9.43}$$

$$B_n := A_n^n.$$

By the definition of P_n, Markov's inequality and Hölder's inequality, we obtain for $Q_n = P_*$ and $Q_n = P_n$

$$Q_n^n(B_n^c) \leqslant n(Q_n\{\Delta|g| > \tfrac{1}{4}n^{1/2}\} + Q_n\{|\bar{r}_n| > \tfrac{1}{4}n\}) = o(n^{-1/2}). \tag{9.44}$$

Hence for $Q_n = P_*$ and $Q_n = P_n$ we have

$$Q_n^n(D(r_{n,\alpha})) = Q_n^n\{x \in B_n\colon \sum_{\nu=1}^{n} \log p_n(x_\nu) \leqslant r'_{n,\alpha}\} + o(n^{-1/2}) \tag{9.45}$$

for some suitably chosen $r'_{n,\alpha} \in R$.

For notational convenience let

$$k_n := \Delta g + n^{-1/2}\bar{r}_n. \tag{9.46}$$

From a Taylor expansion of log we obtain

$$\log p_n = n^{-1/2}k_n - \tfrac{1}{2}n^{-1}k_n^2 + \tfrac{1}{6}n^{-3/2}k_n^3 + n^{-3/2}k_n^3 v(n^{-1/2}k_n), \tag{9.47}$$

where

$$v(y) := \int_0^1 (1-u)^2((1-uy)^{-2}-1)\,du.$$

For $|y| \leqslant \frac{1}{2}$ we have

$$|v(y)| \leqslant 2|y|. \tag{9.48}$$

From (9.46) and (9.47) for $x \in B_n$ we obtain

$$\sum_{\nu=1}^{n} \log p_n(x_\nu) = n^{-1/2}\Delta \sum_{\nu=1}^{n} g(x_\nu) + n^{-1}\Delta^2 \sum_{\nu=1}^{n} (h(x_\nu) - \tfrac{1}{2}g^2(x_\nu)) + \tag{9.49}$$

$$+ n^{-1/2}\Delta^3(\tfrac{1}{3}P_*(g^3) - P_*(gh)) + n^{-3/2}\sum_{\nu=1}^{n} R_n(x_\nu),$$

where

$$R_n = r_n + \tfrac{1}{3}k_n^3 - \tfrac{1}{2}n^{-3/2}r_n^2 + k_n^3 v(n^{-1/2}k_n) -$$
$$-n^{-1/2}\Delta g r_n - n^{-1}\Delta^2 h r_n - \Delta^3\big(gh - P_*(gh) + P_*(g^3)\big) - \tfrac{1}{2}\Delta^4 n^{-1/2}h^2.$$

From Lemmas (9.57) and (9.58) we obtain

(9.50) $$n^{-3/2}\sum_{\nu=1}^{n} R_n(x_\nu)\mathbf{1}_{B_n}(x) = n^{-1/2}o_n(\tfrac{1}{2})$$

with respect to P_* and with respect to P_n, since by (9.48) we have

$$n^{-3/2}\Big|\sum_{\nu=1}^{n} k_n^3(x_\nu)v\big(n^{-1/2}k_n(x_\nu)\big)\Big|\mathbf{1}_{B_n}(x) \leqslant 2n^{-2}\sum_{\nu=1}^{n} k_n^4(x_\nu).$$

As $n^{-3/2}P_*(gr_n) = O(n^{-3/2})$, we infer from (9.37)-(9.40) and Lemma (9.65) that

(9.51) $$P_n^n\{\tilde{g} - \Delta P_*(g^2) - n^{-1/2}\Delta^2 P_*(gh) +$$
$$+ n^{-1/2}\Delta\big(\tilde{h} - \Delta P_*(gh) - \tfrac{1}{2}(g^2)^{\sim} + \tfrac{1}{2}\Delta P_*(g^3)\big) < s\}$$
$$= \Phi(s\sigma_n^{-1}) + n^{-1/2}\varphi(s\sigma_n^{-1})H(s) + o(n^{-1/2})$$

uniformly for $s \in \mathbf{R}$, and

(9.52) $$P_*^n\{\tilde{g} + n^{-1/2}\Delta\big(\tilde{h} - \tfrac{1}{2}(g^2)^{\sim}\big) < s\}$$
$$= \Phi(s\sigma^{-1}) + n^{-1/2}\varphi(s\sigma^{-1})H(s) + o(n^{-1/2})$$

uniformly for $s \in \mathbf{R}$, where $\sigma_n^2 := P_*(g^2) + n^{-1/2}\Delta P_*(g^3)$, and

(9.53) $$H(s) := \sigma^{-3}\big(\tfrac{1}{6}P_*(g^3)(1 - s^2\sigma^{-2}) + (\tfrac{1}{2}P_*(g^3) - P_*(gh))s\Delta\big).$$

Therefore, from (9.45) and (9.50) by Lemma (9.63) it follows that for $Q_n = P_*$ and $Q_n = P_n$

(9.54) $$Q_n^n\big(D_n(r_{n,\alpha})\big) = Q_n^n(C_{n,\alpha}) + o(n^{-1/2}),$$

where

$$C_{n,\alpha} := \{\tilde{g} + n^{-1/2}\Delta\big(\tilde{h} - \tfrac{1}{2}(g^2)^{\sim}\big) < c_{n,\alpha}\}$$

with

$$c_{n,\alpha} := r'_{n,\alpha}\Delta^{-1} + \tfrac{1}{2}\Delta P_*(g^2) + n^{-1/2}\Delta^2\big(P_*(gh) - \tfrac{1}{3}P_*(g^3)\big).$$

As $P_*^n(C_{n,\alpha}) = \alpha + o(n^{-1/2})$, from a uniform version of Lemma 7 in [5], p. 1016, we obtain

(9.55) $$c_{n,\alpha} = \Delta\big(N_\alpha\sigma - \tfrac{1}{6}n^{-1/2}\sigma^{-2}P_*(g^3)(1 - N_\alpha^2)\big) +$$
$$+ \Delta^2\big(P_*(g^2) + n^{-1/2}\sigma^{-1}N_\alpha P_*(gh)\big) + n^{-1/2}\Delta^3\big(2P_*(gh) + \tfrac{1}{2}P_*(g^3)\big).$$

The assertion of the lemma now follows from (9.42), (9.52), (9.54) and (9.55). Relation (9.41) follows immediately from (9.21).

(9.56) Remark. In the case $\Delta < 0$ and $g = -\sigma_{00}^{-1} f_0$, in the same way as in Lemma (9.35) one can derive

$$P_*^n(\varphi_n) \geqslant \pi_n(\Delta, \alpha) + o(n^{-1/2}).$$

The following lemma is an immediate consequence of Lemma 6.3 in [4], p. 152.

(9.57) LEMMA. *Let $\mathfrak{Q}_n$, $n \in N$, be families of p-measures. Let $s \in [0, \infty)$ and $a > \frac{1}{2}$. Let $h_n(\cdot, Q)$: $X \to \boldsymbol{R}$, $Q \in \mathfrak{Q}_n$, $n \in N$, be measurable functions fulfilling*

$$M^*_{(s+1)/a}\big(\{P * h_n(\cdot, Q)\colon n \in N, P, Q \in \mathfrak{Q}_n\}\big).$$

Assume that one of the following conditions is satisfied:

$$a > 1$$

or

$$a \leqslant 1 \quad \text{and} \quad \sup_{P,Q \in \mathfrak{Q}_n} \big|P\big(h_n(\cdot, Q)\big)\big| = o(n^{a-1}).$$

Then there exist $\delta > 0$ and, for every $c > 0$, a constant B depending on

$$\sup_{n \in N} \sup_{P,Q \in \mathfrak{Q}_n} P\big(|h_n(\cdot, Q)|^{(s+1+\delta)/a}\big) \quad \text{and} \quad \sup_{P,Q \in \mathfrak{Q}_n} \big|P\big(h_n(\cdot, Q)\big)\big|$$

such that

$$\sup_{P,Q \in \mathfrak{Q}_n} P^n\Big\{x \in X^n\colon n^{-a}\Big|\sum_{\nu=1}^{n} h_n(\cdot, Q)\Big| > c\Big\} \leqslant B n^{-(s+\delta)}.$$

(9.58) LEMMA. *Let the assumptions of Lemma (9.57) be satisfied for $s = \frac{1}{2}$, $\mathfrak{Q}_n = \{P_*\}$ and $h_n(\cdot, Q) = h_n$. Let P_n, $n \in N$, be a sequence of p-measures admitting a P_*-density (9.36) such that*

(9.59) $$M_3(P_* * g),$$

(9.60) $$M^*_{3/2}(\{P_* * \bar{r}_n\colon n \in N\}).$$

Then

$$n^{-a} \sum_{\nu=1}^{n} h_n(x_\nu) = o_n(\tfrac{1}{2})$$

with respect to P_n.

Proof. Let A_n be determined by (9.43). Let a p-measure on $\mathscr{A}$ be defined by

(9.61) $$Q_n(A) := P_n(A \cap A_n)/P_n(A_n), \quad A \in \mathscr{A}.$$

Since $Q_n(h_n) = P_*(h_n) + O(n^{-1/2})$ if $a \leqslant 1$, and the P_*-density of Q_n is bounded by 3/2, from Lemma (9.57) we obtain

(9.62) $$Q_n^n\{n^{-a}|\sum_{\nu=1}^{n} h_n(x_\nu)| > c\} = o(n^{-1/2}).$$

The assertion now follows from (9.44) and (9.62).

(9.63) LEMMA. *Let $\mathfrak{Q}_n$, $n \in N$, be families of p-measures over $\mathscr{A}$. Let $h_n(\cdot, Q)$: $X^n \to R$ and $g_n(\cdot, Q)$: $X^n \to R$, $n \in N$, $Q \in \mathfrak{Q}_n$, be measurable functions fulfilling*

$$h_n(\cdot, Q) = g_n(\cdot, Q) + n^{-1/2} o_n(\tfrac{1}{2})$$

with respect to $\mathfrak{Q}_n$.

Let $H_n(\cdot, Q)$ and $G_n(\cdot, Q)$ be the distribution functions of $Q^n * h_n(\cdot, Q)$ and $Q^n * g_n(\cdot, Q)$, respectively.

If

(9.64) $$|H_n(s, Q) - H_n(s', Q)| \leqslant c|s - s'| + o(n^{-1/2})$$

uniformly for $s, s' \in R$ and $Q \in \mathfrak{Q}_n$, then

$$G_n(s, Q) = H_n(s, Q) + o(n^{-1/2})$$

uniformly for $s \in R$ and $Q \in \mathfrak{Q}_n$.

(9.64) is in particular fulfilled if $H_n(\cdot, Q)$ admits an Edgeworth expansion of order $n^{-1/2}$, uniformly for $Q \in \mathfrak{Q}_n$.

Proof. Choose c_n, $n \in N$, such that $c_n \downarrow 0$ and

$$Q^n\{n^{1/2}|h_n(\cdot, Q) - g_n(\cdot, Q)| > c_n\} = o(n^{-1/2})$$

uniformly for $Q \in \mathfrak{Q}_n$.

Then from (9.64) we obtain

$$\begin{aligned} G_n(s, Q) &\leqslant Q^n\{h_n(\cdot, Q) < s + n^{-1/2} c_n\} + Q^n\{n^{1/2}|h_n(\cdot, Q) - g_n(\cdot, Q)| \geqslant c_n\} \\ &= H_n(s, Q) + o(n^{-1/2}) \end{aligned}$$

uniformly for $s \in R$ and $Q \in \mathfrak{Q}_n$.

In the same way one can show that $G_n(s, Q) \geqslant H_n(s, Q) + o(n^{-1/2})$.

(9.65) LEMMA. *Let P_n, $n \in N$, be a sequence of p-measures fulfilling (9.36), (9.59), and (9.60). Let h_1: $X \to R$ and h_2: $X \to R$ be measurable functions for which the following regularity conditions are fulfilled:*

(9.66) $$P_*(h_1) = P_*(h_2) = 0,$$

(9.67) $$M_3(P_* * h_1), \quad M_{3/2}(P_* * h_2),$$

(9.68) $$C(P_* * h_1).$$

Then

$$P_n^n\{\tilde{h}_1 - n^{1/2} P_n(h_1) + n^{-1/2}(\tilde{h}_2 - \Delta P_*(gh_2)) < s\}$$
$$= \Phi(s\sigma_n^{-1}) + n^{-1/2}\varphi(s\sigma^{-1}) H(s) + o(n^{-1/2})$$

uniformly for $s \in \mathbf{R}$, *where*

$$\sigma_n^2 := P_*(h_1^2) + n^{-1/2}\Delta P_*(h_1^2 g),$$
$$H(s) := \sigma^{-3}\big(\tfrac{1}{6}P_*(h_1^3)(1 - s^2\sigma^{-2}) - P_*(h_1 h_2)s\big)$$

with $\sigma^2 := P_*(h_1^2)$.

Proof. Let A_n and Q_n be defined by (9.43) and (9.61), respectively. By (9.36), (9.59), (9.60), (9.66) and (9.67) we have

$$Q_n(h_1) - P_n(h_1) = o(n^{-1}), \quad Q_n(h_2) - n^{-1/2}\Delta P_*(gh_2) = o(n^{-1/2}).$$

Thus, from (9.67), (9.68) and from Theorem 1 in [2], p. 650, applied for $\tilde{h}_1 - n^{1/2}Q_n(h_1) + n^{-1/2}(\tilde{h}_2 - n^{1/2}Q_n(h_2))$, we obtain

(9.69) $$Q_n^n\{\tilde{h}_1 - n^{1/2}P_n(h_1) + n^{-1/2}(\tilde{h}_2 - \Delta P_*(gh_2)) < s\}$$
$$= \Phi(s\sigma_n'^{-1}) + n^{-1/2}\varphi(s\sigma_n'^{-1}) H_n(s) + o(n^{-1/2})$$

uniformly for $s \in \mathbf{R}$, where $\sigma_n'^2$ is the variance of $Q_n * h_1$, and

$$H_n(s) := \sigma_n'^{-3}\big(\tfrac{1}{6}Q_n(h_1^3)(1 - s^2\sigma_n'^{-2}) - Q_n(h_1 h_2)s\big).$$

Since $\sigma_n' - \sigma_n = o(n^{-1/2})$ and $Q_n(h_1^{\alpha_1} h_2^{\alpha_2}) \to P_*(h_1^{\alpha_1} h_2^{\alpha_2})$, $n \in N$, for all (α_1, α_2) such that $\alpha_1 + 2\alpha_2 \leqslant 3$, the assertion of the lemma follows from (9.69) and (9.44).

(9.70) LEMMA. *Assume that for some strong neighborhood* U_* *of* P_* *in* $\mathfrak{P}$ *the following regularity conditions are fulfilled:*

(9.71) $$K_{3/2}(\varkappa(P_*), U_*) \quad \text{for } f\colon X \times T \to \mathbf{R},$$

(9.72) $$M_3^*(\{P * f^\alpha(\ ,\varkappa(Q))\colon P, Q \in U_*\}) \quad \text{for } |\alpha| = 1, 2, 3,$$

(9.73) $$L_{3/2}^*(\varkappa(P_*), U_*) \quad \text{for } f^\alpha\colon X \times T \to \mathbf{R} \text{ if } |\alpha| = 3.$$

Then, for $i = 0, \ldots, p$,

$$n^{1/2}(\varkappa_i^{(n)} - \varkappa_i(P)) = \tilde{f}_0(\ ,P) + n^{-1/2} M_i(\tilde{f}^{\boldsymbol{\cdot}}, \tilde{f}^{\boldsymbol{\cdot\cdot}}, P) + n^{-1/2} o_n(\tfrac{1}{2})$$

with respect to $U_{n,\delta}(P_*)$ *for every* $\delta \in (0, 1)$, *where*

(9.74) $$M_i(\tilde{f}^{\boldsymbol{\cdot}}, \tilde{f}^{\boldsymbol{\cdot\cdot}}, P) = -\tfrac{1}{2}A_{ij}F_{jkl}\,\tilde{f}_l(\cdot, P)\,\tilde{f}_k(\cdot, P) + \tilde{f}_j(\ ,P)\,\tilde{f}_i^{(j)}(\cdot, P).$$

Proof. The proof follows the pattern of the proof of Theorem 5 in [1], p. 298ff. The crucial point is to show that

(9.75) $$\|\varkappa^{(n)} - \varkappa(P)\| = o_n(\tfrac{1}{2})$$

with respect to $U_{n,\delta}(P_*)$ for every $\delta \in (0, 1)$.

If we copy the proof in [3], p. 79, for the case $K = \{\varkappa(P_*)\}$, we obtain immediately

(9.76) $$\|\varkappa^{(n)} - \varkappa(P_*)\| = o_n(\tfrac{1}{2})$$

with respect to $U_{n,\delta}(P_*)$ for every $\delta \in (0, 1)$.

Since $P \to \varkappa(P)$ is continuous by General Assumption (8.5), relation (9.76) implies (9.75).

10. Proofs. In order not to overload the paper with technicalities, the proofs are given for fixed Δ. Uniformity in Δ can be obtained by exactly the same reasoning if uniform versions of the lemmas are used.

Proof of Theorem (4.16). (i) By General Assumption (8.5), $P \to \varkappa(P)$ is continuous. Hence condition (4.21) implies the existence of g with $M^*_{3/2}(\{P * g\colon P \in U_*\})$ such that, for some strong neighborhood $U'_* \subset U_*$ of P_*,

(10.1) $$|f^{(ij)}(\cdot, \varkappa(P)) - f^{(ij)}(\ , \varkappa(P_*))| \leqslant |(\varkappa_k(P) - \varkappa_k(P_*))\, f^{(ijk)}(\ , \varkappa(P_*))| + \|\varkappa(P) - \varkappa(P_*)\|^2 g,$$

(10.2) $$|f^{(ijk)}(\cdot, \varkappa(P)) - f^{(ijk)}(\cdot, \varkappa(P_*))| \leqslant \|\varkappa(P) - \varkappa(P_*)\|\, g.$$

Hence it follows easily that $P \to F_{ijk}(P)$, $P \to F_{ij}(P)$ and $P \to A_{ij}(P)$ are continuous at P_* in the strong topology.

Thus the coefficients of the polynomials $M_i(\cdot, \cdot, P)$ defined in (9.74) are continuous at P_*.

(ii) By condition (4.23), for every $P \in U'_*$ there exists a P-linearly independent subsystem $\{f_0(\cdot, P), g_1(\cdot, P), \ldots, g_m(\cdot, P)\}$ of $\{f_i(\cdot, P), i = 0, \ldots, p, f_0^{(j)}(\cdot, P) - \delta_{0j}, j = 0, \ldots, p, k(\cdot, P) - P(k(\cdot, P))\}$ generating the same space and fulfilling

(10.3) $$C(\{P_* * (f_0(\cdot, P), g(\cdot, P))\colon P \in U'_*\}).$$

Without loss of generality we may assume that $f_0(\cdot, P_*)$ and $g_i(\cdot, P_*)$ are P_*-uncorrelated. Otherwise, we replace $g_i(\cdot, P)$ by

$$g'_i(\cdot, P) := g_i(\cdot, P) - P_*(f_0(\cdot, P_*)\, g_i(\cdot, P_*))\, \sigma_{00}^{-1} f_0(\ , P).$$

Notice that (10.3) and the following statements remain valid for $g'_i(\ , P)$. Moreover, there exists a polynomial $M(\ , \cdot, P)$ the coefficients of which are continuous at P_* such that

(10.4) $$M(\tilde{f}_0(\cdot, P), \tilde{g}(\cdot, P), P) = M_0(\tilde{f}^{\bullet}, \tilde{f}^{\bullet\bullet}, P) + \tilde{k}(\cdot, P) \qquad P^n\text{-a.e.}$$

From Lemma (9.70), (4.3), (4.6), and (10.4) we get

$$F_n(\cdot, \varkappa_0(P)) = \tilde{f}_0(\cdot, P) + N_\alpha \sigma_0(P) + {} + n^{-1/2} M(\tilde{f}_0(\cdot, P), \tilde{g}(\cdot, P), P) - n^{-1/2} c_\alpha(P) + n^{-1/2} o_n(\tfrac{1}{2})$$

with respect to $U_{n,\delta}(P_*)$.

By Lemma 3 in [6], p. 245, we see from the choice of c_α (cf. (4.5)) that F_n, $n \in N$, is asymptotically similar of level $\alpha + o(n^{-1/2})$.

Proof of Proposition (4.25) (i).

(α) Let $V(\varkappa(P_*))$ be given by condition (4.30). Then we infer from (9.76) that for every $\delta \in (0, 1)$

$$P^n\{\varkappa^{(n)}(x) \notin V(\varkappa(P_*))\} = o(n^{-1/2}) \tag{10.5}$$

uniformly for $P \in U_{n,\delta}(P_*)$.

Furthermore, it follows from General Assumption (8.5) that there exists a strong neighborhood $U'_* \subset U_*$ such that $\varkappa(P) \in V(\varkappa(P_*))$ for $P \in U'_*$. Thus for $\varkappa^{(n)}(x) \in V(\varkappa(P_*))$ and $P \in U'_*$, by a Taylor expansion of $t \to n^{-1} \sum_{\nu=1}^{n} f^{(ij)}(x_\nu, t)$ about $\varkappa(P)$, we obtain

$$F_{ij}^{(n)} - F_{ij}(P) = n^{-1/2}\, \tilde{f}^{(ij)}(\cdot, \varkappa(P)) + F_{ijk}(P)(\varkappa_k^{(n)} - \varkappa_k(P)) + R'_n(\cdot, P), \tag{10.6}$$

where

$$\begin{aligned} &|R'_n(x, P)| \\ &\leqslant \|\varkappa^{(n)}(x) - \varkappa(P)\| \left\| \left(n^{-1} \sum_{\nu=1}^{n} f^{(ijk)}(x_\nu, \varkappa(P)) - F_{ijk}(P)\right)_{k=0,\dots,p} \right\| + \\ &\qquad + \|\varkappa^{(n)}(x) - \varkappa(P)\|^2\, \tfrac{1}{2} n^{-1} \sum_{\nu=1}^{n} g(x_\nu), \end{aligned} \tag{10.7}$$

g being the function which occurs in $L_2(\varkappa(P_*), U_*)$ for f^α if $|\alpha| = 3$.

By Lemma (9.57), (9.75) and General Assumption (8.5) we have

$$\|\varkappa^{(n)} - \varkappa(P)\| = n^{-1/4-\varepsilon} o_n(\tfrac{1}{2}) \tag{10.8}$$

with respect to $U_{n,\delta}(P_*)$ for every $\delta \in (0, 1)$ and some sufficiently small $\varepsilon > 0$, and

$$\left\| \left(n^{-1/2}\, \tilde{f}^{(ijk)}(\cdot, \varkappa(P))\right)_{k=0,\dots,p} \right\| = n^{-1/4-\varepsilon} o_n(\tfrac{1}{2}) \tag{10.9}$$

with respect to $U_{n,\delta}(P_*)$ for every $\delta \in (0, 1)$.

Thus, by Lemma (9.57), (10.5), (10.6), (10.8), and (10.9), we get

$$R'_n(\cdot, P) = n^{-1/2} o_n(\tfrac{1}{2}) \tag{10.10}$$

with respect to $U_{n,\delta}(P_*)$ for every $\delta \in (0, 1)$.

Let

$$\varphi_{ij}(\cdot, P) := f^{(ij)}(\cdot, \varkappa(P)) - F_{ij}(P) - F_{ijk}(P) f_k(\cdot, P).$$

Using Lemma (9.57), we obtain

$$F_{ij}^{(n)} - F_{ij}(P) = n^{-1/2}\, \tilde{\varphi}_{ij}(\cdot, P) + n^{-1/2} o_n(\tfrac{1}{2}) \tag{10.11}$$

with respect to $U_{n,\delta}(P_*)$ for every $\delta \in (0, 1)$.

In a similar way as above one can show that

$$(10.12)\quad F_{i,j}^{(n)}-F_{i,j}(P) = n^{-1/2}\Big(\big(f^{(i)}f^{(j)}\big)^{\sim}(\cdot,\varkappa(P))+\big(F_{i,jk}(P)+F_{j,ik}(P)\big)\tilde{f}_k(\cdot,P)\Big)+n^{-1/2}o_n(\tfrac{1}{2})$$

with respect to $U_{n,\delta}(P_*)$ for every $\delta\in(0,1)$.

(β) Let

$$C_n := \{x\in X^n\colon F_{ij}^{(n)}(x) \text{ is invertible}\}.$$

As $P\to F_{ij}(P)$ is continuous because of condition (4.29) and the continuity of $P\to\varkappa(P)$, we have

$$C_n^c \subset \{\|(F_{ij}^{(n)}-F_{ij}(P_*))_{i,j=0,\dots,p}\| \geqslant d\}$$

for some $d>0$ and for all P in some neighborhood $U_*''\subset U_*$.

Thus

$$(10.13)\qquad P^n(C_n^c) = o(n^{-1/2})$$

uniformly for all $P\in U_{n,\delta}(P_*)$ for every $\delta\in(0,1)$.

Putting

$$\alpha_{ij}(\cdot,P) := -A_{il}(P)A_{jk}(P)\varphi_{lk}(\cdot,P)$$

we obtain from (10.11)

$$(10.14)\qquad F_{ij}^{(n)}\big(A_{jl}(P)+n^{-1/2}\tilde{\alpha}_{jl}(\cdot,P)\big) = \delta_{il}+n^{-1/2}o_n(\tfrac{1}{2})$$

with respect to $U_{n,\delta}(P_*)$ for every $\delta\in(0,1)$ and, therefore, by (10.13),

$$(10.15)\qquad A_{ij}^{(n)} = A_{ij}(P)+n^{-1/2}\tilde{\alpha}_{ij}(\cdot,P)+n^{-1/2}o_n(\tfrac{1}{2})$$

with respect to $U_{n,\delta}(P_*)$ for every $\delta\in(0,1)$.

From (10.12), (10.15), and a Taylor expansion of $x\to x^{1/2}$ about $x=\sigma_{00}$ we obtain

$$(10.16)\qquad \sigma_0^{(n)} = \sigma_0(P)+n^{-1/2}\tilde{k}(\cdot,P)+n^{-1/2}o_n(\tfrac{1}{2})$$

with respect to $U_{n,\delta}(P_*)$ for every $\delta\in(0,1)$, where $k(\cdot,P)$ is given by (4.27).

P r o o f o f P r o p o s i t i o n (4.25) (ii). The proof is a simple application of Lemma (9.57) and will be omitted.

P r o o f o f T h e o r e m (5.1). The theorem follows immediately from Lemmas (9.22) and (9.35) applied for $P_{n,\Delta}$, $n\in N$, $0<\Delta\leqslant\Delta_0$.

P r o o f o f C o r o l l a r y (5.11). The corollary follows immediately from Theorems (4.16) and (5.1) if we establish that for every Δ $(0<\Delta\leqslant\Delta_0)$ there exists a sequence $P_{n,\Delta}\in\mathfrak{P}$, $n\in N$, fulfilling (5.2)-(5.5). We restrict ourselves to prove the assertion for fixed $\Delta>0$.

By (5.12), there exists $\varepsilon \in (0,1)$ such that $M_{(9+\varepsilon)/2}\big(P_* * f^\alpha(\cdot, \varkappa(P_*))\big)$ is fulfilled for $|\alpha| = 1$. Let

$$\beta := \frac{3+\varepsilon/4}{6+3\varepsilon/4} \in (0, \tfrac{1}{2}) \quad \text{and} \quad k'_{n,i} := f^{(i)} \mathbf{1}_{\{|f^{(i)}| \leq n^\beta\}}.$$

Since $P_*(f^{(i)}) = 0$, we obtain

(10.17) $$P_*(k'_{n,i}) = o(n^{-3/2}).$$

Let, furthermore, $k_{n,i} := k'_{n,i} - P_*(k'_{n,i})$ and let a be defined by (9.7). From (10.17) and a Taylor expansion of $t \to f^{(j)}(\cdot, t)$ about $\varkappa(P_*)$ we obtain

(10.18) $$P_*\Big(\big(f^{(i)} - k_{n,i}\big) f^{(j)}\big(\cdot, \varkappa(P_*) + n^{-1/2} a\big)\Big) = O(n^{-1}).$$

Let F_n be a matrix defined by

$$F_{n,i,j} := P_*\Big(k_{n,i} f^{(j)}\big(\cdot, \varkappa(P_*) + n^{-1/2} \Delta a\big)\Big), \qquad i,j = 0, \ldots, p.$$

By a Taylor expansion and (10.18) we have

$$F_{n,i,j} = F_{i,j} + n^{-1/2} \Delta a_k F_{i,jk} + O(n^{-1}).$$

Thus, F_n is invertible if n is sufficiently large, and the inverse, say B_n, admits the expansion

(10.19) $$B_{n,ij} = B_{ij} + n^{-1/2} e_{ij} + O(n^{-1}),$$

where $(B_{ij})_{i,j=0,\ldots,p}$ is the inverse of $(F_{i,j})_{i,j=0,\ldots,p}$, and

(10.20) $$e_{ij} := -B_{jk} B_{il} F_{l,kp} a_p.$$

Let now $a_{n,j}$, $n \in N$, $j = 0, \ldots, p$, be defined by

$$a_{n,j} := n^{1/2} B_{n,ji} P_*\Big(f^{(i)}\big(\cdot, \varkappa(P_*) + n^{-1/2} \Delta a\big)\Big).$$

From (10.19) we obtain

(10.21) $$a_{n,j} = \Delta \sigma_{00}^{-1} A_{0j} + n^{-1/2} \Delta^2 (e_{ji} F_{ik} a_k + B_{ji} F_{ikl} a_k a_l) + n^{-1} R_{n,j},$$

where $R_{n,j} = O(n^0)$.

As $a_{n,j}$ is bounded, the signed measure P_n, defined by the P_*-density $p_n := 1 + n^{-1/2} a_{n,j} k_{n,j}$, belongs to $\mathfrak{P}$ if n is sufficiently large.

Furthermore, by a simple calculation we obtain

$$P_n\Big(f^{(i)}\big(\cdot, \varkappa(P_*) + n^{-1/2} \Delta a\big)\Big) = 0, \qquad i = 0, \ldots, p,$$

provided n is sufficiently large.

Thus

$$\varkappa(P_n) = \varkappa(P_*) + n^{-1/2} \Delta a.$$

It follows from (10.21) and the definition of $k_{n,j}$ that p_n can be written in the form (5.3) with

$$h := e_{ji}(F_{ik}a_k + B_{ji}F_{ikl}a_k a_l) f^{(j)},$$

$$r_{n,\Delta} := n^{1/2}\Delta\left(n^{1/2}\sigma_{00}^{-1}A_{0j} + \Delta e_{ji}(F_{ik}a_k + B_{ji}F_{ikl}a_k a_l)\right) f^{(j)} \mathbf{1}_{\{|f^{(j)}| > n^{\beta}\}} + \\ + nR_{n,j}K'_{n,j} - na_{n,j}P_*(K'_{n,j}).$$

Condition (5.4) holds trivially.

By the choice of β,

$$\int |f^{(i)}|^{(3/2+\varepsilon)/8} \mathbf{1}_{\{|f^{(i)}| > n^{\beta}\}} dP_* = O(n^{-(3/2+\varepsilon)/8}),$$

$$\int f^{(i)2} \mathbf{1}_{\{|f^{(i)}| > n^{\beta}\}} dP_* = o(n^{-1}).$$

Hence condition (5.5) is fulfilled.

Proof of Corollary (5.15). Let $\Delta > 0$ and $a_i := -\Lambda_{00}^{-1}\Lambda_{0i}$, $i = 0, \ldots, p$. Then for sufficiently large $n \in N$ we have $\theta_* + n^{-1/2}\Delta a \in \Theta$, and the sequence $P_n := P_{\theta^* + n^{-1/2}\Delta a}$ fulfills (5.2).

We have

(10.22) $$p(\cdot, \theta^* + n^{-1/2}\Delta a)/p(\cdot, \theta^*)$$
$$= 1 + n^{-1/2}\Delta a_i p^{(i)}(\cdot, \theta^*)/p(\cdot, \theta^*) + \tfrac{1}{2}n^{-1}\Delta^2 a_i a_j p^{(ij)}(\cdot, \theta^*)/p(\cdot, \theta^*) + \\ + n^{-1}\Delta^2 a_i a_j \int_0^1 \left[(1-u)\left(p^{(ij)}(\cdot, \theta^* + un^{-1/2}\Delta a) - p^{(ij)}(\cdot, \theta^*)\right)/p(\cdot, \theta^*)\right] du.$$

Hence (5.3)-(5.5) follow easily by conditions (5.12) and (5.16).

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