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LAWS OF LARGE NUMBERS OF ERDÖS-RÉNYI'S TYPE FOR NON-STATIONARY RANDOM FIELDS

BY

AUGUST MICHAŁ ZAPAŁA (LUBLIN)

Abstract. Let $K_r \subset R^r$ be the r-dimensional cartesian product of the set of positive integers and let $\{X_{\bar{k}},\ \bar{k} \in K_r\}$ be a random field – a collection of independent, not necessarily identically distributed random variables with mean zero. Under appropriate additional assumptions we derive for $\{X_{\bar{k}},\ \bar{k} \in K_r\}$ strong limit theorems of the same type as the Erdös-Rényi law of large numbers. Our results are based on the large deviation theorem of Petrov extended to random fields.

1. Introduction. Let K_r be the set of r-tuples

$$k = (k(1), k(2), \ldots, k(r))$$

with $k(j)$, $1 \leqslant j \leqslant r$, taken from the set $\mathcal{N}$ of all positive integers and let

$$K_r^0 = \{\bar{k} = (k(1), k(2), \ldots, k(r)):\ k(j) \in \mathcal{N} \cup \{0\},\ 1 \leqslant j \leqslant r\}, \quad r \geqslant 1.$$

The sets K_r and K_r^0 can be partially ordered by the relation

$$\bar{k} \leqslant \bar{m} \Leftrightarrow k(j) \leqslant m(j) \text{ for } j = 1, 2, \ldots, r.$$

Write

$$|\bar{k}| = \prod_{j=1}^{r} k(j),$$

$$\bar{k} \pm \bar{m} = (k(1) \pm m(1), k(2) \pm m(2), \ldots, k(r) \pm m(r)),$$

$$[t\bar{k}] = ([tk(1)], [tk(2)], \ldots, [tk(r)]), \quad t \geqslant 0,$$

where $[x]$ denotes the greatest integer less than or equal to x,

$$\bar{m}\bar{k} = (m(1)k(1), m(2)k(2), \ldots, m(r)k(r)),$$

$$(\bar{a}, \bar{b}\rangle = \{\bar{k} \in K_r:\ \bar{a} + \bar{1} \leqslant \bar{k} \leqslant \bar{b},\ \bar{a} \in K_r^0,\ \bar{b} \in K_r\}.$$

In what follows, $\bar{k} \to \infty$ means that $k(j) \to \infty$ for $j = 1, 2, \ldots, r$.

We shall study a random field $\{X_{\bar{k}}, \bar{k} \in K_r\}$, consisted of independent random variables indexed by K_r, with zero means, defined on the same probability space $(\Omega, \mathscr{F}, P)$. Let us put

$$S(\bar{a}, \bar{b}\rangle = \sum_{\bar{k} \in (\bar{a}, \bar{b}\rangle} X_{\bar{k}}$$

and

$$\sum(\bar{N}, \bar{K}_N^{(\alpha)}) = \max_{\bar{0} \leqslant \bar{n} \leqslant \bar{N} - \bar{K}_N^{(\alpha)}} \frac{S(\bar{n}, \bar{n} + \bar{K}_N^{(\alpha)}\rangle}{\sqrt{ES^2(\bar{n}, \bar{n} + \bar{K}_N^{(\alpha)}\rangle \sum \psi(\varrho(\bar{n}, \bar{K}_N^{(\alpha)}))}},$$

where $\bar{K}_N^{(\alpha)} \leqslant \bar{N}$ and $\sum \psi(\varrho(\bar{n}, \bar{K}_N^{(\alpha)}))$, $\alpha = 1, 2$, will be defined precisely in the sequel.

To define $\sum \psi(\varrho(\bar{n}, \bar{K}_N^{(\alpha)}))$ consider real functions

$$\psi_j \colon \mathscr{R}_+ \to \mathscr{R}_+, \quad 1 \leqslant j \leqslant r,$$

satisfying the following conditions:

(1) ψ_j, $1 \leqslant j \leqslant r$, are continuous and monotonically increasing to infinity;

(2) $\lim_{x \to \infty} \frac{\psi_j(x)}{\psi_j(x+1)} = c_j$, $0 < c_j \leqslant 1$, $1 \leqslant j \leqslant r$;

(3) $\lim_{x \to \infty} \frac{\psi_j(x)}{\log x} = \infty$, $1 \leqslant j \leqslant r$;

(4) the set

$$W_{\Delta''} = \left[\{\bar{k}\} \subset K_r \colon \lim_{|\bar{k}| \to \infty} \frac{\sum_{j=1}^{r} \psi_j(\Delta'' k(j))}{|\bar{k}|} = 0 \right],$$

where Δ'' is a positive constant, is nonempty.

In addition, let us put

$$R_N^{(\alpha)}(j) = \psi_j^{-1}(2\lambda^{-2} c_j^{[-\Delta^{(\alpha)}]} \log N(j)), \quad 1 \leqslant j \leqslant r,$$

where λ and $\Delta^{(\alpha)}$ $(\alpha = 1, 2)$ are positive constants,

$$K_N^{(\alpha)}(j) = [R_N^{(\alpha)}(j)/\Delta^{(\alpha)}],$$

$$\bar{K}_N^{(\alpha)} = (K_N^{(\alpha)}(1), K_N^{(\alpha)}(2), \ldots, K_N^{(\alpha)}(r)),$$

$$\bar{p}(p(j) = s) = (p(1), p(2), \ldots, p(j-1), s, p(j+1), \ldots, p(r)),$$

$$m_j(\bar{n}, \bar{K}_N^{(1)}) = \min_{\substack{1 \leqslant p(i) \leqslant K_N^{(1)}(i) \\ i \neq j}} \sigma^2 S(\bar{n} + (\bar{p} - \bar{1})(p(j) - 1 = 0), \bar{n} + \bar{p}(p(j) = K_N^{(1)}(j))\rangle,$$

$$M_j(\bar{n}, \bar{K}_N^{(2)}) = \max_{\substack{1 \leqslant p(i) \leqslant K_N^{(2)}(i) \\ i \neq j}} \sigma^2 S(\bar{n} + (\bar{p} - \bar{1})(p(i) - 1 = 0), \bar{n} + \bar{p}(p(j) = K_N^{(2)}(j))\rangle,$$

and

$$\sum \psi\big(\varrho(\bar{n}, \bar{K}_{\bar{N}}^{(1)})\big) = \sum_{j=1}^{r} \psi_j\big(m_j(\bar{n}, \bar{K}_{\bar{N}}^{(1)})\big),$$

$$\sum \psi\big(\varrho(\bar{n}, \bar{K}_{\bar{N}}^{(2)})\big) = \sum_{j=1}^{r} \psi_j\big(M_j(\bar{n}, \bar{K}_{\bar{N}}^{(2)})\big).$$

Let us observe that in the case where $\{X_{\bar{k}}, \bar{k} \in K_r\}$ are independent and identically distributed random variables with $\sigma^2 X_{\bar{k}} = \sigma^2$ we have

$$m_j(\bar{n}, \bar{K}_{\bar{N}}^{(1)}) = K_{\bar{N}}^{(1)}(j)\sigma^2, \quad M_j(\bar{n}, \bar{K}_{\bar{N}}^{(2)}) = K_{\bar{N}}^{(2)}(j)\sigma^2.$$

To describe the asymptotic behaviour of sums of the type $\sum(\bar{N}, \bar{K}_{\bar{N}}^{(\alpha)})$ we need a result giving estimates for large deviation probabilities of lattice indexed sums of independent random variables.

THEOREM 1. *Let $\{X_{\bar{k}}, \bar{k} \in K_r\}$ be a random field of independent random variables with $\mathrm{E}X_{\bar{k}} = 0$ and $\sigma^2 X_{\bar{k}} = \sigma_{\bar{k}}^2 < \infty$. Suppose that there exists $H > 0$ such that for $|z| < H$ there exist moment generating functions $\mathrm{E}\exp(zX_{\bar{k}})$ for all $\bar{k} \in K_r$, and let us put*

$$L_{\bar{k}}(z) = \log \mathrm{E}\exp(zX_{\bar{k}}), \quad L_{\bar{k}}(0) = 0.$$

If there exists $b_{\bar{k}}$, $\bar{k} \in K_r$, such that

(5) $$|L_{\bar{k}}(z)| \leqslant b_{\bar{k}} \quad \text{for } |z| < H, \bar{k} \in K_r,$$

(6) $$\limsup_{|\bar{k}| \to \infty} \frac{1}{|\bar{k}|} \sum_{\bar{i} \leqslant \bar{k}} b_{\bar{i}}^{3/2} < \infty,$$

and there exists a positive constant $\Delta' > 0$ such that, for $\bar{k} \in K_r$,

(7) $$\frac{1}{|\bar{k}|} \sum_{\bar{i} \leqslant \bar{k}} \sigma_{\bar{i}}^2 \geqslant \Delta',$$

and if $x_{\bar{n}} \geqslant 0$, $x_{\bar{n}} = o(\sqrt{|\bar{n}|})$ as $|\bar{n}| \to \infty$, then

(8) $$\frac{1 - F_{\bar{n}}(x_{\bar{n}})}{1 - \Phi(x_{\bar{n}})} = \exp\left\{\frac{x_{\bar{n}}^3}{\sqrt{|\bar{n}|}} g_{\bar{n}}\left(\frac{x_{\bar{n}}}{\sqrt{|\bar{n}|}}\right)\right\}\left[1 + O\left(\frac{x_{\bar{n}}+1}{\sqrt{|\bar{n}|}}\right)\right] \quad \text{as } |\bar{n}| \to \infty,$$

where

$$F_{\bar{n}}(x_{\bar{n}}) = P\left[S(\bar{0}, \bar{n}\rangle < x_{\bar{n}}\sqrt{\sigma^2 S(\bar{0}, \bar{n}\rangle}\right],$$

Φ denotes the standard normal distribution function, while $g_{\bar{n}}(t)$ are (known) power series convergent uniformly with respect to $\bar{n}$ for all sufficiently small $t \geqslant 0$.

Theorem 1 can be proved following Petrov's considerations of [4], p. 270-280.

2. The Erdös-Rényi law of large numbers for directed sets of independent random variables. Using Theorem 1 and some ideas of [2] we can prove the following result:

THEOREM 2. *Let* $\{X_{\bar{k}}, \bar{k} \in K_r\}$ *be a random field of independent random variables with* $\mathrm{E}X_{\bar{k}} = 0$, $\mathrm{E}X_{\bar{k}}^2 = \sigma_{\bar{k}}^2 < \infty$, *and let* $L_{\bar{k}}(z) = \log \mathrm{E} \exp(zX_{\bar{k}})$, *where* $L_{\bar{k}}(0) = 0$.

Suppose that there exist positive constants H *and* $b_{\bar{k}}$, $\bar{k} \in K_r$, *such that*

$$|L_{\bar{k}}(z)| \leqslant b_{\bar{k}} \quad \text{for } |z| < H, \tag{9}$$

$$\limsup_{|\bar{k}| \to \infty} \left[\sup_{\bar{n} \in K_r^0} \frac{1}{|\bar{k}|} \sum_{\bar{i}=\bar{n}+\bar{1}}^{\bar{n}+\bar{k}} b_{\bar{i}}^{3/2} \right] < \infty, \tag{10}$$

and there exists a constant $\Delta' > 0$ *such that for* $|\bar{k}| > k_1$, *where* $\bar{k} \in K_r$ *and* k_1 *is a positive number,*

$$\inf_{\bar{n} \in K_r^0} \frac{1}{|\bar{k}|} \sum_{\bar{i}=\bar{n}+\bar{1}}^{\bar{n}+\bar{k}} \sigma_{\bar{i}}^2 \geqslant \Delta'. \tag{11}$$

If ψ_j, $1 \leqslant j \leqslant r$, *are real functions satisfying conditions* (1)-(4), *then for any given* $\lambda > 0$ *and* $0 < \Delta^{(1)} \leqslant \Delta'$, *and for* $\bar{N} \to \infty$ *in such a way that* $\{\bar{K}_{\bar{N}}^{(1)}\} \in W_{\Delta''}$, *we have*

$$\limsup_{\bar{N} \to \infty} \sum (\bar{N}, \bar{K}_{\bar{N}}^{(1)}) \leqslant \lambda \text{ a.s.}, \tag{12}$$

and for any given $\lambda > 0$ *and* $\Delta^{(2)} \geqslant \Delta'' > 0$, *and for* $\bar{N} \to \infty$ *in such a way that* $\{\bar{K}_{\bar{N}}^{(2)}\} \in W_{\Delta''}$, *we obtain*

$$\liminf_{\bar{N} \to \infty} \sum (\bar{N}, \bar{K}_{\bar{N}}^{(2)}) \geqslant \lambda \text{ a.s.} \tag{13}$$

Remark. If (9)-(11) are satisfied, then there exists a constant Δ'', $0 < \Delta'' < \infty$, such that

$$\sup_{\bar{n} \in K_r^0} \frac{1}{|\bar{k}|} \sum_{\bar{i}=\bar{n}+\bar{1}}^{\bar{n}+\bar{k}} \sigma_{\bar{i}}^2 \leqslant \Delta'' \quad \text{for } \bar{k} \in K_r. \tag{14}$$

COROLLARY 1. *If* $\{X_{\bar{k}}, \bar{k} \in K_r\}$ *are independent and identically distributed random variables with* $\Delta^{(1)} = \Delta^{(2)} = \sigma^2 X_{\bar{k}}$, *then* $\bar{K}_{\bar{N}}^{(1)} = \bar{K}_{\bar{N}}^{(2)} (= \bar{K}_{\bar{N}})$ *and*

$$\sum \psi(\varrho(\bar{n}, \bar{K}_{\bar{N}}^{(1)})) = \sum \psi(\varrho(\bar{n}, \bar{K}_{\bar{N}}^{(2)})) = \sum \psi(\varrho(\bar{n}, \bar{K}_{\bar{N}})),$$

$$\lim_{\bar{N} \to \infty} \sum (\bar{N}, \bar{K}_{\bar{N}}) = \lambda \text{ a.s.}$$

COROLLARY 2 (Book [2]). *If* $r = 1$, *then*

$$\limsup_{N \to \infty} \sum (N, K_N^{(1)}) \leqslant \lambda \text{ a.s.},$$

$$\liminf_{N \to \infty} \sum (N, K_N^{(2)}) \geqslant \lambda \text{ a.s.}$$

COROLLARY 3. *Let $\{X_{\bar{k}}, \bar{k} \in K_r\}$ be a random field of independent and identically distributed random variables with $EX_{\bar{k}} = 0, EX_{\bar{k}}^2 = 1, \bar{k} \in K_r$, such that* (9) *is satisfied.*

If $\psi_j(x) = x^{(2-t_j)/t_j}$, where $1 < t_j < 2, 1 \leqslant j \leqslant r$, and if $\Delta^{(1)} = \Delta^{(2)} = 1$, then $\bar{K}_{\bar{N}}^{(1)} = \bar{K}_{\bar{N}}^{(2)}$ and

$$\lim_{\bar{N}\to\infty}\left[\max_{\bar{0}\leqslant\bar{n}\leqslant\bar{N}-\bar{K}_{\bar{N}}}\frac{S(\bar{n},\bar{n}+\bar{K}_{\bar{N}}\rangle}{\sqrt{|\bar{K}_{\bar{N}}|\sum_{j=1}^{r}(K_{\bar{N}}(j))^{(2-t_j)/t_j}}}\right] = \lambda \text{ a.s.}$$

For $r = 1$ Corollary 3 reduces to Theorem 2.1 of [1].

Suppose now that we change the asymptotic behaviour of $\{\bar{K}_{\bar{N}}^{(1)}\}$ by the following additional condition:

There exist $k_1, k_2, \ldots, k_s$ $(1 \leqslant k_1 < k_2 < \ldots < k_s \leqslant r)$ such that for all k_i, $1 \leqslant i \leqslant s$,

$$\frac{\psi_{k_i}(R_{\bar{N}}^{(1)}(k_i))}{\log N(k_i)} \to \infty \quad \text{as } N(k_i) \to \infty, \tag{15}$$

where $\{R_{\bar{N}}^{(1)}(k_i)\}, 1 \leqslant i \leqslant s$, are arbitrary increasing sequences of real numbers, indexed by $N(k_i)$, such that $R_{\bar{N}}^{(1)}(k_i) \to \infty$ and $R_{\bar{N}}^{(1)}(k_i)/N(k_i) \to 0$ as $N(k_i) \to \infty$ for $i = 1, 2, \ldots, s$, while $K_{\bar{N}}^{(1)}(k)$, $k = 1, 2, \ldots, r$, is defined as previously, i.e.

$$K_{\bar{N}}^{(1)}(k_i) = [R_{\bar{N}}^{(1)}(k_i)/\Delta^{(1)}], \quad 1 \leqslant i \leqslant s,$$

$$K_{\bar{N}}^{(1)}(k) = [\psi_k^{-1}(2\lambda^{-2}c_k^{[-\Delta^{(1)}]}\log N(k))/\Delta^{(1)}] \quad \text{for } k \neq k_i, i = 1, 2, \ldots, s.$$

Write

$$\bar{N}' = (N(k_1), N(k_2), \ldots, N(k_s)), \quad \bar{N}'' = (N(j_1), N(j_2), \ldots, N(j_{r-s})),$$

where $1 \leqslant j_1 < \ldots < j_{r-s} \leqslant r$, $j_m \neq k_n$, $1 \leqslant n \leqslant s$, $1 \leqslant m \leqslant r-s$.

Now we can prove the following result:

THEOREM 3. *Let $\{X_{\bar{k}}, \bar{k} \in K_r\}$ be a random field of independent random variables with mean zero satisfying* (9)-(11) *and let $\psi_j, 1 \leqslant j \leqslant r$, be real functions satisfying* (1)-(4). *Furthermore, suppose that condition* (15) *is satisfied.*

Then, for an arbitrary $\lambda > 0$ and $0 < \Delta^{(1)} \leqslant \Delta'$, and for $\bar{N} \to \infty$ in such a way that $\{\bar{K}_{\bar{N}}^{(1)}\} \in W_{\Delta''}$, (12) *is true and, moreover, for $\bar{N}' \to \infty$ with $|\bar{N}''|$ bounded in such a way that $\{\bar{K}_{\bar{N}}^{(1)}\} \in W_{\Delta''}$, we have*

$$\lim \textstyle\sum(\bar{N}, \bar{K}_{\bar{N}}^{(1)}) = 0 \text{ a.s.} \tag{16}$$

COROLLARY 4. *If $r = 1$ and $\psi(K_N)/\log N \to \infty$ as $N \to \infty$, then*

$$\lim_{N\to\infty} \textstyle\sum(N, K_N^{(1)}) = 0 \text{ a.s.}$$

This is Theorem 4.1 of [2].

Let us consider now the case where (15) is replaced by the following condition:

There exist $k_1, k_2, \ldots, k_s$ $(1 \leqslant k_1 < \ldots < k_s \leqslant r)$ such that for all k_i, $1 \leqslant i \leqslant s$,

(17) $$\frac{\psi_{k_i}(R_N^{(2)}(k_i))}{\log N(k_i)} \to 0 \quad \text{as } N(k_i) \to \infty,$$

where $\{R_N^{(2)}(k_i)\}$, $1 \leqslant i \leqslant s$, are arbitrary increasing sequences of real numbers, indexed by $N(k_i)$, such that $R_N^{(2)}(k_i) \to \infty$ as $N(k_i) \to \infty$ for $i = 1, 2, \ldots, s$ and, as previously,

$$K_N^{(2)}(k_i) = [R_N^{(2)}(k_i)/\Delta^{(2)}], \quad 1 \leqslant i \leqslant s,$$

$$K_N^{(2)}(k) = [\psi_k^{-1}(2\lambda^{-2} c_k^{[-\Delta^{(2)}]} \log N(k))/\Delta^{(2)}] \quad \text{for } k \neq k_i,\ i = 1, 2, \ldots, s.$$

THEOREM 4. *Let $\{X_{\bar{k}}, \bar{k} \in K_r\}$ be a random field of independent random variables with mean zero satisfying* (9)-(11) *and let ψ_j, $1 \leqslant j \leqslant r$, be real functions satisfying* (1)-(4). *Furthermore, suppose that condition* (17) *holds.*

Then, for an arbitrary $\lambda > 0$ and $\Delta^{(2)} \geqslant \Delta''$, and for $\bar{N} \to \infty$ in such a way that $\{\bar{K}_N^{(2)}\} \in W_{\Delta''}$, (13) *is true and, moreover, for $\bar{N}' \to \infty$ with $|\bar{N}''|$ bounded in such a way that $\{\bar{K}_N^{(2)}\} \in W_{\Delta''}$, we have*

(18) $$\lim \sum (\bar{N}, \bar{K}_N^{(2)}) = \infty \quad a.s.$$

COROLLARY 5. *If $r = 1$ and $\psi(K_N)/\log N \to 0$ as $N \to \infty$, then*

$$\lim_{N \to \infty} \sum (N, K_N^{(2)}) = \infty \quad a.s.$$

This is Theorem 4.2 of [2].

3. Proofs. In what follows C will denote positive constants, in general different.

Proof of Theorem 2. Inequality (14) can be proved by the same arguments as in [4], p. 271, applied to the random field $\{X_{\bar{k}}, \bar{k} \in K_r\}$.

To prove (12) let us put

(19) $$m_{\bar{j}}(k) = \min\{m \in \mathcal{N}: j(k) \leqslant \psi_k^{-1}(2\lambda^{-2} c_k^{[-\Delta^{(1)}]} \log m)/\Delta^{(1)}\},$$

(20) $$M_{\bar{j}}(k) = \max\{m \in \mathcal{N}: \psi_k^{-1}(2\lambda^{-2} c_k^{[-\Delta^{(1)}]} \log m)/\Delta^{(1)} < j(k)+1\},$$

where $\bar{j} \in K_r$. We need only to consider such $\bar{j} \in K_r$ for which $m_{\bar{j}}(k) \leqslant M_{\bar{j}}(k)$, $k = 1, 2, \ldots, r$, since if $m_{\bar{j}}(k) > M_{\bar{j}}(k)$, then $\bar{K}_N^{(1)}$ cannot take the value $\bar{j}$. Note that for all $\bar{N} \in K_r$ such that $m_{\bar{j}}(k) \leqslant N(k) \leqslant M_{\bar{j}}(k)$, $k = 1, 2, \ldots, r$, the numbers

$$ES^2(\bar{n}, \bar{n}+\bar{K}_N^{(1)}\rangle \sum \psi(\varrho(\bar{n}, \bar{K}_N^{(1)})) = ES^2(\bar{n}, \bar{n}+\bar{j}\rangle \sum \psi(\varrho(\bar{n}, \bar{j}))$$

depend only on $\bar{n}$ and $\bar{j}$.

Put

(21) $$Z(\bar{M}_{\bar{j}}) = \max_{\bar{0}\leqslant\bar{n}\leqslant\bar{M}_{\bar{j}}-\bar{K}^{(1)}_{\bar{M}_{\bar{j}}}} \frac{S(\bar{n}, \bar{n}+\bar{K}^{(1)}_{\bar{M}_{\bar{j}}}\rangle}{\sqrt{ES^2(\bar{n}, \bar{n}+\bar{K}^{(1)}_{\bar{M}_{\bar{j}}}\rangle \sum \psi\big(\varrho(\bar{n}, \bar{K}^{(1)}_{\bar{M}_{\bar{j}}})\big)}}.$$

It is easy to see that for all $\bar{N}\in K_r$ such that $m_{\bar{j}}(k) \leqslant N(k) \leqslant M_{\bar{j}}(k)$, $k = 1, 2, \ldots, r$, the inequality $\sum(\bar{N}, \bar{K}^{(1)}_{\bar{N}}) \leqslant Z(\bar{M}_{\bar{j}})$ is true. Hence, for any given $\varepsilon > 0$,

(22) $$P\{\bigcup_{\substack{\bar{N}\in K_r\\ \bar{K}^{(1)}_{\bar{N}}=\bar{j}\in K_r}} [\textstyle\sum(\bar{N}, \bar{K}^{(1)}_{\bar{N}}) > \lambda+\varepsilon]\} \leqslant P[Z(\bar{M}_{\bar{j}}) > \lambda+\varepsilon].$$

By Theorem 1 we have

(23) $$P[Z(\bar{M}_{\bar{j}}) > \lambda+\varepsilon]$$

$$\leqslant \sum_{\bar{0}\leqslant\bar{n}\leqslant\bar{M}_{\bar{j}}-\bar{K}^{(1)}_{\bar{M}_{\bar{j}}}} P\left[\frac{S(\bar{n}, \bar{n}+\bar{K}_{\bar{M}_{\bar{j}}}\rangle}{[ES^2(\bar{n}, \bar{n}+\bar{K}^{(1)}_{\bar{M}_{\bar{j}}}\rangle \sum \psi\big(\varrho(\bar{n}, \bar{K}^{(1)}_{\bar{M}_{\bar{j}}})\big)]^{1/2}} > \lambda+\varepsilon\right]$$

$$\leqslant C \sum_{\bar{0}\leqslant\bar{n}\leqslant\bar{M}_{\bar{j}}-\bar{K}^{(1)}_{\bar{M}_{\bar{j}}}} \exp\left\{-\frac{1}{2}(\lambda+\varepsilon)^2 \sum \psi\big(\varrho(\bar{n}, \bar{K}^{(1)}_{\bar{M}_{\bar{j}}})\big) + \right.$$

$$\left. + \frac{(\lambda+\varepsilon)^3[\sum \psi(\varrho(\bar{n}, \bar{K}^{(1)}_{\bar{M}_{\bar{j}}}))]^{3/2}}{|\bar{K}^{(1)}_{\bar{M}_{\bar{j}}}|^{1/2}} g_{\bar{n},\bar{K}^{(1)}_{\bar{M}_{\bar{j}}}}\left(\frac{[\sum \psi(\varrho(\bar{n}, \bar{K}^{(1)}_{\bar{M}_{\bar{j}}}))]^{1/2}}{|\bar{K}^{(1)}_{\bar{M}_{\bar{j}}}|^{1/2}}(\lambda+\varepsilon)\right)\right\} \times$$

$$\times \frac{1}{(\lambda+\varepsilon)[\sum \psi(\varrho(\bar{n}, \bar{K}^{(1)}_{\bar{M}_{\bar{j}}}))]^{1/2}}$$

for sufficiently large $|\bar{j}|$. Taking into account assumption (4) and inequality (14), we get

$$\frac{\sum \psi\big(\varrho(\bar{n}, \bar{K}^{(1)}_{\bar{M}_{\bar{j}}})\big)}{|\bar{K}^{(1)}_{\bar{M}_{\bar{j}}}|} \leqslant \frac{\sum_{i=1}^{r} \psi_i\big(\Delta'' \bar{K}^{(1)}_{\bar{M}_{\bar{j}}}(i)\big)}{|\bar{K}^{(1)}_{\bar{M}_{\bar{j}}}|} \to 0 \quad \text{as } |\bar{M}_{\bar{j}}| \to \infty.$$

But series $g_{\bar{n},\bar{K}^{(1)}_{\bar{M}_{\bar{j}}}}(t)$ are uniformly convergent with respect to $\bar{K}^{(1)}_{\bar{M}_{\bar{j}}}$ and $\bar{n}\in K_r$ for all sufficiently small $t \geqslant 0$, so we can write (23) in the form

(24) $$P[Z(\bar{M}_{\bar{j}}) > \lambda+\varepsilon] \leqslant C \sum_{\bar{0}\leqslant\bar{n}\leqslant\bar{M}_{\bar{j}}-\bar{K}^{(1)}_{\bar{M}_{\bar{j}}}} \exp\left\{-\frac{(\lambda+\varepsilon)^2}{2(1+\delta)} \sum \psi\big(\varrho(\bar{n}, \bar{K}^{(1)}_{\bar{M}_{\bar{j}}})\big)\right\},$$

where δ is a positive number such that $0 < \delta < \varepsilon/\lambda$.

Now, by assumption (11) and condition (2), we can choose $\bar{N}_1 = \bar{N}_1(\delta)$ such that, for $\bar{N} \geqslant \bar{N}_1$,

$$(25)\quad \frac{1}{2}(\lambda+\varepsilon)^2 \sum \psi\big(\varrho(\bar{n}, \bar{K}_{\bar{N}}^{(1)})\big)$$

$$\geqslant \frac{1}{2}\lambda^2\Big(1+\frac{\varepsilon}{\lambda}\Big)^2 \Big\{\sum_{j=1}^{r} \prod_{i=1}^{-[-\Delta^{(1)}]} \frac{\psi_j(\Delta^{(1)}K_{\bar{N}}^{(1)}(j)+i-1)}{\psi_j(\Delta^{(1)}K_{\bar{N}}^{(1)}(j)+i)} \times$$

$$\times \psi_j(\Delta^{(1)}K_{\bar{N}}^{(1)}(j)-[-\Delta^{(1)}])\Big\}$$

$$\geqslant \frac{1}{2}\lambda^2\Big(1+\frac{\varepsilon}{\lambda}\Big)^2 \Big\{\sum_{j=1}^{r} \frac{c_j^{-[-\Delta^{(1)}]}}{1+\delta}\psi_j(R_{\bar{N}}^{(1)}(j))\Big\} \geqslant (1+\delta)\log|\bar{N}|.$$

By similar considerations we get

$$(26)\quad \sum \psi\big(\varrho(\bar{n}, \bar{K}_{\bar{N}}^{(1)})\big) \geqslant \frac{2}{\lambda^2(1+\delta)}\log|\bar{N}| \quad \text{for } \bar{N} \geqslant \bar{N}_2 = \bar{N}_2(\delta).$$

Moreover, for all values of $\bar{N} \in K_r$ such that $\bar{K}_{\bar{N}}^{(1)} = \bar{j} \in K_r$, we have

$$\psi_k(\Delta^{(1)}j(k)) \leqslant 2\lambda^{-2}c_k^{[-\Delta^{(1)}]}\log N(k) < \psi_k\big(\Delta^{(1)}(j(k)+1)\big), \quad k = 1, 2, \ldots, r.$$

Hence one can obtain

$$(27)\quad \exp\Big\{\frac{\lambda^2}{2}\sum_{k=1}^{r} c_k^{-[-\Delta^{(1)}]}\psi_k(\Delta^{(1)}j(k))\Big\} \leqslant |\bar{N}|$$

$$\leqslant \exp\Big\{\frac{\lambda^2}{2}\sum_{k=1}^{r} c_k^{-[-\Delta^{(1)}]}\psi_k\big(\Delta^{(1)}(j(k)+1)\big)\Big\}.$$

Noting that, by (11), for sufficiently large $\bar{j}$ the inequality $m_k(\bar{n}, \bar{K}_{\bar{M}_{\bar{j}}}^{(1)}) \geqslant \Delta^{(1)}j(k)$ holds, and using (27) with (24) we get

$$(28)\quad P[Z(\bar{M}_{\bar{j}}) > \lambda+\varepsilon] \leqslant C|\bar{M}_{\bar{j}}|\exp\Big\{-\frac{(\lambda+\varepsilon)^2}{2(1+\delta)}\sum_{k=1}^{r}\psi_k(\Delta^{(1)}j(k))\Big\}$$

$$\leqslant C\exp\Big\{\frac{\lambda^2}{2}\Big[\sum_{k=1}^{r} c_k^{-[-\Delta^{(1)}]}\psi_k\big(\Delta^{(1)}(j(k)+1)\big)-(1+\delta)\psi_k(\Delta^{(1)}j(k))\Big]\Big\}.$$

But, by (2),

$$\psi_k\big(\Delta^{(1)}(j(k)+1)\big) = \prod_{i=1}^{-[-\Delta^{(1)}]} \frac{\psi_k\big(\Delta^{(1)}(j(k)+1)-i+1\big)}{\psi_k\big(\Delta^{(1)}(j(k)+1)-i\big)}\psi_k\big(\Delta^{(1)}(j(k)+1)+[-\Delta^{(1)}]\big)$$

$$\leqslant \frac{1}{(c_k-\eta_k)^{-[-\Delta^{(1)}]}}\psi_k(\Delta^{(1)}j(k)),$$

where $\eta_k > 0$ can be arbitrarily small for sufficiently large $j(k)$, $k = 1, 2, \ldots, r$. Therefore, we can find $\tau > 0$ such that, for sufficiently large $\bar{j}$,

$$P[Z(\bar{M}_{\bar{j}}) > \lambda+\varepsilon] \leqslant C\exp\{-\tau \sum_{k=1}^{r} \psi_k(\Delta^{(1)}j(k))\}. \tag{29}$$

Taking into account (3) we conclude that

$$\sum_{\bar{j}\in K_r} P[Z(\bar{M}_{\bar{j}}) > \lambda+\varepsilon] < \infty. \tag{30}$$

Hence, the Borel-Cantelli lemma and (22) allow us to write

$$\limsup_{\bar{N}\to\infty} \sum(\bar{N}, \bar{K}_{\bar{N}}^{(1)}) \leqslant \lambda+\varepsilon \text{ a.s.}$$

Thus (12) is proved since we can take $\varepsilon \to 0$.

To prove (13) we take $\varepsilon > 0$ such that $\lambda-\varepsilon > 0$ and we set $\delta > 0$ such that $-\delta-(1-\varepsilon/\lambda)^2(1+\delta)^2 \geqslant -(1-2\delta_1)$ for some $\delta_1 > 0$. One can note that

$$\begin{aligned} &P[\textstyle\sum(\bar{N}, \bar{K}_{\bar{N}}^{(2)}) < \lambda-\varepsilon] \\ &\leqslant P\left\{\frac{S(\bar{m}\bar{K}_{\bar{N}}^{(2)}, (\bar{m}+\bar{1})\bar{K}_{\bar{N}}^{(2)}\rangle}{\sqrt{\mathrm{E}S^2(\bar{m}\bar{K}_{\bar{N}}^{(2)}, (\bar{m}+\bar{1})\bar{K}_{\bar{N}}^{(2)}\rangle \sum\psi(\varrho(\bar{m}\bar{K}_{\bar{N}}^{(2)}, \bar{K}_{\bar{N}}^{(2)}))}} < \lambda-\varepsilon,\right. \\ &\qquad\qquad \left. 0 \leqslant m(k) \leqslant \left[\frac{N(k)}{K_{\bar{N}}^{(2)}(k)}\right]-1,\ 1 \leqslant k \leqslant r\right\} \\ &= \prod_{0\leqslant m(k)\leqslant\left[\frac{N(k)}{K_{\bar{N}}^{(2)}(k)}\right]-1} P\left\{\frac{S(\bar{m}\bar{K}_{\bar{N}}^{(2)}, (\bar{m}+\bar{1})\bar{K}_{\bar{N}}^{(2)}\rangle}{\sqrt{\mathrm{E}S^2(\bar{m}\bar{K}_{\bar{N}}^{(2)}, (\bar{m}+\bar{1})\bar{K}_{\bar{N}}^{(2)}\rangle \sum\psi(\varrho(\bar{m}\bar{K}_{\bar{N}}^{(2)}, \bar{K}_{\bar{N}}^{(2)}))}} < \lambda-\varepsilon\right\}. \end{aligned} \tag{31}$$

By Theorem 1, we get

$$\begin{aligned} &P\left[\frac{S(\bar{m}\bar{K}_{\bar{N}}^{(2)}, (\bar{m}+\bar{1})\bar{K}_{\bar{N}}^{(2)}\rangle}{[\mathrm{E}S^2(\bar{m}\bar{K}_{\bar{N}}^{(2)}, (\bar{m}+\bar{1})\bar{K}_{\bar{N}}^{(2)}\rangle]^{1/2}} \geqslant (\lambda-\varepsilon)[\textstyle\sum\psi(\varrho(\bar{m}\bar{K}_{\bar{N}}^{(2)}, \bar{K}_{\bar{N}}^{(2)}))]^{1/2}\right] \\ &= \frac{1}{(2\pi)^{1/2}} \frac{1}{(\lambda-\varepsilon)[\sum\psi(\varrho(\bar{m}\bar{K}_{\bar{N}}^{(2)}, \bar{K}_{\bar{N}}^{(2)}))]^{1/2}} \times \\ &\quad \times \exp\left\{-\frac{1}{2}(\lambda-\varepsilon)^2\textstyle\sum\psi(\varrho(\bar{m}\bar{K}_{\bar{N}}^{(2)}, \bar{K}_{\bar{N}}^{(2)}))+\frac{(\lambda-\varepsilon)^3}{|\bar{K}_{\bar{N}}^{(2)}|^{1/2}} \times\right. \\ &\quad \left.\times[\textstyle\sum\psi(\varrho(\bar{m}\bar{K}_{\bar{N}}^{(2)}, \bar{K}_{\bar{N}}^{(2)}))]^{3/2} g_{\bar{m}\bar{K}_{\bar{N}}^{(2)}, \bar{K}_{\bar{N}}^{(2)}}\left(\frac{(\lambda-\varepsilon)[\sum\psi(\varrho(\bar{m}\bar{K}_{\bar{N}}^{(2)}, \bar{K}_{\bar{N}}^{(2)}))]^{1/2}}{|\bar{K}_{\bar{N}}^{(2)}|^{1/2}}\right)\right\}\times \\ &\quad \times\left[1+O\left(\frac{(\lambda-\varepsilon)[\sum\psi(\varrho(\bar{m}\bar{K}_{\bar{N}}^{(2)}, \bar{K}_{\bar{N}}^{(2)}))]^{1/2}}{|\bar{K}_{\bar{N}}^{(2)}|^{1/2}}\right)\right]. \end{aligned} \tag{32}$$

Inequality (14) and assumption (4) imply that

$$\frac{(\lambda-\varepsilon)\sqrt{\sum\psi\left(\varrho\left(\bar{m}\bar{K}_{\bar{N}}^{(2)},\bar{K}_{\bar{N}}^{(2)}\right)\right)}}{\sqrt{|\bar{K}_{\bar{N}}^{(2)}|}}\to 0 \quad \text{as } |\bar{N}|\to\infty$$

uniformly with respect to $\bar{m}$. Moreover, the series $g_{\bar{m}\bar{K}_{\bar{N}}^{(2)},\bar{K}_{\bar{N}}^{(2)}}(t)$ converge uniformly with respect to $\bar{K}_{\bar{N}}^{(2)}$ and $\bar{m}$ for sufficiently small $t\geqslant 0$. Thus, for sufficiently large $\bar{N}\in K_r$, equality (32) can be replaced by

$$\text{(33)}\quad P\left[\frac{S\left(\bar{m}\bar{K}_{\bar{N}}^{(2)},(\bar{m}+\bar{1})\bar{K}_{\bar{N}}^{(2)}\right\rangle}{\sqrt{\mathrm{E}S^2\left(\bar{m}\bar{K}_{\bar{N}}^{(2)},(\bar{m}+\bar{1})\bar{K}_{\bar{N}}^{(2)}\right\rangle}}\geqslant(\lambda-\varepsilon)\sqrt{\sum\psi\left(\varrho\left(\bar{m}\bar{K}_{\bar{N}}^{(2)},\bar{K}_{\bar{N}}^{(2)}\right)\right)}\right]$$
$$\geqslant\frac{C}{\sqrt{\sum\psi\left(\varrho\left(\bar{m}\bar{K}_{\bar{N}}^{(2)},\bar{K}_{\bar{N}}^{(2)}\right)\right)}}\exp\left\{-\frac{1}{2}(\lambda-\varepsilon)^2(1+\delta)\sum\psi\left(\varrho\left(\bar{m}\bar{K}_{\bar{N}}^{(2)},\bar{K}_{\bar{N}}^{(2)}\right)\right)\right\}.$$

Furthermore, for sufficiently large $\bar{N}\in K_r$ we have

$$\sum\psi\left(\varrho\left(\bar{m}\bar{K}_{\bar{N}}^{(2)},\bar{K}_{\bar{N}}^{(2)}\right)\right)\leqslant\sum_{k=1}^{r}\psi_k\left(\Delta^{(2)}K_{\bar{N}}^{(2)}(k)+[-\Delta^{(2)}]\right)$$
$$\leqslant\sum_{k=1}^{r}(c_k+\eta_k)^{-[-\Delta^{(2)}]}\psi_k\left(R_{\bar{N}}^{(2)}(k)\right)\leqslant\frac{2(1+\delta)}{\lambda^2}\log|\bar{N}|,$$

as η_k can be arbitrarily small for sufficiently large $K_{\bar{N}}^{(2)}(k)$, $1\leqslant k\leqslant r$. Therefore, the term on the right-hand side of (33) for sufficiently large $\bar{N}\in K_r$ can be estimated from below by

$$C\frac{1}{\sqrt{\log|\bar{N}|}}\exp\left\{-\left(1-\frac{\varepsilon}{\lambda}\right)^2(1+\delta)^2\log|\bar{N}|\right\}\geqslant|\bar{N}|^{-\delta-(1-\varepsilon/\lambda)^2(1+\delta)^2}.$$

Hence, on account of (31), for sufficiently large $\bar{N}\in K_r$ we obtain

(34)
$$P\left[\sum(\bar{N},\bar{K}_{\bar{N}}^{(2)})<\lambda-\varepsilon\right]\leqslant\exp\left\{\prod_{k=1}^{r}\left[\frac{N(k)}{K_{\bar{N}}^{(2)}(k)}\right]\ln\left(1-|\bar{N}|^{-\delta-(1-\varepsilon/\lambda)^2(1+\delta)^2}\right)\right\}$$
$$\leqslant\exp\left\{-|\bar{N}|^{-\delta-(1-\varepsilon/\lambda)^2(1+\delta)^2}\prod_{k=1}^{r}\left[\frac{N(k)}{K_{\bar{N}}^{(2)}(k)}\right]\right\}.$$

Moreover, condition (3) implies that, for sufficiently large $\bar{N}$,

$$\left[\frac{N(k)}{K_{\bar{N}}^{(2)}(k)}\right]\geqslant\left(N(k)\right)^{(1-\delta_1)},\quad 1\leqslant k\leqslant r.$$

Therefore, basing on (33), we have

$$\text{(35)}\qquad P\left[\sum(\bar{N},\bar{K}_{\bar{N}}^{(2)})<\lambda-\varepsilon\right]\leqslant\exp\{-|\bar{N}|^{\delta_1}\},$$

which by the Borel-Cantelli lemma proves that

$$\liminf_{\bar{N}\to\infty} \sum(\bar{N}, \bar{K}_{\bar{N}}^{(2)}) \geqslant \lambda-\varepsilon \text{ a.s.}$$

Letting $\varepsilon \to 0$ we get assertion (13), completing the proof of Theorem 2.

Proof of Theorem 3. Let $\varepsilon > 0$ be an arbitrarily fixed real number. Using Theorem 1 and choosing sufficiently large $\bar{N}'$, we get

$$(36)\quad P\left[\sum(\bar{N}, \bar{K}_{\bar{N}}^{(1)}) \geqslant \varepsilon\right] \leqslant C \sum_{\bar{0}\leqslant\bar{n}\leqslant\bar{N}-\bar{K}_{\bar{N}}^{(1)}} \frac{1}{\varepsilon\left[\sum\psi\left(\varrho(\bar{n}, \bar{K}_{\bar{N}}^{(1)})\right)\right]^{1/2}} \exp\left\{-\frac{1}{2}\varepsilon^2\sum\psi\left(\varrho(\bar{n}, \bar{K}_{\bar{N}}^{(1)})\right)+ \frac{\varepsilon^3\left[\sum\psi\left(\varrho(\bar{n}, \bar{K}_{\bar{N}}^{(1)})\right)\right]^{3/2}}{|\bar{K}_{\bar{N}}^{(1)}|^{1/2}} g_{\bar{n},\bar{K}_{\bar{N}}^{(1)}}\left(\frac{\varepsilon\left[\sum\psi\left(\varrho(\bar{n}, \bar{K}_{\bar{N}}^{(1)})\right)\right]^{1/2}}{|\bar{K}_{\bar{N}}^{(1)}|^{1/2}}\right)\right\}.$$

Moreover, under assumption (4) and by (14) we have

$$(37)\quad \frac{\sum\psi\left(\varrho(\bar{n}, \bar{K}_{\bar{N}}^{(1)})\right)}{|\bar{K}_{\bar{N}}^{(1)}|} \leqslant \frac{\sum_{k=1}^{r}\psi_k\left(\Delta'' K_{\bar{N}}^{(1)}(k)\right)}{|\bar{K}_{\bar{N}}^{(1)}|} \to 0 \quad \text{as } \bar{N}' \to \infty.$$

Taking into account the fact that $g_{\bar{n},\bar{K}_{\bar{N}}^{(1)}}(t)$ converges uniformly with respect to $\bar{n}$ and $\bar{K}_{\bar{N}}^{(1)}$ for sufficiently small $t \geqslant 0$ and using (37), we conclude that

$$(38)\quad \frac{\varepsilon^3\left[\sum\psi\left(\varrho(\bar{n}, \bar{K}_{\bar{N}}^{(1)})\right)\right]^{3/2}}{|\bar{K}_{\bar{N}}^{(1)}|^{1/2}} g_{\bar{n},\bar{K}_{\bar{N}}^{(1)}}\left(\frac{\varepsilon\left[\sum\psi\left(\varrho(\bar{n}, \bar{K}_{\bar{N}}^{(1)})\right)\right]^{1/2}}{|\bar{K}_{\bar{N}}^{(1)}|^{1/2}}\right) \leqslant \frac{1}{4}\varepsilon^2\sum\psi\left(\varrho(\bar{n}, \bar{K}_{\bar{N}}^{(1)})\right).$$

We note now that, by (11), (3) and (15),

$$(39)\quad \frac{1}{4}\varepsilon^2\sum\psi\left(\varrho(\bar{n}, \bar{K}_{\bar{N}}^{(1)})\right) \geqslant \frac{1}{4}\varepsilon^2\sum_{k=1}^{r} c_k^{-[-\Delta^{(1)}]}\psi_k\left(R_{\bar{N}}^{(1)}(k)\right) \geqslant 3\log|\bar{N}'|.$$

Using (38) and (39) we obtain

$$(40)\quad P\left[\sum(\bar{N}, \bar{K}_{\bar{N}}^{(1)}) \geqslant \varepsilon\right] \leqslant C \sum_{\bar{0}\leqslant\bar{n}\leqslant\bar{N}-\bar{K}_{\bar{N}}^{(1)}} \exp\left\{-\frac{1}{4}\varepsilon^2\sum\psi\left(\varrho(\bar{n}, \bar{K}_{\bar{N}}^{(1)})\right)\right\} \leqslant C|\bar{N}|\exp\{-3\log|\bar{N}'|\} = C|\bar{N}''|\,|\bar{N}'|^{-2}.$$

But $|\bar{N}''|$ is bounded, so

$$\sum_{\bar{N}} P\left[\sum(\bar{N}, \bar{K}_{\bar{N}}^{(1)}) \geqslant \varepsilon\right] \leqslant \sum_{\bar{N}} C|\bar{N}''|\,|\bar{N}'|^{-2} < \infty,$$

which, by the Borel-Cantelli lemma and the fact that $\varepsilon > 0$ is arbitrary, implies

$$\limsup \sum(\bar{N}, \bar{K}_N^{(1)}) \leqslant 0 \text{ a.s.} \tag{41}$$

as $\bar{N}' \to \infty$ and $|\bar{N}''|$ remains bounded.

Now one can note that the random variables $\{-X_{\bar{k}}, \bar{k} \in K_r\}$ satisfy all the assumptions of Theorem 3. Therefore, by (41), we have

$$\limsup \max_{0 \leqslant n \leqslant N - K_N^{(1)}} \frac{-S(\bar{n}, \bar{n}+\bar{K}_N^{(1)}\rangle}{\sqrt{ES^2(\bar{n}, \bar{n}+\bar{K}_N^{(1)}\rangle \sum \psi(\varrho(\bar{n}, \bar{K}_N^{(1)}))}} \leqslant 0 \text{ a.s.},$$

i.e.

$$\liminf \min_{\bar{0} \leqslant \bar{n} \leqslant \bar{N} - \bar{K}_N^{(1)}} \frac{S(\bar{n}, \bar{n}+\bar{K}_N^{(1)}\rangle}{\sqrt{ES^2(\bar{n}, \bar{n}+\bar{K}_N^{(1)}\rangle \sum \psi(\varrho(\bar{n}, \bar{K}_N^{(1)}))}} \geqslant 0 \text{ a.s.}$$

as $\bar{N}' \to \infty$ and $|\bar{N}''|$ is bounded. The last inequality implies that

$$\liminf \sum(\bar{N}, \bar{K}_N^{(1)}) \geqslant 0 \text{ a.s.}$$

as $\bar{N}' \to \infty$ and $|\bar{N}''|$ is bounded, which together with (41) gives (16).

The proof of (12) is similar to the proof of the same assertion in Theorem 2, so we omit details.

Proof of Theorem 4. Let us fix $T > 0$ and assume that $\bar{N}' \to \infty$ and $|\bar{N}''|$ is bounded. In addition, assume that

$$0 \leqslant m(k) \leqslant \max\left\{\left[\frac{N(k)}{K_N^{(2)}(k)}\right] - 1, 0\right\}, \quad k = 1, 2, \ldots, r.$$

Taking into account (14) and assumption (4), we note that

$$\frac{\sum \psi(\varrho(\bar{m}\bar{K}_N^{(2)}, \bar{K}_N^{(2)}))}{|\bar{K}_N^{(2)}|} \to 0 \quad \text{as } \bar{N}' \to \infty \tag{42}$$

uniformly in $\bar{m}$.

Hence, following considerations leading us to (32) and (33), by Theorem 1, we obtain

$$P\left[\frac{S(\bar{m}\bar{K}_N^{(2)}, (\bar{m}+\bar{1})\bar{K}_N^{(2)}\rangle}{\sqrt{ES^2(\bar{m}\bar{K}_N^{(2)}, (\bar{m}+\bar{1})\bar{K}_N^{(2)}\rangle}} \geqslant T\sqrt{\sum \psi(\varrho(\bar{m}\bar{K}_N^{(2)}, \bar{K}_N^{(2)}))}\right] \tag{43}$$

$$\geqslant \frac{C}{\sqrt{\sum \psi(\varrho(\bar{m}\bar{K}_N^{(2)}, \bar{K}_N^{(2)}))}} \exp\left\{-\frac{1}{2} T^2(1+\delta) \sum \psi(\varrho(\bar{m}\bar{K}_N^{(2)}, \bar{K}_N^{(2)}))\right\},$$

where $\delta > 0$ is an arbitrary constant and $\bar{N}' \geqslant \bar{N}'(\delta)$. Moreover, by (14) and under assumption (17), for sufficiently large $\bar{N}'$ we have

$$\sum \psi\left(\varrho\left(\bar{m}\bar{K}_{\bar{N}}^{(2)}, \bar{K}_{\bar{N}}^{(2)}\right)\right) \leqslant \sum_{i=1}^{s} \psi_{k_i}\left(\Delta'' K_{\bar{N}}^{(2)}(k_i)\right) + C \sum_{\substack{k \neq k_i \\ 1 \leqslant i \leqslant s}} 2\lambda^{-2} c_k^{[-\Delta(2)]} \log N(k)$$

$$\leqslant \frac{1}{T^2(1+\delta)} \log|\bar{N}'| + C \log|\bar{N}''|.$$

Hence (43) is bounded from below by

$$\frac{C}{(\log|\bar{N}'|)^{1/2}} |\bar{N}'|^{-1/2} \geqslant |\bar{N}'|^{-\delta-1/2}$$

for sufficiently large $\bar{N}' \geqslant \bar{N}'(\delta)$.

We note now that condition (3) implies the inequality $\psi_{k_i}(x) \geqslant (1/\delta)\log x$ for an arbitrary $\delta > 0$ and sufficiently large x. Therefore, from (17) it follows that

$$\frac{K_{\bar{N}}^{(2)}(k_i)}{[N(k_i)]^{\delta}} \to 0 \quad \text{as } N(k_i) \to \infty,\ i = 1, 2, \ldots, s,$$

so

$$\prod_{j=1}^{r} \left[\frac{N(j)}{K_{\bar{N}}^{(2)}(j)}\right] \geqslant |\bar{N}'|^{1-\delta}.$$

Using (31) and estimates given above, we get the inequality

$$P\left[\sum(\bar{N}, \bar{K}_{\bar{N}}^{(2)}) < T\right] \leqslant \exp\left\{\prod_{j=1}^{r} \left[\frac{N(j)}{K_{\bar{N}}^{(2)}(j)}\right] \ln\left(1 - |\bar{N}'|^{-\delta-1/2}\right)\right\}$$

$$\leqslant \exp\left\{-|\bar{N}'|^{1/2-2\delta}\right\}.$$

But $|\bar{N}''|$ is bounded, so for $0 < \delta < 1/4$ we obtain

$$\sum_{\bar{N}} \exp\left\{-|\bar{N}'|^{1/2-2\delta}\right\} < \infty,$$

which by the Borel-Cantelli lemma implies that

$$\liminf \sum(\bar{N}, \bar{K}_{\bar{N}}^{(2)}) \geqslant T \text{ a.s.}$$

as $\bar{N}' \to \infty$ and $|\bar{N}''|$ is bounded. Letting $T \to \infty$ we get (18).

The proof of (13) is essentially the same as in Theorem 2.

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Institute of Mathematics
M. Curie-Skłodowska University
ul. Nowotki 10, 20-031 Lublin, Poland

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