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ON PAUL LÉVY'S ARC SINE LAW AND SHIGA–WATANABE'S TIME INVERSION RESULT

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Abstract. Let $((X_t), P)$ be a symmetric real-valued H-self-similar diffusion starting at 0. We characterize the distributions of A_t, the time spent on $(0, \infty)$ before time t, and g_t, the time of the last visit to 0 before t. This gives a simple new proof to well-known results including P. Lévy's arc sine law for Brownian motion and Brownian bridge and similar results for symmetrized Bessel processes. Our focus is more on simplicity of proofs than on novelty of results. Section 3 contains a generalization of T. Shiga's and S. Watanabe's theorem on time inversion for Bessel processes. We show that their result holds also for symmetrized Bessel processes.

0. Introduction. P. Lévy showed in [8] that for a Brownian motion starting at 0 both

$$A_1 = \int_0^1 I_{\{B_u > 0\}} du \quad \text{and} \quad g_1 = \sup\{s < 1 \mid B_s = 0\}$$

are arc sine distributed. This result has been extended mainly in two directions: on one hand for Lévy processes (see [5]), on the other hand for self-similar Markov processes (see [1], [4], [6], and [10]). A_1 remains beta distributed in many cases if (B_t) is replaced by a more general Lévy process (see [5]) whereas g_1 is beta distributed if (B_t) is replaced by any self-similar Markov process (see Dynkin [4] and Lamperti [6]). In [1] the distribution of A_1 was calculated for symmetrized Bessel processes (which are $\frac{1}{2}$-self-similar); it turned out *not* to be a beta distribution.

In this note we calculate the moments of g_1 (Theorem 1 in Section 2). This gives a new proof of the fact that g_1 is beta distributed for any Bessel process (cf. [4] and [6]). Thereafter we use this to calculate the moments of A_1. We first show, by quite elementary methods, the connection between A_1 and g_1 (see Lemma 4 in Section 1). Under an independence assumption (A), which we make, this gives also a connection between the moments of A_1 and g_1. However, to show that (A) is valid for all symmetrized Bessel processes (Theorem 2 in

Section 4) we need a generalization of T. Shiga's and S. Watanabe's time inversion result (Proposition 1 in Section 3). We believe that this result might have some interest on its own, independently of the rest of the paper. Except in the first section we are most of the time working with symmetrized Bessel processes but the results are also valid for Bessel and Brownian bridges and for symmetric H-self-similar diffusions on the whole real line, $H > 0$ (see Remarks 5 and 6).

1. Notation and basic ideas. Throughout this paper $((X_t), P)$ denotes a real-valued process starting at 0 with continuous paths satisfying the following properties:

1. Symmetry, i.e. $(X_t) \sim P = (-X_t) \sim P$, and that $P(X_t = 0) \equiv 0$ for $t > 0$.
2. $((X_t), P)$ is a strong Markov process.
3. $((X_t), P)$ is H-self-similar under P, i.e.

$$(a^{-H} X_{at}) \stackrel{d}{=} (X_t) \text{ under } P \quad \text{for all } a > 0.$$

Examples: Brownian motion and more general symmetrized Besel processes with index $\nu \in (-1, 0)$ starting at 0. Here $H = 1/2$.

By a *symmetrized Bessel process with index* ν we mean a unique symmetric diffusion on the whole real line which on $(0, \infty)$ behaves like a Bessel diffusion with the same index ν (see [1]). They form exactly the class of diffusions fulfilling the properties 1, 2 and 3 in the case $H = 1/2$. Brownian motion is a special case, corresponding to the index $\nu = -1/2$. The processes fulfilling the properties 1, 2 and 3 for $H \neq 1/2$ can be obtained similarly from H-self-similar diffusions on $(0,\infty)$. See also Section 3, Remark 2 in Section 2 and Remark 5 in Section 4.

The following notation will be used repeatedly. Define for all $t > 0$

$$A_t := \int_0^t 1_{\{X_u > 0\}} du, \qquad d^t := \inf\{s > t \mid X_s = 0\}, \qquad g_t := \sup\{s < t \mid X_s = 0\},$$

and

$$\tau_0 := \inf\{s > 0 \mid X_s = 0\}.$$

If $X_s \neq 0$ for all $s > 0$, then d^t and $\tau_0 := \infty$, $g_t := 0$.

Remark 1. Obviously, the property 3 implies that A_t, d^t and g_t are equal in law to tA_1, td^1 and tg_1, respectively.

LEMMA 1.

$$A_t^n = n! \int_0^t \int_0^{t_n} \ldots \int_0^{t_2} 1_{\{X_{t_1} > 0, \ldots, X_{t_n} > 0\}} dt_1 \ldots dt_n$$

for all $t > 0$ *and* $n \geqslant 1$.

Proof. We use induction based on the formula $A_t^n = n \int_0^t A_s^{n-1} dA_s$. ∎

Notice that neither of the properties 1, 2 or 3 were used. For each $n \geqslant 1$ we thus have by Fubini's theorem

$$\mu_n := E[A_1^n] = n! \int_0^1 \int_0^{t_n} \dots \int_0^{t_2} P(X_{t_1} > 0, \dots, X_{t_n} > 0)\, dt_1 \dots dt_n.$$

Being bounded the distribution of A_1 is determined by its moments. In order to compute these we shall use the following simple identity. Note that only the property 1 is needed.

LEMMA 2.

$$(*) \qquad \mu_1 = \tfrac{1}{2} \text{ and } 2\mu_n = \sum_{k=1}^{n-1} \binom{n}{k} \cdot (-1)^k \cdot \mu_k + 1 \quad \text{for every odd } n \geqslant 3.$$

Proof. The property 1 shows that A_1 and $1-A_1$ are identically distributed under P. The binomial expansion of $(1-A_1)^n$ immediately gives $(*)$. ∎

The next result, which makes use of the properties 1 and 2, gives an expression for the distribution of g_t and establishes a link between the distributions of g_t and A_t.

LEMMA 3.

$$P(g_t \leqslant s) = 4P(X_s > 0, X_t > 0) - 1 \quad \text{for all } 0 < s < t.$$

Proof. Let $0 < s < t$ be given. Using the strong Markov property at time point d^s and symmetry we get

$$\begin{aligned} &4P(X_s > 0, X_t > 0) = 4P(X_t > 0) - 4P(X_s < 0, X_t > 0) \\ &= 2 - 4P(X_s < 0, d^s < t, X_t > 0) = 2 - 4E[P(X_{t-u} > 0)_{u=d^s}, d^s < t, X_s < 0] \\ &= 2 - 2P(X_s < 0, d^s < t) = 2 - \big(P(X_s < 0, d^s < t) + P(X_s > 0, d^s < t)\big) \\ &= 2 - P(d^s < t) = 1 + P(d^s > t) = 1 + P(g_t \leqslant s). \ \blacksquare \end{aligned}$$

Using this we immediately get

$$\begin{aligned} \mu_2 &= 2\int_0^1 \int_0^s P(X_t > 0, X_s > 0)\, dt\, ds = \int_0^1 \int_0^s \big(\tfrac{1}{2} + \tfrac{1}{2} P(g_s \leqslant t)\big)\, dt\, ds \\ &= \tfrac{1}{4} + \tfrac{1}{2} E\Big[\int_0^1 \int_0^s 1_{\{g_s \leqslant t\}}\, dt\, ds\Big] = \tfrac{1}{4} + \tfrac{1}{2} E\Big[\int_0^1 (s - g_s)\, ds\Big] = \tfrac{1}{2} - \tfrac{1}{2} \int_0^1 E[g_s]\, ds. \end{aligned}$$

From now on the self-similarity (property 3) will be fundamental. As noticed in Remark 1 we have for all $n \geqslant 1$ and $t > 0$

$$E[g_t] = t \cdot E[g_1] \quad \text{and} \quad E[A_t^n] = t^n \cdot E[A_1^n].$$

Inserting above shows that the first three moments of A_1 are determined by

$$\mu_1 = \tfrac{1}{2}, \quad \mu_2 = \tfrac{1}{2} - \tfrac{1}{4} E[g_1] \quad \text{and} \quad \mu_3 = \tfrac{3}{2}\mu_2 - \tfrac{3}{2}\mu_1 + \tfrac{1}{2} = \tfrac{1}{2} - \tfrac{3}{8} E[g_1].$$

In order to compute the higher moments we need the following lemma:

LEMMA 4.

$$E[A_1^n] = E[A_1^{n-1}] - \tfrac{1}{2}E[A_{g_1}^{n-1}] = E[A_1^{n-1}] - \tfrac{1}{2}E\big[g_1^{n-1}\big(\int_0^1 \mathbf{1}_{\{X_{ug_1}>0\}}\,du\big)^{n-1}\big]$$

for all $n \geq 1$.

Proof. It suffices to prove the first equality, the second one then follows by substitution. Arguing like in Lemma 3 we have for all $n \geq 1$ and all $0 < t_1 < \ldots < t_n < 1$

$$\begin{aligned}
&2P(X_{t_1} > 0, \ldots, X_{t_{n-1}} > 0) - 2P(X_{t_1} > 0, \ldots, X_{t_n} > 0)\\
&= 2P(X_{t_1} > 0, \ldots, X_{t_{n-1}} > 0, X_{t_n} < 0)\\
&= 2P(X_{t_1} > 0, \ldots, X_{t_{n-1}} > 0, d^{t_{n-1}} < t_n, X_{t_n} < 0)\\
&= P(X_{t_1} > 0, \ldots, X_{t_{n-1}} > 0, d^{t_{n-1}} < t_n)\\
&= P(X_{t_1} > 0, \ldots, X_{t_{n-1}} > 0) - P(X_{t_1} > 0, \ldots, X_{t_{n-1}} > 0, d^{t_{n-1}} > t_n)\\
&= P(X_{t_1} > 0, \ldots, X_{t_{n-1}} > 0) - P(X_{t_1} > 0, \ldots, X_{t_{n-1}} > 0, g_{t_n} \leq t_{n-1})\\
&= P(X_{t_1} > 0, \ldots, X_{t_{n-1}} > 0) - P(X_{t_1/t_n} > 0, \ldots, X_{t_{n-1}/t_n} > 0, g_1 \leq t_{n-1}/t_n),
\end{aligned}$$

and therefore

$$\begin{aligned}
E[A_1^n] &= n!\int_0^1\int_0^{t_n}\ldots\int_0^{t_2} P(X_{t_1} > 0, \ldots, X_{t_n} > 0)\,dt_1\ldots dt_n\\
&= \tfrac{1}{2}E[A_1^{n-1}] +\\
&\quad + \frac{n!}{2}\int_0^1 t_n^{n-1}\int_0^1\ldots\int_0^{u_2} P(X_{u_1} > 0, \ldots, X_{u_{n-1}} > 0, g_1 \leq u_{n-1})\,du_1\ldots du_{n-1}\,dt_n\\
&= \tfrac{1}{2}E[A_1^{n-1}] +\\
&\quad + \frac{(n-1)!}{2}\int_0^1\ldots\int_0^{u_2} P(X_{u_1} > 0, \ldots, X_{u_{n-1}} > 0, g_1 \leq u_{n-1})\,du_1\ldots du_{n-1}\\
&= \tfrac{1}{2}E[A_1^{n-1}] + \tfrac{1}{2}E\big[\int_0^1 \mathbf{1}_{\{g_1 \leq s\}}\,dA_s^{n-1}\big] = \tfrac{1}{2}E[A_1^{n-1}] + \tfrac{1}{2}E[\mathbf{1}_{\{g_1 \leq 1\}}\cdot A_1^{n-1}] -\\
&\quad - \tfrac{1}{2}E\big[\int_0^1 A_s^{n-1}\,d\mathbf{1}_{\{g_1 \leq s\}}\big] = E[A_1^{n-1}] - \tfrac{1}{2}E[A_{g_1}^{n-1}]. \quad \blacksquare
\end{aligned}$$

Under the following assumption (A), which will be characterized later in Section 4,

(A) $\quad g_1$ and $\int_0^1 \mathbf{1}_{\{X_{ug_1}>0\}}\,du = \int_0^1 \mathbf{1}_{\{g_1^{-H}X_{ug_1}>0\}}\,du$ are independent, the identity between the first and last terms in Lemma 4 can be rewritten as

$$(**)\qquad \mu_n = \mu_{n-1} - \tfrac{1}{2}\tilde{\mu}_{n-1} E[g_1^{n-1}] \quad \text{for all } n \geqslant 1,$$

where

$$\tilde{\mu}_n = E[\tilde{A}_1^n] := E[(\int_0^1 \mathbf{1}_{\{X_{ug_1} > 0\}}\, du)^n] \quad \text{for } n \geqslant 0.$$

This formula contains a lot of information. To see this define $(\tilde{X}_t) := (X_{tg_1})$. We see that $((\tilde{X}_t), P)$ is a continuous process starting at 0 and satisfying the property 1. Thus $(\tilde{\mu}_n)_{n \geqslant 1}$ fulfills (*). Combining (*), (**) and the fact that μ_1, $\tilde{\mu}_0$ and $\tilde{\mu}_1$ are (trivially) known we see that in order to recursively compute μ_n and $\tilde{\mu}_n$ for all n it is enough to calculate the moments of g_1. Thus under the assumption (A) the distribution of g_1 determines that of both A_1 and $\tilde{A}_1$.

2. The distribution of g_1. As concluded above it is important to be able to compute the moments of g_1 under P. A first step in this direction was already taken in Lemma 3 and using this result we shall now deduce the following known result:

THEOREM 1. *Let $((X_t), P)$ be a symmetrized Bessel process of index $\nu \in (-1, 0)$ starting at 0, that is, a symmetric diffusion on the real line which on $(0, \infty)$ behaves like an ordinary Bessel process of index ν. Then*

$$g_1 \stackrel{d}{=} \text{beta}(-\nu, \nu+1) \text{ under } P,$$

i.e. the distribution of g_1 under P is absolutely continuous with respect to the Lebesgue measure with density

$$t \to \tfrac{1}{\pi}\sin(\pi|\nu|) \cdot t^{-1-\nu} \cdot (1-t)^{\nu} \quad \text{for } t \in (0, 1).$$

In order to prove Theorem 1 we need the following

LEMMA 5. *We have $g_1 \stackrel{d}{=} 1/d^1$ under P.*

Proof. Obviously, $P(g_1 \leqslant t) = P(d^t \geqslant 1)$. Using the self-similarity property 3 (see Remark 1) this becomes equal to $P(td^1 \geqslant 1) = P(1/d^1 \leqslant t)$. ∎

Using Lemma 5 and a well-known formula related to the gamma function we obtain for the moments of g_1

$$E[g_1^n] = \frac{1}{(n-1)!}\int_0^\infty t^{n-1} E[\exp(-td^1)]\, dt \quad \text{for } n \geqslant 1.$$

Write (X_t, P_x) for a symmetrized Bessel process of index ν for some $\nu \in (-1, 0)$, starting at $x \in \mathbf{R}$. The Markov property and self-similarity imply for all $t > 0$

$$\begin{aligned} E_0[\exp(-td^1)] &= e^{-t} E_0[E_{X_1}[\exp(-t\tau_0)]] \\ &= e^{-t} E_0[E_{t^H X_1}[\exp(-\tau_0)]] = e^{-t} E_0[E_{X_t}[\exp(-\tau_0)]]. \end{aligned}$$

Rewriting $f(x) = E_x[\exp(-\tau_0)]$ for $x \in \mathbf{R}$ we have

$$E_0[g_1^n] = \frac{1}{(n-1)!}\int_0^\infty t^{n-1} e^{-t} E_0[f(X_t)]\,dt \qquad \text{for each } n \geqslant 1.$$

Proof of Theorem 1. Let $((X_t), (P_x)_{x\in\mathbf{R}})$ be a symmetrized Bessel process of index ν for some $\nu \in (-1, 0)$. Let further $G^1(\cdot,\cdot)$ and $(p_t(\cdot,\cdot))_{t>0}$ denote continuous versions of the corresponding 1-Green function and the transition density with respect to the speed measure, i.e.

$$G^1(x, y) = \int_0^\infty e^{-t}\cdot p_t(x, y)\,dt \qquad \text{for all } x, y \in \mathbf{R}.$$

It is known that

$$p_t(0, 0) = \frac{|\nu|}{\Gamma(1+\nu)}\cdot 2^{-\nu} t^{-1-\nu} \qquad \text{for } t > 0,$$

and therefore

$$G^1(0, 0) = \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\cdot 2^{-\nu}.$$

The well-known theory (see [2]) shows that $f = c\cdot G^1(\cdot, 0)$, where the constant c is determined by the equation $f(0) = 1$, i.e.

$$c = 2^{\nu}\cdot\frac{\Gamma(n-\nu)}{\Gamma(1+\nu)}.$$

Using the Chapman–Kolmogorov equation we get

$$E_0[f(X_t)] = c\,E_0\Big[\int_0^\infty e^{-u}\cdot p_u(X_t, 0)\,du\Big] = c\int_0^\infty e^{-u}\int p_u(z, 0)\cdot p_t(0, z)\,dz\,du$$

$$= c\int_0^\infty e^{-u}\cdot p_{t+u}(0, 0)\,du = ce^t\int_t^\infty e^{-u}\cdot p_u(0, 0)\,du$$

and inserting this above we obtain

$$E_0[g_1^n] = \frac{c}{(n-1)!}\int_0^\infty t^{n-1}\Big\{\int_t^\infty e^{-u}\cdot p_u(0, 0)\,du\Big\}\,dt = \frac{c}{n!}\int_0^\infty t^n e^{-t}\cdot p_t(0, 0)\,dt$$

$$= c\frac{|\nu|}{\Gamma(1+\nu)}\cdot 2^{-\nu}\frac{1}{n!}\int_0^\infty t^{n-\nu-1} e^{-t}\,dt = \frac{1}{n!}|\nu|\frac{\Gamma(n-\nu)}{\Gamma(1+\nu)} = \frac{1}{n!}\frac{\Gamma(n-\nu)}{\Gamma(-\nu)}$$

for every $n \geqslant 1$. This is exactly the n-th moment of a beta$(-\nu, \nu+1)$ distribution. Writing now $\mathbf{P} = \mathbf{P}_0$ we can conclude that the result is proved. ∎

Remark 2. Using Theorem 1 we can calculate the moments of g_1 corresponding to any symmetric H-self-similar diffusion satisfying the basic assumptions 1, 2 and 3. Everything depends obviously only on the radial process, which is an H-self-similar diffusion on $[0, \infty)$. Now let (r_t^ν, P) denote a Bessel process on $[0, \infty)$ of index ν starting at 0. It is known (see [6] and [7]) that any H-self-similar diffusion (R_t^ν, P) on $[0, \infty)$ starting at 0 is identical in law to a process of the form

$$R_t^\nu = (r_{\sigma^2 t}^\nu)^{2H}$$

for some (unique) ν and $\sigma^2 > 0$. This is due to the fact that (R_t^ν, P) on $(0, \infty)$ is governed by the differential operator

$$A = 2\sigma^2 H^2 r^{2-(1/H)} \frac{d^2}{dr^2} + 2\sigma^2 H(H+\nu) r^{1-(1/H)} \frac{d}{dr}.$$

Using this relation and the following identities it is clear how results for Bessel processes can be used to give general results (assume for simplicity $\sigma^2 = 1$):

$$A_1^{R^\nu} = \int_0^1 I_{\{R_s^\nu > 0\}}\, ds = \int_0^1 I_{\{(r_s^\nu)^{2H} > 0\}}\, ds = \int_0^1 I_{\{r_s^\nu > 0\}}\, ds = A_1^{r^\nu}.$$

Similarly,

$$\begin{aligned} g_1^{R^\nu} &= \sup\{s < 1 \mid R_1^\nu = 0\} = \sup\{s < 1 \mid (r_s^\nu)^{2H} = 0\} \\ &= \sup\{s < 1 \mid r_s^\nu = 0\} = g_1^{r^\nu}. \end{aligned}$$

Remark 3. According to the result of Lamperti [7], 0 is a regular boundary point for (R_t^ν) (that is, the process can hit 0 and can be started from 0) iff

$$\tfrac{1}{2}(1-(1/H))(4H^2) < 2H(H+\nu) < 2H^2 \qquad \text{or, equivalently,} \qquad -1 < \nu < 0,$$

which is exactly the case when 0 is a regular boundary point for (r_t^ν).

3. On a result of T. Shiga and S. Watanabe. In this section we prove a time inversion result for symmetrized Bessel processes, analogous to the result which Shiga and Watanabe [12] showed for ordinary Bessel processes on $[0, \infty)$. This is needed in Section 4 to show that (A) is valid for any symmetrized Bessel process of index $\nu \in (-1, 0)$. Their result states:

(3.1) If (r_t, P) is a Bessel process on $[0, \infty)$ starting at 0, then (r_t) and $(tr_{1/t})$ are equivalent diffusions under P.

It is well known that a similar result is true for Brownian motion. We shall prove the following generalization:

PROPOSITION 1. *Let (X_t, P) be a symmetrized Bessel process of index $\nu \in (-1, 0)$ starting at 0. Then (X_t) and $(tX_{1/t})$ are equivalent diffusions under P.*

Proof. Let $\nu \in (-1, 0)$ be given and let (r_t, Q_0) be a Bessel process on $[0, \infty)$ of index ν starting at 0. Denoting by $(e_n)_{n \geq 1}$ an ordering of the excursions from 0 for (r_t, Q_0) it is clear (compare with [11], exercise (2.16), p. 449) that (X_t) under P is identical in law to $(Z_t, \tilde{Q})$, which is defined by

$$Z_t = Y_n e_n(t-s_n) \quad \text{if } t \in S_n,$$

$\tilde{Q} = Q_0 \times Q$, $((Y_n)_{n\geq 1}, Q)$ is a Rademacher sequence and $S_n = [s_n, s_n')$ is the excursion interval corresponding to e_n for each n. Notice that $Z_t = Y_n r_t$ if $t \in S_n$ and that $\bigcup S_n = [0, \infty)$ since $r_0 = 0$. Obviously,

$$tZ_{1/t} = Y_n te_n((1/t)-s_n) = Y_n tX_{1/t} \quad \text{if } 1/t \in S_n.$$

According to [12], (r_t) and $(tr_{1/t})$ have the same distribution under Q_0. Now, let $0 < t_1 < \ldots < t_n$ be given. To prove that (X_t) and $(tX_{1/t})$ or, equivalently, (Z_t) and $(tZ_{1/t})$ have the same finite-dimensional distributions we need to verify that for a given set of subintervals $I_1, \ldots, I_n$ of $[0, \infty)$ and elements $i_1, \ldots, i_n$ in $\{-1, 1\}$ the following two probabilities are equal:

$$P(|Z_{t_k}| \in I_k, \operatorname{sign} Z_{t_k} = i_k, k = 1, \ldots, n)$$

and

$$P(|t_k Z_{1/t_k}| \in I_k, \operatorname{sign} t_k Z_{1/t_k} = i_k, k = 1, \ldots, n).$$

We shall consider only the case $n = 2$ because this case includes all the difficulties (except the combinatorial ones) which occur in the general case. Write

$$P(|Z_{t_k}| \in I_k, \operatorname{sign} Z_{t_k} = i_k, k = 1, 2) = p_1 + p_2,$$

where p_1 and p_2 mean

$$p_1 = P\big(|Z_{t_k}| \in I_k, \operatorname{sign} Z_{t_k} = i_k, k = 1, 2;\ t_1, t_2 \text{ belong to different excursions of } (r_t)\big)$$

and

$$p_2 = P\big(|Z_{t_k}| \in I_k, \operatorname{sign} Z_{t_k} = i_k, k = 1, 2;\ t_1, t_2 \text{ belong to the same excursion of } (r_t)\big).$$

Due to the independence, p_1 is equal to

$$\tfrac{1}{4} Q_0\big(r_{t_k} \in I_k, k = 1, 2;\ t_1, t_2 \text{ belong to different excursions of } (r_t)\big)$$
$$= \tfrac{1}{4} Q_0\big(r_{t_k} \in I_k, k = 1, 2; \exists s \in (t_1, t_2) \text{ such that } r_s = 0\big),$$

and p_2 is equal to

$$Q(i_1, i_2) Q_0\big(r_{t_k} \in I_k, k = 1, 2;\ t_1, t_2 \text{ belong to the same excursion of } (r_t)\big)$$
$$= Q(i_1, i_2) Q_0\big(r_{t_k} \in I_k, k = 1, 2; r_s > 0\ \forall s \in (t_1, t_2)\big),$$

where

$$Q(i_1, i_2) = 1/2 \text{ if } i_1 = i_2 \quad \text{and} \quad Q(i_1, i_2) = 0 \text{ if } i_1 \neq i_2.$$

Using the result of Shiga and Watanabe (3.1) we obtain

$$Q_0\big(r_{t_k}\in I_k,\ k=1,2;\ \exists s\in(t_1,t_2) \text{ such that } r_s=0\big)$$
$$=Q_0\big(t_k r_{1/t_k}\in I_k,\ k=1,2;\ \exists s\in(t_1,t_2) \text{ such that } sr_{1/s}=0\big)$$
$$=Q_0\big(t_k r_{1/t_k}\in I_k,\ k=1,2;\ \exists 1/s\in(1/t_2,1/t_1) \text{ such that } r_{1/s}=0\big)$$
$$=Q_0\big(t_k r_{1/t_k}\in I_k,\ k=1,2;\ \exists u\in(1/t_2,1/t_1) \text{ such that } r_u=0\big)$$
$$=Q_0\big(t_k r_{1/t_k}\in I_k,\ k=1,2;\ 1/t_1,\ 1/t_2 \text{ belong to different excursions of } (r_s)\big),$$

which, using the independence, implies that p_1 is equal to

$$P\big(|t_k Z_{1/t_k}|\in I_k,\ \operatorname{sign} t_k Z_{1/t_k}=i_k,\ k=1,2;\ 1/t_1,\ 1/t_2 \text{ belong to different excursions of } (r_t)\big).$$

Similarly, we can show that p_2 is equal to

$$P\big(|t_k Z_{1/t_k}|\in I_k,\ \operatorname{sign} t_k Z_{1/t_k}=i_k,\ k=1,2;\ 1/t_1,\ 1/t_2 \text{ belong to the same excursion of } (r_t)\big).$$

Adding up we get

$$P(|Z_{t_k}|\in I_k,\ \operatorname{sign} Z_{t_k}=i_k,\ k=1,2)=P(|t_k Z_{1/t_k}|\in I_k,\ \operatorname{sign} t_k Z_{1/t_k}=i_k,\ k=1,2),$$

which implies that the two-dimensional marginal distributions of (Z_t) and $(tZ_{1/t})$ are the same. As remarked above, the same kind of argument shows that all the finite-dimensional distributions of (Z_t) and $(tZ_{1/t})$ are the same. Thus (X_t) and $(tX_{1/t})$ have the same distribution under P. Finally, since (X_t, P) is continuous and Markovian with respect to a Feller semigroup, the continuity and the strong Markov property of $(tX_{1/t})$ under P are proved by using standard arguments.

4. The independence assumption (A). In this section we shall use the result of Section 3 and show that the assumption (A) is valid:

THEOREM 2. *Let* $((X_t), P)$ *be a symmetrized Bessel process of index* $\nu\in(-1,0)$ *starting at* 0. *Then* (A) *is satisfied.*

Proof. Define $(Y_t):=(tX_{1/t})$. As is well known for Brownian motion and proved in Proposition 1 (Section 3) in the general case (X_t) and (Y_t) are equivalent diffusions under P. Define $T:=d^1(Y)$. Then T is a finite stopping time and $P(Y_T=0)=1$. The strong Markov property implies that the distribution of $(Y_{T+t})_{t\geq 0}$ under $P(\cdot\mid T=a)$ equals the distribution of $(Y_t)_{t\geq 0}$ under P for any $a>1$. Therefore, since (Y_t) under P, and thus also (Y_{a+t}) under $P(\cdot\mid T=a)$ is $\frac{1}{2}$-self-similar, we have for any $a>1$

$$(Y_{a+t})_{t\geq 0}\overset{d}{=}(a^{-1/2}Y_{a+at})_{t\geq 0} \text{ under } P(\cdot\mid T=a),$$

which implies that

$$\big((1-t)\,Y_{a+t/(1-t)}\big)_{0\leqslant t\leqslant 1} \quad \text{and} \quad \big(a^{-1/2}(1-t)\,Y_{a+(at)/(1-t)}\big)_{0\leqslant t\leqslant 1}$$

have the same distribution under $P(\cdot \mid T=a)$ for all $a>1$. Consider the law of

$$\big((1-t)\,Y_{a+t/(1-t)}\big)_{0\leqslant t\leqslant 1} \text{ under } P(\cdot \mid T=a).$$

For all $a>1$ this is the law of $(Y_{a+at})_{t\geqslant 0}$, under $P(\cdot \mid T=a)$, conditioned to hit 0 at $t=1$ ('bridge of Y'). This statement is straightforward in the case of Brownian motion; for the general Bessel case see [9], Theorem 5.8, p. 324. Rewriting in terms of (X_t) shows that the law of

$$(a^{1/2}X_{u/a})_{0\leqslant u\leqslant 1} \text{ under } P(\cdot \mid T=a)$$

is the same for all $a>1$. But $T=d^1(Y)=g_1^{-1}$, and so the distribution of

$$(g_1^{-1/2}X_{ug_1})_{0\leqslant u\leqslant 1} \text{ under } P(\cdot \mid T=a)=P(\cdot \mid g_1=a^{-1})$$

does not depend on a, which means that g_1 and $(g_1^{-1/2}\cdot X_{ug_1})_{0\leqslant u\leqslant 1}$ are independent under P. This immediately proves the statement. ∎

Remark 4. If (X_t) is a Brownian motion, then $(g_1^{-1/2}X_{ug_1})_{0\leqslant u\leqslant 1}$ is a Brownian bridge (see [11]). Similarly, we can construct bridges from symmetrized Bessel processes. Except the moments of A_t we can use (*) and (**) to calculate the moments of $\tilde{A}_t$, which correspond to the bridge process $(g_1^{-1/2}X_{ug_1})$.

Remark 5. The results of Proposition 1 and Theorem 2 are also valid in the general H-self-similar case. Obviously, any H-self-similar symmetric diffusion can be constructed from the radial excursions in the same way as the symmetrized Bessel process is constructed from the ordinary Bessel process; and using Remark 2 of Section 2 and the H-self-similarity we can show that (R_t) and $(t^{2H}R_{1/t})$ are equivalent, where (R_t) is an H-self-similar diffusion on $[0,\infty)$ starting at 0. Using these two facts it is easily seen that the arguments used in the proof of Proposition 1 and Theorem 2 are applicable in the general case.

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