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DISCRETE PROBABILITY MEASURES ON 2×2 STOCHASTIC MATRICES AND A FUNCTIONAL EQUATION ON [0, 1]

BY

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Abstract. In this paper, we consider the following natural problem: suppose μ_1 and μ_2 are two probability measures with finite supports $S(\mu_1)$, $S(\mu_2)$, respectively, such that $|S(\mu_1)| = |S(\mu_2)|$ and $S(\mu_1) \cup S(\mu_2) \subset 2\times 2$ stochastic matrices, and μ_1^n (the n-th convolution power of μ_1 under matrix multiplication), as well as μ_2^n, converges weakly to the same probability measure λ, where $S(\lambda) \subset 2\times 2$ stochastic matrices with rank one. Then when does it follow that $\mu_1 = \mu_2$? What if $S(\mu_1) = S(\mu_2)$? In other words, can two different random walks, in this context, have the same invariant probability measure? Here, we consider related problems.

1. Introduction: Statement of the problem. Let μ_1 be a probability measure on 2×2 stochastic matrices such that its support $S(\mu_1)$, consisting of n points, is given by

$$S(\mu_1) = \{A_1, A_2, \ldots, A_n\},$$

where $A_i = (x_i, y_i)$ denotes the stochastic matrix whose first column is (x_i, y_i), $0 < x_i < 1$, $0 < y_i < 1$ and $x_i > y_i$. The matrix (t, t) will be denoted simply by t. Then, it is well-known that the convolution iterates μ_1^n, defined by

$$\mu_1^{n+1}(\mathscr{B}) = \int \mu_1^n \{y\colon\ yx \in \mathscr{B}\}\, \mu_1(dx),$$

converge weakly to a probability measure λ, whose support consists of 2×2 stochastic matrices with identical rows. Thus, the elements in $S(\mu_1)$ can be represented by points below the diagonal in the unit square, and the elements in $S(\lambda)$ can be represented by points on the diagonal. Considering λ as a probability measure on the unit interval [0, 1], let G be the distribution function of λ. Then since λ is uniquely determined by the convolution equation

$$\lambda * \mu = \lambda, \tag{1.1}$$

the function G is uniquely determined by the functional equation

$$G(x) = \sum_{i=1}^{n} p_i\, G\left(\frac{x-y_i}{x_i-y_i}\right), \tag{1.2}$$

where $p_i = \mu(A_i)$, $0 < p_i < 1$, $p_1+p_2+\ldots+p_n = 1$. Writing $g(x) = G(Lx+t_1)$, where $t_i = \lim_{n\to\infty} A_i^n$, $L \equiv t_n - t_1$, $t_1 < t_2 < \ldots < t_n$, it is easily verified that (1.2) becomes

$$g(x) = \sum_{i=1}^{n} p_i g(a_i x - \alpha_i a_i + \alpha_i), \tag{1.3}$$

where $0 \leqslant x \leqslant 1$, $1/a_i \equiv x_i - y_i$, $\alpha_i \equiv (t_i - t_1)/(t_n - t_1)$. It is easily shown that $g(x) > 0$ for $x > 0$ and $g(x) < 1$ for $x < 1$.

In this paper, we study the problem concerning when the limit λ determines uniquely the probability measure μ_1. This problem was earlier examined in [2] in the case when $n = 2$. See also [1]; and [3], p. 159.

Such problems come up in a natural manner in the theory of iterated function systems in the context of fractals/attractors. In that context, the measure μ in (1.1) happens to be the distribution that induces the random walk with values in a set of stochastic matrices, and the measure λ in (1.1) is the distribution that uniquely determines the attractor corresponding to the random walk induced by μ. The problem is whether two different systems can give rise to the same attractor.

In terms of the functional equation (1.3), the problem can be stated as follows: If the function g in (1.3) also satisfies the equation

$$g(x) = \sum_{i=1}^{n} p_i' g(a_i' x - \alpha_i' a_i' + \alpha_i'), \tag{1.4}$$

where the quantities p_i', a_i', α_i' are corresponding to another probability measure μ_2 (with exactly the same meanings as before) such that μ_2^n also converges weakly to the same probability measure λ, then when can we conclude that $\mu_1 = \mu_2$ or, in other words, for each $i \geqslant 1$, $p_i = p_i'$ and $(x_i, y_i) = (x_i', y_i')$? The theorem in the next section is an attempt to answer this question.

2. Main result: A theorem.

THEOREM 2.1. *Let μ_1 and μ_2 be two probability measures each with an n-point support such that*

$$S(\mu_1) = \{A_1, A_2, \ldots, A_n\}, \qquad S(\mu_2) = \{A_1', A_2', \ldots, A_n'\},$$

where $A_i = (x_i, y_i)$, $A_i' = (x_i', y_i')$, $x_i - y_i > 0$, and $x_i' - y_i' > 0$. Suppose that both μ_1^n and μ_2^n converge weakly, as $n \to \infty$, to the same probability measure λ.

Let $t_i = y_i/[1-(x_i-y_i)]$ and $t_i' = y_i'/[1-(x_i'-y_i')]$ so that

$$\lim_{n\to\infty} A_i^n = (t_i, t_i) \quad \text{and} \quad \lim_{n\to\infty} A_i'^n = (t_i', t_i').$$

Suppose that the following conditions hold:

(i) *for* $1 < i \leqslant n$, $t_1 = t_1' < \min\{t_i, t_i'\}$;

(ii) *the map* $x \to t_1 \cdot x$ *is one-to-one on* $S(\mu_1) \cup S(\mu_2)$.

Then $\mu_1 = \mu_2$.

[Let us remark that condition (ii) means that if (a, b) and (c, d) are two different points in $S(\mu_1)\cup S(\mu_2)$, then $t_1(a-b)+b \neq t_1(c-d)+d$. Geometrically, this means that if we consider the points $P=(t_1, t_1)$, $A=(1, 0)$, $A_i=(x_i, y_i)$ and $A_i'=(x_i', y_i')$ in the unit square, then the line through A_i (respectively, A_i') parallel to the line PA does *not* contain any of the points A_j, $j\neq i$ (respectively, A_j', $j \neq i$), and A_i', $1\leqslant i\leqslant n$ (respectively, A_i, $1\leqslant i\leqslant n$). Let us also remark that the theorem remains true if we replace conditions (i) and (ii) above by the following conditions:

(i′) for $1\leqslant i<n$, $t_n=t_n'>\max\{t_i, t_i'\}$;

(ii′) the map $x\to t_n\cdot x$ is one-to-one on $S(\mu_1)\cup S(\mu_2)$.]

Proof. The function g corresponding to λ satisfies the equations

$$g(x)=\sum_{i=1}^{n} p_i g(a_i x-\alpha_i a_i+\alpha_i), \tag{2.1}$$

$$g(x)=\sum_{j=1}^{n} p_j' g(a_j' x-\alpha_j' a_j'+\alpha_j'). \tag{2.2}$$

Substituting one into the other, from (2.1) and (2.2) we have

$$\begin{aligned} g(x) &= \sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j' g(a_i a_j' x-\alpha_i a_i a_j'+\alpha_i a_j'-\alpha_j' a_j'+\alpha_j') \\ &= \sum_{j=1}^{n}\sum_{i=1}^{n} p_j' p_i g(a_i a_j' x-\alpha_j' a_j' a_i+a_i\alpha_j'-\alpha_i a_i+\alpha_i). \end{aligned}$$

Writing $h(x)\equiv g(x)-p_1 p_1' g(a_1 a_1' x)-p_n p_n' g(a_n a_n' x-a_n a_n'+1)$, then we have

$$\begin{aligned} h(x) &= \sum\sum p_i p_j' g(a_i a_j' x-a_j'\alpha_i a_i+a_j'\alpha_i-\alpha_j' a_j'+\alpha_j') \\ &= \sum\sum p_i p_j' g(a_i a_j' x-\alpha_j' a_j' a_i+a_i\alpha_j'-\alpha_i a_i+\alpha_i), \end{aligned} \tag{2.3}$$

where the summations in both expressions above are for $i=1$ to $i=n$, $j=1$ to $j=n$ such that $(i, j)\neq(1, 1)$ and $(i, j)\neq(n, n)$.

Now notice that

$$a_i a_j' x-a_j'\alpha_i a_i+a_j'\alpha_i-\alpha_j' a_j'+\alpha_j' \leqslant 0 \quad \text{iff} \quad x\leqslant \alpha_i\left(1-\frac{1}{a_i}\right)+\frac{\alpha_j'}{a_i}\left(1-\frac{1}{a_j'}\right)$$

and

$$a_i a_j' x-\alpha_j' a_j' a_i+a_i\alpha_j'-\alpha_i a_i+\alpha_i \leqslant 0 \quad \text{iff} \quad x\leqslant \alpha_j'\left(1-\frac{1}{a_j'}\right)+\frac{\alpha_i}{a_j'}\left(1-\frac{1}{a_i}\right).$$

Since $g(x) > 0$ for $x > 0$, we have

$$\min\left\{\alpha_i\left(1-\frac{1}{a_i}\right)+\frac{\alpha_j'}{a_i}\left(1-\frac{1}{a_j'}\right):\ (i,j)\neq(1,1),\ (i,j)\neq(n,n)\right\}$$
$$=\min\left\{\alpha_j'\left(1-\frac{1}{a_j'}\right)+\frac{\alpha_i}{a_j'}\left(1-\frac{1}{a_i}\right):\ (i,j)\neq(1,1),\ (i,j)\neq(n,n)\right\}.$$

Note that $\alpha_1 = \alpha_1' = 0$, and that

$$\min\left\{\frac{\alpha_j'}{a_1}\left(1-\frac{1}{a_j'}\right):\ j\neq 1\right\}<\min\left\{\alpha_j'\left(1-\frac{1}{a_j'}\right)+\frac{\alpha_i}{a_j'}\left(1-\frac{1}{a_i}\right):\ j\neq 1,\ 1\leqslant i\leqslant n\right\};$$

also,

$$\min\left\{\frac{\alpha_i}{a_1'}\left(1-\frac{1}{a_i}\right):\ i\neq 1\right\}<\min\left\{\alpha_i\left(1-\frac{1}{a_i}\right)+\frac{\alpha_j'}{a_i}\left(1-\frac{1}{a_j'}\right):\ i\neq 1,\ 1\leqslant j\leqslant n\right\}.$$

This means that

$$\min\left\{\frac{\alpha_i}{a_1'}\left(1-\frac{1}{a_i}\right):\ i\neq 1\right\}=\min\left\{\frac{\alpha_j'}{a_1}\left(1-\frac{1}{a_j'}\right):\ j\neq 1\right\}. \tag{2.4}$$

Since $g(x) < 1$ for $x < 1$, instead of considering the "minimum" if we considered the "maximum" above, we would obtain similarly (after some calculations)

$$\max\left\{1-\frac{1-\alpha_i}{a_n'}\left(1-\frac{1}{a_i}\right):\ i\neq n\right\}=\max\left\{1-\frac{1-\alpha_j'}{a_n}\left(1-\frac{1}{a_j'}\right):\ j\neq n\right\},$$

so that

$$\min\left\{\frac{1-\alpha_i}{a_n'}\left(1-\frac{1}{a_i}\right):\ i\neq n\right\}=\min\left\{\frac{1-\alpha_j'}{a_n}\left(1-\frac{1}{a_j'}\right):\ j\neq n\right\}. \tag{2.5}$$

Let us now make the following observation. The points A_1, A_i and A_j' are the points (x_1, y_1), (x_i, y_i) and (x_j', y_j'), respectively.

Note that the condition

$$\alpha_i(1-1/a_i)=\alpha_j'(1-1/a_j') \tag{2.6}$$

is equivalent to the condition that the points $t_1 A_i$ and $t_1 A_j'$ are identical.

Similarly, the condition

$$(2.7)\qquad (1-\alpha_i)(1-1/a_i) = (1-\alpha_j')(1-1/a_j)$$

is equivalent to the condition that the points $t_n A_i$ and $t_n A_j'$ are identical.

Now suppose that we are given that the points A_1 and A_1' are the same and that there are no points $A_i \in S(\mu_1)$ and $A_j' \in S(\mu_2)$ such that the condition (2.6) holds unless they are the same. Then, of course, it follows from (2.4) that the points $(x_{i_0}, y_{i_0}) = (x_{j_0}', y_{j_0}')$ for some $i_0 > 1$ and $j_0 > 1$. But notice that

$$(2.8)\qquad g(x) = \begin{cases} p_1 g(a_1 x), & 0 \leqslant x \leqslant a_{i_0}(1-1/a_{i_0}), \\ p_1' g(a_1' x), & 0 \leqslant x \leqslant \alpha_{j_0}'(1-1/a_{j_0}'), \end{cases}$$

and also

$$(2.9)\qquad g(x) = \begin{cases} p_1 g(a_1 x) + p_{i_0} g\big(a_{i_0}[x-\alpha_{i_0}(1-1/a_{i_0})]\big), & 0 \leqslant x \leqslant B_1, \\ p_1' g(a_1' x) + p_{j_0}' g\big(a_{j_0}'[x-\alpha_{j_0}'(1-1/a_{i_0})]\big), & 0 \leqslant x \leqslant B_1', \end{cases}$$

where

$$B_1 = \min\{\alpha_i(1-1/a_i)\colon\ i \neq 1,\ i \neq i_0\},$$

$$B_1' = \min\{\alpha_j'(1-1/a_j')\colon\ j \neq 1,\ j \neq j_0\}.$$

Since $g(x) > 0$ for $x > 0$ and since $A_1 = A_1'$ and $A_{i_0} = A_{j_0}'$, it follows from (2.8) and (2.9) that

$$(2.10)\qquad p_1 = p_1', \quad p_{i_0} = p_{j_0}'.$$

As a result, $a_{i_0} = a_{j_0}'$ and $\alpha_{i_0} = \alpha_{j_0}'$. Now we can go back to (2.3) and subtract appropriate terms from both sides. Writing

$$\begin{aligned} h_1(x) &\equiv h(x) - p_1 p_{j_0}' g(a_1 a_{j_0}' x - \alpha_{j_0}' a_{j_0}' + \alpha_{j_0}') \\ &\quad - p_{i_0} p_{j_0}' g(a_{i_0} a_{j_0}' x - a_{j_0}'\alpha_{i_0} a_{i_0} + a_{j_0}'\alpha_{i_0} - \alpha_{j_0}' a_{j_0}' + \alpha_{j_0}') \\ &\equiv h(x) - p_{i_0} p_1' g(a_{i_0} a_1' x - \alpha_{i_0} a_{i_0} + \alpha_{i_0}) \\ &\quad - p_{j_0}' p_{i_0} g(a_{i_0} a_{j_0}' x - \alpha_{j_0}' a_{j_0}' a_{i_0} + a_{i_0}\alpha_{j_0}' - \alpha_{i_0} a_{i_0} + \alpha_{i_0}) \end{aligned}$$

we then have

$$\begin{aligned} (2.11)\qquad h_1(x) &= \sum\sum p_i p_j' g(a_i a_j' x - a_j'\alpha_i a_i + a_j'\alpha_i - \alpha_j' a_j' + \alpha_j') \\ &= \sum\sum p_i p_j' g(a_i a_j' x - \alpha_j' a_j' a_i + a_i\alpha_j' - \alpha_i a_i + \alpha_i), \end{aligned}$$

where the summation in both expressions on the right are for $i = 1$ to $i = n$, and $j = 1$ to $j = n$ such that $(i, j) \neq (1, 1)$, $(i, j) \neq (n, n)$, $(i, j) \neq (i_0, j_0)$, $(i, j) \neq (1, j_0)$ and $(i, j) \neq (i_0, 1)$.

Again, following the same analysis as before, we now have

$$(2.12)\qquad \min\left\{\frac{\alpha_i}{a_1'}\left(1-\frac{1}{a_i}\right)\colon\ i \neq 1,\ i \neq i_0\right\} = \min\left\{\frac{\alpha_j'}{a_1}\left(1-\frac{1}{a_j'}\right)\colon\ j \neq 1,\ j \neq j_0\right\}.$$

It follows from (2.12) that there exist i_1 and j_1 such that $i_1 \notin \{1, i_0\}$, $j_1 \notin \{1, j_0\}$ and $A_{i_1} = A'_{j_1}$. Like in (2.9) and (2.10), we can again show that $p_{i_1} = p'_{j_1}$. The induction process continues and it follows that $\mu_1 = \mu_2$. ∎

Let us now give an example to show that in Theorem 2.1 the condition that the map $x \to t_1 \cdot x$ is one-to-one cannot be removed.

EXAMPLE 2.2. Consider the probability measures μ_1 and μ_2 such that

$$S(\mu_1) = \{A_1, A_2, A_3\},$$

where

$$A_1 = (3/8, 5/24), \quad A_2 = (5/6, 1/6), \quad A_3 = (19/24, 5/8)$$

with

$$\mu_1(A_1) = 1/6, \quad \mu_1(A_2) = 2/3, \quad \mu_1(A_3) = 1/6,$$

and

$$S(\mu_2) = \{A'_1, A'_2, A'_3\},$$

where

$$A'_1 = A_1 = (3/8, 5/24), \quad A'_2 = (14/15, 2/15), \quad A'_3 = (91/120, 29/40)$$

with

$$\mu_2(A'_1) = 1/6, \quad \mu_2(A'_2) = 4/5, \quad \mu_2(A'_3) = 1/30.$$

It is easily verified that

$$\lim_{n\to\infty} A_1^n = t_1 \equiv (1/4, 1/4).$$

Notice that

$$t_1 \cdot A_2 = t_1 \cdot A'_2 = (1/3, 1/3)$$

so that the map $x \to t_1 \cdot x$ is not one-to-one on $S(\mu_1) \cup S(\mu_2)$.

Let $\lambda_1 = (w) \lim_{n\to\infty} \mu_1^n$. Then the function g_1 corresponding to λ_1 satisfies the equation

$$(2.13) \qquad g_1(x) = \sum_{i=1}^{3} p_i g_1(a_i x - \alpha_i a_i + \alpha_i) = \tfrac{1}{6} g_1(6x) + \tfrac{2}{3} g_1(\tfrac{3}{2}x - \tfrac{1}{4}) + \tfrac{1}{6} g_1(6x - 5)$$

for $0 \leqslant x \leqslant 1$. Similarly, if $\lambda_2 = (w) \lim_{n\to\infty} \mu_2^n$, then the function g_2 corresponding to λ_2 satisfies the equation

$$(2.14) \qquad g_2(x) = \sum_{i=1}^{3} p'_i g_2(a'_i x - \alpha'_i a'_i + \alpha'_i) = \tfrac{1}{6} g_2(6x) + \tfrac{2}{3} g_2(\tfrac{5}{4}x - \tfrac{5}{24}) + \tfrac{1}{30} g_1(30x - 29)$$

for $0 \leqslant x \leqslant 1$.

Observing that g_1 and g_2 both satisfy

$$g_1(x) = 1, \quad g_2(x) = 1 \quad \text{for } x \geqslant 1,$$

and

$$g_1(x) = 0 = g_2(x) \quad \text{for } x \leqslant 0,$$

it follows immediately that the function

$$g(x) = x, \quad 0 \leqslant x \leqslant 1,$$

satisfies both (2.13) and (2.14). Thus, though $\mu_1 \neq \mu_2$, it follows that $\lambda_1 = \lambda_2$.

The following result, though it seems limited, does not seem to be trivial.

THEOREM 2.3. *Suppose μ_1 and μ_2 are two probability measures on 2×2 stochastic matrices such that both $S(\mu_1)$ and $S(\mu_2)$ consist of n points. Suppose that*

(i) $(w) \lim_{n\to\infty} \mu_1^n = (w) \lim_{n\to\infty} \mu_2^n = \lambda$;

(ii) $\mu_1(A_i) = \mu_2(A_i') = p_i$, $1 \leqslant i \leqslant n$, *where* $S(\mu_1) = \{A_1, A_2, \ldots, A_n\}$ *and* $S(\mu_2) = \{A_1', A_2', \ldots, A_n'\}$.

We assume: $t_i < t_{i+1}$, $t_i' < t_{i+1}'$, $1 \leqslant i \leqslant n-1$. *Then, if* $n \leqslant 4$, *then* $\mu_1 = \mu_2$.

Proof. We prove only the case where $n = 4$, which is not trivial.

Let g be the function (as defined earlier) corresponding to λ so that g satisfies the equations

$$g(x) = \sum_{i=1}^{4} p_i g(a_i x - \alpha_i a_i + \alpha_i) = \sum_{i=1}^{4} p_i g(a_i' x - \alpha_i' a_i' + \alpha_i'), \tag{2.15}$$

where a_i, a_i', α_i, α_i' have the same meanings as in (1.3) and (1.4). [We are assuming here, for simplicity, $t_1 < t_2 < t_3 < t_4$, where $t_i = \lim_{n\to\infty} A_i^n$, and also $t_1' < t_2' < t_3' < t_4'$, where $t_i' = \lim_{n\to\infty} A_i'^n$.]

It follows from (2.15), since $\alpha_1 = \alpha_1' = 0$ and $\alpha_4 = \alpha_4' = 1$, that

$$g(x) = p_1 g(a_1 x) = p_1 g(a_1' x) \tag{2.16}$$

if

$$0 \leqslant x \leqslant \min\{\alpha_i(1-1/a_i),\ \alpha_i'(1-1/a_i'):\ 2 \leqslant i \leqslant 4\},$$

and that

$$\begin{aligned} g(x) &= 1 - p_4 + p_4 g\big(a_4[x-(1-1/a_4)]\big) \\ &= 1 - p_4 + p_4 g\big(a_4'[x-(1-1/a_4')]\big) \end{aligned} \tag{2.17}$$

if

$$x \geqslant \max\{1/a_i + \alpha_i(1-1/a_i),\ 1/a_i' + \alpha_i'(1-1/a_i'):\ 1 \leqslant i \leqslant 3\}.$$

Notice that (2.16) implies that, for some $\delta > 0$,

$$g(a_1 x) = g(a_1' x), \qquad 0 \leqslant x \leqslant \delta.$$

Thus, if $a_1 > a_1'$, then

$$g(a_1' x) = g\left(a_1 \cdot \frac{a_1'}{a_1} x\right) = g\left(a_1' \cdot \frac{a_1'}{a_1} x\right) = g\left(a_1 \cdot \left(\frac{a_1'}{a_1}\right)^2 x\right)$$

$$= g\left(a_1 \cdot \left(\frac{a_1'}{a_1}\right)^m x\right) \qquad \text{for } m > 1,$$

so that, for $0 < x < \delta$,

$$g(a_1' x) = 0,$$

which contradicts the fact that for $x > 0$, $g(x) > 0$. Thus, $a_1 \leqslant a_1'$. Similarly, $a_1' \leqslant a_1$ and, consequently, $a_1 = a_1'$.

It follows from (2.17) that there exists $\delta > 0$ such that

$$g\big(a_4(x-1)+1\big) = g\big(a_4'(x-1)+1\big)$$

for $1-\delta \leqslant x \leqslant 1$. Writing y for $x-1$ and putting $h(y) = g(y+1)$, we obtain

$$h(a_4 y) = h(a_4' y), \qquad -\delta \leqslant y \leqslant 0.$$

Noting that $g(x) < 1$ for $x < 1$ and $g(1) = 1$ so that $h(y) < 1$ for $y < 0$ and $h(0) = 1$, we will again get a contradiction unless $a_4 = a_4'$. Thus, it follows that $A_1 = A_1'$ and $A_4 = A_4'$, since $t_1 = t_1'$ and $t_4 = t_4'$.

Now we infer from (2.15) that

$$p_2 g\big(a_2[x-\alpha_2(1-1/a_2)]\big) + p_3 g\big(a_3[x-\alpha_3(1-1/a_3)]\big) = p_2 g\big(a_2'[x-\alpha_2'(1-1/a_2')]\big) + p_3 g\big(a_3'[x-\alpha_3'(1-1/a_3')]\big) \tag{2.18}$$

for $0 \leqslant x \leqslant 1$.

It follows from (2.18) that

$$\min\{\alpha_2(1-1/a_2),\ \alpha_3(1-1/a_3)\} = \min\{\alpha_2'(1-1/a_2'),\ \alpha_3'(1-1/a_3')\} \tag{2.19}$$

and

$$\max\{1/a_2+\alpha_2(1-1/a_2),\ 1/a_3+\alpha_3(1-1/a_3)\} = \max\{1/a_2'+\alpha_2'(1-1/a_2'),\ 1/a_3'+\alpha_3'(1-1/a_3')\}. \tag{2.20}$$

First, we consider the following:

Case 1:

$$\alpha_2(1-1/a_2) = \alpha_2'(1-1/a_2'). \tag{2.21}$$

If

$$\alpha_2(1-1/a_2) < \alpha_3(1-1/a_3) \quad \text{and} \quad \alpha_2'(1-1/a_2') < \alpha_3'(1-1/a_3'),$$

then it follows from (2.18) that there exists $\delta > 0$ such that

$$g(a_2 y) = g(a_2' y), \quad 0 \leqslant y \leqslant \delta. \tag{2.22}$$

It follows from (2.22) that $a_2 = a_2'$ and therefore, from (2.21), $\alpha_2 = \alpha_2'$, and $A_2 = A_2'$. From (2.18) it follows that $A_3 = A_3'$.

Therefore, we assume that

$$\alpha_2(1-1/a_2) = \alpha_2'(1-1/a_2') = \alpha_3(1-1/a_3) < \alpha_3'(1-1/a_3'). \tag{2.23}$$

Since $\alpha_2 < \alpha_3$, we have $1/a_2 < 1/a_3$. Then from (2.20) we obtain

$$1/a_3+\alpha_3(1-1/a_3) = \max\{1/a_2'+\alpha_2'(1-1/a_2'),\ 1/a_3'+\alpha_3'(1-1/a_3')\}.$$

If

$$1/a_3+\alpha_3(1-1/a_3) = 1/a_3'+\alpha_3'(1-1/a_3') > 1/a_2'+\alpha_2'(1-1/a_2'), \tag{2.24}$$

then, by (2.18), for

$$\max\{1/a_2+\alpha_2(1-1/a_2),\ 1/a_2'+\alpha_2'(1-1/a_2')\} \leqslant x < 1/a_3+\alpha_3(1-1/a_3),$$

we have

$$p_2+p_3\, g\big(a_3[x-\alpha_3(1-1/a_3)]\big) = p_2+p_3\, g\big(a_3'[x-\alpha_3'(1-1/a_3')]\big)$$

or

$$g\big(a_3[x-\alpha_3(1-1/a_3)]\big) = g\big(a_3'[x-\alpha_3'(1-1/a_3')]\big).$$

It follows that there exists $\delta > 0$ such that

$$h(a_3 y) = h(a_3' y), \quad -\delta \leqslant y \leqslant 0,$$

where $h(0) = 1$ and $h(y) < 1$ for $y < 0$. It follows that $a_3 = a_3'$, and therefore $\alpha_3 = \alpha_3'$ and $A_3 = A_3'$. Consequently, from (2.18), we get $A_2 = A_2'$. Therefore, we assume that

$$1/a_3+\alpha_3(1-1/a_3) = 1/a_2'+\alpha_2'(1-1/a_2') > 1/a_3'+\alpha_3'(1-1/a_3'). \tag{2.25}$$

Then, from (2.23) and (2.15) we get

$$A_2' = A_3, \tag{2.26}$$

and from (2.18) we obtain

$$(p_2-p_3)\, g\big(a_3[x-\alpha_3(1-1/a_3)]\big) \\ = p_2\, g\big(a_2[x-\alpha_2(1-1/a_2)]\big)-p_3\, g\big(a_3'[x-\alpha_3'(1-1/a_3')]\big). \tag{2.27}$$

By (2.25), for

$$1/a_3+\alpha_3(1-1/a_3) > x \geqslant \max\{1/a_2+\alpha_2(1-1/a_2),\ 1/a_3'+\alpha_3'(1-1/a_3')\}$$

we then have

$$(p_2-p_3)g(a_3[x-\alpha_3(1-1/a_3)]) = p_2-p_3,$$

which implies that

$$p_2 = p_3.$$

Consequently, from (2.27) we get

$$g(a_2[x-\alpha_2(1-1/a_2)]) = g(a'_3[x-\alpha'_3(1-1/a'_3)])$$

for $0 \leqslant x \leqslant 1$. It follows that

$$A_2 = A'_3. \tag{2.28}$$

Then from (2.26) and (2.28) we obtain

$$A'_2 = A_3, \quad A_2 = A'_3,$$

which is a contradiction since $t_1 < t_2 < t_3 < t_4$ and $t'_1 < t'_2 < t'_3 < t'_4$. Thus, we must assume that

$$\begin{aligned}1/a_3+\alpha_3(1-1/a_3) &= 1/a'_2+\alpha'_2(1-1/a'_2) \\ &= 1/a'_3+\alpha'_3(1-1/a'_3) > 1/a_2+\alpha_2(1-1/a_2).\end{aligned} \tag{2.29}$$

By (2.18), then there exists $\delta > 0$ such that

$$\begin{aligned}&(p_2-p_3)g(a_3[x-\alpha_3(1-1/a_3)]) \\ &\quad = p_2-p_3 g(a'_3[x-\alpha'_3(1-1/a'_3)]) \geqslant p_2-p_3\end{aligned} \tag{2.30}$$

for $1-\delta \leqslant x \leqslant 1$. It follows that $p_2 \leqslant p_3$.

Also, from (2.18) (see also (2.23)), for $x < \alpha'_3(1-1/a'_3)$ we get

$$p_2 g(a_2[x-\alpha_2(1-1/a_2)]) = (p_2-p_3)g(a'_2[x-\alpha'_2(1-1/a'_2)]),$$

so that $p_2 \geqslant p_3$. Hence $p_2 = p_3$. This gives in (2.30):

$$g(a'_3[x-\alpha'_3(1-1/a'_3)]) = 1$$

for $x < 1/a'_3+\alpha'_3(1-1/a_3)$, a contradiction. Thus, we cannot assume (2.23).

Similarly, we cannot assume

$$\alpha_2(1-1/a_2) = \alpha'_2(1-1/a'_2) = \alpha'_3(1-1/a'_3) < \alpha_3(1-1/a_3).$$

Thus, we must assume, if possible,

$$\alpha_2(1-1/a_2) = \alpha'_2(1-1/a'_2) = \alpha_3(1-1/a_3) = \alpha'_3(1-1/a'_3).$$

Then $1/a_2 < 1/a_3$ and $1/a'_2 < 1/a'_3$. From (2.20) we get

$$1/a_3+\alpha_3(1-1/a_3) = 1/a'_3+\alpha'_3(1-1/a'_3).$$

It follows that $A_3 = A'_3$. From (2.18) we obtain $A_2 = A'_2$.

Case 2: $\alpha_2(1-1/a_2) = \alpha'_3(1-1/a'_3)$.
In this case,

$$\alpha'_3(1-1/a'_3) \leqslant \alpha'_2(1-1/a'_2),$$

so that, since $\alpha'_3 > \alpha'_2$,

$$1-1/a'_3 < 1-1/a'_2 \quad \text{or} \quad 1/a'_3 > 1/a'_2.$$

This means that

$$1/a'_3+\alpha'_3(1-1/a'_3) > 1/a'_2+\alpha'_2(1-1/a'_2)$$

since

$$\alpha'_2(1-1/a'_2)-\alpha'_3(1-1/a'_3) < \alpha'_2(1-1/a'_2)-\alpha'_3(1-1/a'_3)$$
$$= \alpha'_2(1/a'_3-1/a'_2) < 1/a'_3-1/a'_2.$$

Thus, since

$$\max\{1/a_2+\alpha_2(1-1/a_2),\ 1/a_3+\alpha_3(1-1/a_3)\}$$
$$= \max\{1/a'_3+\alpha'_3(1-1/a'_3),\ 1/a'_2+\alpha'_2(1-1/a'_2)\},$$

it follows that we must have one of the following two possibilities:
(i) $1/a_2+\alpha_2(1-1/a_2) = 1/a'_3+\alpha'_3(1-1/a'_3)$;
(ii) $1/a_3+\alpha_3(1-1/a_3) = 1/a'_3+\alpha'_3(1-1/a'_3)$.

In the first situation, since $\alpha_2(1-1/a_2) = \alpha'_3(1-1/a'_3)$, we have $a_2 = a'_3$ and $\alpha_2 = \alpha'_3$. This, of course, means that $t_2 = t'_3$ and

$$y_2 = t_2[1-1/a_2] = t'_3[1-1/a'_3] = y'_3,$$

so that $A_2 = A'_3$. Then we have

$$(2.31) \quad (p_2-p_3)g\big(a_2[x-\alpha_2(1-1/a_2)]\big)$$
$$= p_2\,g\big(a'_2[x-\alpha'_2(1-1/a'_2)]\big)-p_3\,g\big(a_3[x-\alpha_3(1-1/a_3)]\big).$$

This means that

$$1/a_2+\alpha_2(1-1/a_2) = 1/a_3+\alpha_3(1-1/a_3),$$

for if

$$1/a_3+\alpha_3(1-1/a_3) < 1/a_2+\alpha_2(1-1/a_2),$$

then

$$B \equiv \max\{1/a_3+\alpha_3(1-1/a_3),\ 1/a'_2+\alpha'_2(1-1/a'_2)\} < 1/a_2+\alpha_2(1-1/a_2),$$

so that for $B \leqslant x < 1/a_2+\alpha_2(1-1/a_2)$ the right-hand side of (2.31) is p_2-p_3, so that

$$g\big(a_2[x-\alpha_2(1-1/a_2)]\big) = 1,$$

which contradicts that $g(y) < 1$ for $y < 1$. Thus, we can now assume that

(2.32) $$D \equiv 1/a_2 + \alpha_2(1-1/a_2) = 1/a_3 + \alpha_3(1-1/a_3) = 1/a'_3 + \alpha'_3(1-1/a'_3).$$

Note that, for $1/a'_2 + \alpha'_2(1-1/a'_2) \leqslant x < D$, we have

$$(p_2 - p_3)\, g\big(a_2[x - \alpha_2(1-1/a_2)]\big) = p_2 - p_3\, g\big(a_3[x-\alpha_3(1-1/a_3)]\big) \geqslant p_2 - p_3,$$

which means that $p_2 \leqslant p_3$.

Now, if we have

$$\alpha_2(1-1/a_2) < \alpha_3(1-1/a_3),$$

and also

$$\alpha'_3(1-1/a'_3) < \alpha'_2(1-1/a'_2),$$

then for

$$\alpha_2(1-1/a_2) = \alpha'_3(1-1/a'_3) < x < \min\{1/a_3 + \alpha_3(1-1/a_3),\ 1/a'_2 + \alpha'_2(1-1/a'_2)\}$$

we obtain

(2.33) $$p_2\, g\big(a_2[x-\alpha_2(1-1/a_2)]\big) = p_3\, g\big(a'_3[x-\alpha'_3(1-1/a'_3)]\big).$$

Since $A_2 = A'_3$, we have $p_2 = p_3$.

If $\alpha_2(1-1/a_2) = \alpha_3(1-1/a_3)$, then it follows from (2.32) that $A_2 = A_3$, which is not possible.

Also, if

$$\alpha'_3(1-1/a'_3) = \alpha'_2(1-1/a'_2) \quad (= \alpha_2(1-1/a_2)),$$

then either one of these is equal to $\alpha_3(1-1/a_3)$, in which case $A_2 = A_3$ (a contradiction), or each one is less than $\alpha_3(1-1/a_3)$, so that for $\alpha_2(1-1/a_2) < x < \alpha_3(1-1/a_3)$ we have

$$(p_2-p_3)\, g\big(a_2[x-\alpha_2(1-1/a_2)]\big) = p_2\, g\big(a'_2[x-\alpha'_2(1-1/a'_2)]\big),$$

which means that $p_2 \geqslant p_3$. Thus, in this case, $p_2 = p_3$, so that

$$g\big(a'_2[x-\alpha'_2(1-1/a'_2)]\big) = 0$$

for $x > \alpha'_2(1-1/a'_2)$, a contradiction.

Thus, we have $p_2 = p_3$, $A_2 = A'_3$ and, consequently, from (2.18) we obtain

$$g\big(a_3[x-\alpha_3(1-1/a_3)]\big) = g\big(a'_2[x-\alpha'_2(1-1/a'_2)]\big),$$

which is a contradiction, since

$$1/a_3+\alpha_3(1-1/a_3) < 1/a'_2+\alpha'_2(1-1/a'_2)$$

in this case.

Thus, the only possibility is that (i) does not occur and (ii) occurs, that is

$$1/a_2+\alpha_2(1-1/a_2) < 1/a_3+\alpha_3(1-1/a_3) = 1/a'_3+\alpha'_3(1-1/a'_3).$$

Let

$$\max\{1/a_2+\alpha_2(1-1/a_2),\ 1/a'_2+\alpha'_2(1-1/a'_2)\}$$
$$\leqslant x < M \equiv 1/a_3+\alpha_3(1-1/a_3) = 1/a'_3+\alpha'_3(1-1/a'_3).$$

Then, we infer from (2.18) that there exists $\delta > 0$ such that

$$g(a_3[x-\alpha_3(1-1/a_3)]) = g(a'_3[x-\alpha'_3(1-1/a'_3)])$$

for $M-\delta \leqslant x \leqslant M$. Writing $y \equiv x-M$ and $g(y+1) \equiv h(y)$, we have

$$h(a_3 y) = h(a'_3 y), \qquad -\delta \leqslant y \leqslant 0.$$

Note that $h(0) = 1$, and if $y < 0$, then $h(y) < 1$. For any $y < 0$, if $a_3 > a'_3$, then

$$h(a_3 y) = h\left(a_3\cdot\frac{a'_3}{a_3}y\right) = h\left(a_3\cdot\frac{(a'_3)^2}{a_3^2}y\right) = h\left(a_3\left(\frac{a'_3}{a_3}\right)^n y\right) \to 1,$$

a contradiction. Thus, $a_3 \leqslant a'_3$. Similarly, $a_3 \geqslant a'_3$, so that $a_3 = a'_3$, $\alpha_3 = \alpha'_3$. Hence $A_3 = A'_3$.

It follows from (2.18) that

$$g(a_2[x-\alpha_2(1-1/a_2)]) = g(a'_2[x-\alpha'_2(1-1/a'_2)])$$

for $0 \leqslant x \leqslant 1$. Thus,

$$\alpha_2(1-1/a_2) = \alpha'_2(1-1/a'_2)$$

and

$$1/a_2+\alpha_2(1-1/a_2) = 1/a'_2+\alpha'_2(1-1/a'_2).$$

It follows that $A_2 = A'_2$. ∎

We can also prove the following theorem:

THEOREM 2.4. *Suppose μ_1 and μ_2 are two probability measures on 2×2 stochastic matrices such that $S(\mu_1)$ and $S(\mu_2)$ consist of n points. Suppose that*

(i) $(w)\lim_{n\to\infty}\mu_1^n = (w)\lim_{n\to\infty}\mu_2^n = \lambda$;

(ii) $S(\mu_1) = S(\mu_2) = \{A_1, A_2, \ldots, A_n\}$.

We assume: $t_i < t_{i+1}$. *Then, if* $n \leqslant 4$, $\mu_1 = \mu_2$.

We omit the proof (which is simpler than that of Theorem 2.3).

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