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STRONG LIMIT THEOREMS FOR GENERAL RENEWAL PROCESSES

BY

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Abstract. An approach is discussed to derive strong limit theorems for general renewal processes from the corresponding asymptotics of the underlying renewal sequence. Neither independence nor stationarity of increments is required. In certain situations, just the dualities between the renewal processes and their defining sequences in combination with some regularity conditions on the normalizing constants are sufficient for the proofs. There are other cases, however, in which the duality arguments do not apply, and where other techniques have to be developed. Finally, there are also examples, in which an inversion of the limit theorems under consideration cannot work at all.

1. INTRODUCTION

Consider a so-called *renewal sequence*, i.e. a sequence of partial sums $\{S_n, n \geqslant 0\}$, $S_0 = 0$, of independent identically distributed (i.i.d.) nonnegative random variables $\{X_n, n \geqslant 1\}$ with $0 < EX_1 = a < \infty$. Define their corresponding *renewal process* $\{N(t), t \geqslant 0\}$ as

$$N(t) = \sum_{n=1}^{\infty} I\{S_n \leqslant t\}, \quad t \geqslant 0. \tag{1.1}$$

A possible interpretation is that X_n represents the time between the $(n-1)$-st and n-th replacement (renewal) of (say) a machine part, so that $N(t)$ counts the number of replacements (renewals) up to time t. It is well known that many limit theorems for the counting process $\{N(t), t \geqslant 0\}$ are consequences of their corresponding counterparts for the renewal sequence via the following "duality":

$$\{N(t) = n\} \Leftrightarrow \{S_n \leqslant t, S_{n+1} > t\} \quad \text{for all } t \geqslant 0 \text{ and } n \in N_0. \tag{1.2}$$

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For example, the strong law of large numbers (SLLN)

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{a} \text{ almost surely (a.s.)} \tag{1.3}$$

is immediate from the SLLN

$$\lim_{n\to\infty} \frac{S_n}{n} = a \text{ a.s.,}$$

whereas other limit theorems such as the law of the iterated logarithm (LIL)

$$\limsup_{t\to\infty} \frac{N(t)-t/a}{\sqrt{2t\log\log t}} = \sqrt{\frac{\sigma^2}{a^3}} \text{ a.s.,} \tag{1.4}$$

and the central limit theorem (CLT)

$$\frac{N(t)-t/a}{\sqrt{t\sigma^2/a^3}} \xrightarrow{D} Z \quad \text{as } t\to\infty, \tag{1.5}$$

where Z is a standard normal random variable and $0 < \sigma^2 = \mathrm{Var}(X_1) < \infty$, require more sophisticated tools.

In case of nonnegative and i.i.d. summands $\{X_n, n \geqslant 1\}$ it has been shown by Gut et al. [6] that there are certain equivalences between limit theorems for partial sums and renewal processes. For a comprehensive study of renewal processes and random sums confer Gut [5], in which also the "general" case of S_n (possibly having negative summands with positive expectation) has been treated in further detail. For some recent equivalence statements in this "general" i.i.d. case confer Frolov et al. [3].

Much less is known about renewal processes for which either identical distribution or independence, or nonnegativity, or all of these assumptions are dropped. The aim of this paper is to develop a general approach to deriving limit theorems for "renewal processes" from their corresponding counterparts for the underlying "partial sum sequence". Indeed, it is not necessary to assume any structure of the defining sequence $\{S_n, n \geqslant 0\}$; one can just start from an appropriate limit theorem there.

Note, however, that certain regularity assumptions are sometimes crucial for the applicability of a duality argument. There are situations in which a limit theorem for the renewal process is almost immediate from its partial sum counterpart. But there are other cases where the desired inversion requires more sophisticated techniques. Finally, there are also examples in which a duality argument does not work at all because either the partial sum sequence satisfies a certain limit theorem, but not so its corresponding renewal process, or vice versa. We restrict our attention to *strong limit theorems* for renewal processes. A similar approach applies to *weak limit theorems* and to *renewal functions*, i.e. the expected number of renewals, but will be exploited elsewhere.

In order to avoid confusion with the i.i.d. situation, we change notation from now on and assume that $\{Z_n, n \geqslant 0\}$ is a *general* sequence of real-valued random variables. We then define a *general renewal process* $\{N(t), t \geqslant 0\}$ pointwise as

$$N(t) = \sum_{n=1}^{\infty} I\{Z_n \leqslant t\}, \quad t \geqslant 0. \tag{1.6}$$

In case of $Z_n \to \infty$ a.s., as $n \to \infty$, $N(t)$ is finite a.s. for every $t \geqslant 0$, since only a finite number of summands in (1.6) are nonzero. Along with $\{N(t), t \geqslant 0\}$ we introduce two other *general renewal processes*, that is

$$\begin{aligned} M(t) &= \sup\{n \geqslant 0\colon \max(Z_0, Z_1, \ldots, Z_n) \leqslant t\} \\ &= \sum_{n=1}^{\infty} I\{\max(Z_0, Z_1, \ldots, Z_n) \leqslant t\}, \quad t \geqslant 0, \end{aligned} \tag{1.7}$$

$\sup\emptyset = 0$, i.e. $M(t)+1$ is the *first-passage time* of the sequence $\{Z_n, n \geqslant 0\}$ from the set $(-\infty, t]$, and

$$L(t) = \sup\{n \geqslant 0\colon Z_n \leqslant t\} = \sum_{n=1}^{\infty} I\{\inf(Z_n, Z_{n+1}, \ldots) \leqslant t\}, \quad t \geqslant 0, \tag{1.8}$$

i.e. $L(t)+1$ is the *last-exit time* of $\{Z_n, n \geqslant 0\}$ from $(-\infty, t]$.

Formally speaking, both $\{M(t)\}$ and $\{L(t)\}$ are particular cases of $\{N(t)\}$ with Z_n replaced by $\max(Z_0, Z_1, \ldots, Z_n)$ and $\inf(Z_n, Z_{n+1}, \ldots)$, respectively, but they will play a special role in the proofs below because of the dualities

$$\{M(t) \geqslant n\} \Leftrightarrow \{\max(Z_0, Z_1, \ldots, Z_n) \leqslant t\}, \tag{1.9}$$

$$\{M(t) = n\} \Leftrightarrow \{\max(Z_0, Z_1, \ldots, Z_n) \leqslant t, Z_{n+1} > t\}; \tag{1.10}$$

and

$$\{L(t) \geqslant n\} \Leftrightarrow \{\inf(Z_n, Z_{n+1}, \ldots) \leqslant t\}, \tag{1.11}$$

$$\{L(t) = n\} \Leftrightarrow \{Z_n \leqslant t, \inf(Z_{n+1}, Z_{n+2}, \ldots) > t\}. \tag{1.12}$$

Note that we have no such nice properties for $N(t)$, however it is obvious that, for any $t \geqslant 0$,

$$I\{\max(Z_0, Z_1, \ldots, Z_n) \leqslant t\} \leqslant I\{Z_n \leqslant t\} \leqslant I\{\inf(Z_n, Z_{n+1}, \ldots) \leqslant t\},$$

so that

$$M(t) \leqslant N(t) \leqslant L(t). \tag{1.13}$$

Moreover, the following inequalities hold true for finite $M(t)$ and $L(t)$, respectively:

$$Z_{M(t)} \leqslant t < Z_{M(t)+1}, \tag{1.14}$$

$$Z_{L(t)} \leqslant t < Z_{L(t)+1}. \tag{1.15}$$

Since $Z_n < +\infty$ a.s. for all n, it is obvious that

$$M(t) = \min\{L(t), M(t), N(t)\} \to +\infty \text{ a.s.}$$

Remark 1.1. If $0 = Z_0 \leqslant Z_1 \leqslant Z_2 \leqslant \ldots$, then obviously

$$M(t) = N(t) = L(t).$$

But if (say) $Z_n > Z_{n+1}$ for some n, then for $Z_{n+1} \leqslant t < Z_n$

$$I\{\max(Z_0, Z_1, \ldots, Z_{n+1}) \leqslant t\} = 0, \quad I\{Z_{n+1} \leqslant t\} = 1,$$

so $M(t) < N(t)$, and also

$$I\{Z_n \leqslant t\} = 0, \quad I\{\inf(Z_n, Z_{n+1}, \ldots) \leqslant t\} = 1,$$

so $N(t) < L(t)$.

The paper is organized as follows: In Section 2, strong laws of large numbers are presented for general renewal processes including rates of convergence statements such as Marcinkiewicz–Zygmund-type results. Usefulness of this general approach is demonstrated via a series of examples in Section 3, including renewal sequences of independent, but nonidentically distributed summands, martingales and mixing sequences, weighted sums and nonlinear renewal processes, and others.

2. STRONG LIMIT THEOREMS

2.1. Strong laws of large numbers. Assuming a strong law of large numbers for $\{Z_n, n \geqslant 0\}$, for the general renewal processes we obtain immediately corresponding results from inequalities (1.14) and (1.15) if the normalizing sequence satisfies certain regularity conditions.

THEOREM 2.1. *Assume*

$$Z_n/a_n \to 1 \text{ a.s.,} \tag{2.1}$$

where $\{a_n, n \geqslant 1\}$ *is a nonrandom sequence such that* $a_n \to \infty$ *as* $n \to \infty$ *and*

$$a_{n+1}/a_n \to 1. \tag{2.2}$$

Then, as $t \to \infty$,

$$a_{M(t)}/t \to 1 \text{ a.s.,} \tag{2.3}$$

$$a_{L(t)}/t \to 1 \text{ a.s.} \tag{2.4}$$

Moreover, if $\{a_n, n \geqslant 1\}$ *is nondecreasing, then also*

$$a_{N(t)}/t \to 1 \text{ a.s.} \tag{2.5}$$

Proof. In view of (1.14) and $M(t) \to \infty$ a.s., $t \to \infty$,

$$\frac{Z_{M(t)}}{a_{M(t)}} \leqslant \frac{t}{a_{M(t)}} < \frac{Z_{M(t)+1}}{a_{M(t)+1}} \frac{a_{M(t)+1}}{a_{M(t)}},$$

where both the left-hand side and the right-hand side of these inequalities tend to 1 a.s. because of (2.1) and (2.2). This proves (2.3).

Similarly, (2.4) follows from (1.15), (2.1), and (2.2).

If $\{a_n, n \geqslant 1\}$ is nondecreasing, (1.13) implies $a_{M(t)} \leqslant a_{N(t)} \leqslant a_{L(t)}$, so that (2.5) is immediate from (2.3) and (2.4). ∎

Now, if $\{a_n, n \geqslant 0\}$ is strictly increasing, let $\{a(t): t \geqslant 0\}$ be its extension, i.e. $a(n) = a_n$ for all $n = 0, 1, 2, \ldots$, such that

(2.6) $a(\cdot)$ is continuous and strictly increasing with $a(t) \to \infty$, $t \to \infty$.

Define

$$a^{-1}(u) = \inf\{t: a(t) = u\}, \quad u > u_0 = a_0.$$

Obviously, $a^{-1}(\cdot)$ is also continuous and strictly increasing, with $a^{-1}(u) \to \infty$ as $u \to \infty$.

Assume

$$\lim_{\varepsilon \downarrow 0} \limsup_{t \to \infty} \left| \frac{a^{-1}\big((1 \pm \varepsilon) t\big)}{a^{-1}(t)} - 1 \right| = 0. \tag{2.7}$$

Then the following strong laws hold true:

Corollary 2.1. *Assume* (2.1) *and* (2.2), (2.6), (2.7). *Then, as* $t \to \infty$,

$$M(t)/a^{-1}(t) \to 1 \ \text{a.s.}, \tag{2.8}$$

$$L(t)/a^{-1}(t) \to 1 \ \text{a.s.}, \tag{2.9}$$

$$N(t)/a^{-1}(t) \to 1 \ \text{a.s.} \tag{2.10}$$

Proof. From (2.3) we obtain

$$a\big(M(t)\big)/t = a_{M(t)}/t \to 1 \ \text{a.s.}, \quad t \to \infty.$$

So, for any $0 < \varepsilon < 1$,

$$t(1-\varepsilon) < a\big(M(t)\big) < t(1+\varepsilon) \quad \text{for } t \geqslant t_0 = t_0(\varepsilon, \omega).$$

By the monotonicity of $a^{-1}(t)$, we have

$$a^{-1}\big(t(1-\varepsilon)\big) < M(t) < a^{-1}\big(t(1+\varepsilon)\big)$$

if $t \geqslant t_0$. In view of (2.7), this gives (2.8).

The proof of (2.9) and (2.10) follows by the same arguments. ∎

Remark 2.1. There are situations in which the strong laws of Corollary 2.1 cannot be derived from their corresponding counterparts of Theorem 2.1, since e.g. condition (2.2) is violated. Consider, for example, partial sums $S_n = X_1 + \ldots + X_n$ of an i.i.d. sequence $\{X_n, n \geqslant 1\}$ of positive random variables with $EX_1 = 1$. Choose $Z_n = S_{2^n}$, $n \geqslant 1$. Then, with $a_n = 2^n$, of course

$$Z_n/a_n \to 1 \text{ a.s.}, \quad n \to \infty,$$

but neither $a_{n+1}/a_n \to 1$ nor

$$\frac{Z_{n+1} - Z_n}{a_n} = \frac{X_{2^n+1} + \ldots + X_{2^{n+1}}}{2^n} \to 0 \text{ a.s.},$$

so that the arguments used in the proof of Theorem 2.1 do not apply here.

So, the growth condition (2.2) on the normalizing sequence $\{a_n, n \geqslant 1\}$ is crucial for deriving the strong laws of Corollary 2.1 from their counterparts in Theorem 2.1. However, this condition can be avoided, and thus the regularity assumptions can be weakened by applying a totally different technique of proof. Such a method was introduced in Klesov and Steinebach [9] for the case of renewal processes constructed from random walks with multidimensional time.

THEOREM 2.2. *Assume* (2.1), (2.6), *and* (2.7). *Then, as* $t \to \infty$, *assertions* (2.8)–(2.10) *retain.*

Proof. Let us first consider

$$N(t) = \sum_{k=1}^{\infty} I\{Z_k \leqslant t\}.$$

Now, for any $0 < \varepsilon < 1$, with $n = [a^{-1}(t)]$ and $m^{\pm} = [a^{-1}(t(1 \pm \varepsilon))]$,

$$\begin{aligned} N(t) - n &= -\sum_{k=1}^{n} I\{Z_k > t\} + \sum_{k=n+1}^{\infty} I\{Z_k \leqslant t\} \\ &= -\sum_{k=1}^{m^-} I\{Z_k > t\} - \sum_{k=m^-+1}^{n} I\{Z_k > t\} \\ &\quad + \sum_{k=n+1}^{m^+} I\{Z_k \leqslant t\} + \sum_{k=m^++1}^{\infty} I\{Z_k \leqslant t\} \\ &= -N_1(t) - N_2(t) + N_3(t) + N_4(t). \end{aligned} \tag{2.11}$$

On observing that, for $k \leqslant m^- \leqslant a^{-1}((1-\varepsilon)t)$,

$$\frac{t}{a_k} \geqslant \frac{t}{a(a^{-1}((1-\varepsilon)t))} = \frac{1}{1-\varepsilon} > 1,$$

but, in view of (2.1), $Z_k/a_k > 1/(1-\varepsilon)$ for almost finitely many k (a.s.), it is obvious that

(2.12) $$N_1(t)/a^{-1}(t) \to 0 \text{ a.s.}, \quad t \to \infty.$$

A similar argument shows that also

(2.13) $$N_4(t)/a^{-1}(t) \to 0 \text{ a.s.}, \quad t \to \infty.$$

Next, in view of (2.7) we have

(2.14) $$0 \leqslant \frac{N_2(t)}{a^{-1}(t)} \leqslant \frac{a^{-1}(t) - a^{-1}((1-\varepsilon)t) + 1}{a^{-1}(t)} \to 0 \quad \text{as } t \to \infty \text{ and } \varepsilon \downarrow 0,$$

and similarly

(2.15) $$0 \leqslant \frac{N_3(t)}{a^{-1}(t)} \leqslant \frac{a^{-1}((1+\varepsilon)t) - a^{-1}(t) + 1}{a^{-1}(t)} \to 0 \quad \text{as } t \to \infty \text{ and } \varepsilon \downarrow 0.$$

A combination of (2.11)–(2.15) proves (2.10), since

$$\frac{N(t) - [a^{-1}(t)]}{a^{-1}(t)} = \frac{N(t)}{a^{-1}(t)} - 1 + o(1) \quad \text{as } t \to \infty.$$

For the proof of (2.8) and (2.9) we note that, from $Z_n/a_n \to 1$ and $a_n \uparrow \infty$, $n \to \infty$, it also follows that

$$\frac{\max(Z_0, Z_1, \ldots, Z_n)}{a_n} \to 1 \quad \text{and} \quad \frac{\inf(Z_n, Z_{n+1}, \ldots)}{a_n} \to 1 \quad \text{as } n \to \infty.$$

This renders the same arguments possible for $M(t)$ and $L(t)$ as applied to $N(t)$. ∎

Remark 2.2. Unfortunately, there are also situations in which the inversion techniques applied in Theorems 2.1, 2.2 and Corollary 2.1 cannot work at all. Consider, for instance, a max-scheme of i.i.d. random variables $\{X_n, n \geqslant 1\}$ with distribution function $F(t) = \boldsymbol{P}(X_1 \leqslant t)$, $t \in \boldsymbol{R}$. For $Z_n = \max(X_1, \ldots, X_n)$, $n \geqslant 1$, $Z_0 = 0$, the corresponding renewal processes $\{M(t), t \geqslant 0\}$, $\{N(t), t \geqslant 0\}$, and $\{L(t), t \geqslant 0\}$ coincide. Moreover, for any $t \geqslant 0$, $N(t)$ has a geometric distribution, i.e.

$$\boldsymbol{P}(N(t) = n) = \boldsymbol{P}(\max(X_1, \ldots, X_n) \leqslant t, X_{n+1} > t) = F^n(t)(1 - F(t)),$$
$$\boldsymbol{P}(N(t) \geqslant n) = F^n(t), \quad n = 0, 1, \ldots$$

Therefore, if $F(t) < 1$ for all $t \geqslant 0$, then for all fixed $x \geqslant 0$

$$\boldsymbol{P}(N(t) > x/(1-F(t))) = \boldsymbol{P}(N(t) \geqslant [x/(1-F(t))] + 1)$$
$$= \exp\{x(\log F(t))/(1-F(t)) + O(1)\log F(t)\}.$$

Since $\log(1-x)/x \to -1$ as $x \to 0$, the right-hand side tends to $\exp\{-x\}$ as $t \to \infty$. Hence

$$N(t)(1-F(t)) \xrightarrow{D} E \quad \text{as } t \to \infty,$$

where E has an exponential Exp(1)-distribution. In view of this fact it is impossible that

$$N(t)/b(t) \to 1 \text{ a.s.}, \quad t \to \infty,$$

for any (nonrandom) normalizing family $\{b(t),\, t > 0\}$. Because otherwise, for each $\varepsilon > 0$,

$$P(N(t) > (1+\varepsilon)\, b(t)) \to 0 \quad \text{as } t \to \infty$$

requires $b(t)(1-F(t)) \to \infty$ as $t \to \infty$, by the consideration above. This, however, in turn implies

$$P(N(t) \leqslant (1-\varepsilon)\, b(t)) \to 1 \quad \text{as } t \to \infty,$$

so that not even a *weak* law of large numbers applies to $\{N(t),\, t \geqslant 0\}$.

Nevertheless, the underlying "renewal sequence" $\{Z_n,\, n \geqslant 1\}$ may satisfy a *strong* law of large numbers. For example, in the case of an Exp(1)-distribution, i.e. $F(t) = 1-e^{-t}$ for $t \geqslant 0$, and $F(t) = 0$, otherwise, it is well known (cf. Galambos [4]) that

$$\frac{Z_n}{\log n} = \frac{\max(X_1, \ldots, X_n)}{\log n} \to 1 \text{ a.s.}, \quad n \to \infty.$$

Note that, in the latter case, all assumptions of Theorems 2.1, 2.2 and Corollary 2.1 are fulfilled with $a_n = \log n$, $a(t) = \log t$, $a^{-1}(t) = e^t$, with the exception of (2.7). So, the latter condition cannot be dropped in general.

Another example would be $F(t) = \Phi(t)$, $t \in \boldsymbol{R}$, a standard normal distribution function, in which case

$$\frac{Z_n}{\sqrt{2\log n}} = \frac{\max(X_1, \ldots, X_n)}{\sqrt{2\log n}} \to 1 \text{ a.s.}, \quad n \to \infty$$

(cf. Galambos [4]). Here $a^{-1}(t) = \exp\{t^2/2\}$ also violates assumption (2.7).

Consequently, there are (renewal) sequences $\{Z_n,\, n \geqslant 0\}$ satisfying an SLLN for which their corresponding renewal processes $\{N(t),\, t \geqslant 0\}$, $\{M(t),\, t \geqslant 0\}$, and $\{L(t),\, t \geqslant 0\}$ do not have any (nondegenerate) strong limiting behaviour.

Remark 2.3. Just for the sake of completeness we should like to mention that there are also cases in which the renewal process satisfies an SLLN, but not so its sequence of renewal times. Consider, for example, a nonhomogeneous Poisson process $\{N(t),\, t \geqslant 0\}$ with cumulative intensity function $\{\lambda(t),\, t \geqslant 0\}$, i.e. $\lambda(t) = \boldsymbol{E}N(t)$, $t \geqslant 0$. If e.g. $\lambda(t)$ is continuous and strictly increasing to infinity, it is well known that

$$\{N(t),\ t \geqslant 0\} \overset{\mathscr{D}}{=} \{\tilde{N}(\lambda(t)),\ t \geqslant 0\},$$

where $\{\tilde{N}(t),\ t \geqslant 0\}$ is a homogeneous Poisson process with renewal times $\tilde{S}_0 = 0$, $\tilde{S}_n = X_1 + \ldots + X_n$, $n \geqslant 1$, based on an i.i.d. sequence $\{X_n,\ n \geqslant 1\}$ of Exp(1)-random variables.

Choose

$$\lambda(t) = \begin{cases} \log t, & t \geqslant e, \\ t/e, & 0 \leqslant t \leqslant e. \end{cases}$$

Then, from the SLLN for $\{\tilde{N}(t),\ t \geqslant 0\}$, as $t \to \infty$,

$$N(t)/\log t = \tilde{N}(\log t)/\log t \to 1 \ \text{a.s.}$$

But, since $Z_n = \exp(\tilde{S}_n)$ are the renewal times of $\{N(t),\ t \geqslant 0\}$, in view of the LIL for the partial sums $\{\tilde{S}_n,\ n \geqslant 1\}$ it follows that

$$Z_n/e^n = \exp\{\tilde{S}_n - n\}$$

oscillates between 0 and $+\infty$ a.s., $n \to \infty$.

2.2. Convergence rate results. It may also be interesting to collect general conditions under which convergence rate statements apply to the laws of large numbers in Theorems 2.1, 2.2 and Corollary 2.1.

THEOREM 2.3. *Assume, as* $n \to \infty$,

$$(Z_n - a_n)/b_n \to 0 \ \textit{a.s.}, \tag{2.16}$$

where

$$a_n \to \infty, \quad \textit{but}\ \ a_n - a_{n-1} = o(b_n), \tag{2.17}$$

$$0 < b_n \to \infty, \quad \textit{but}\ \ b_n = o(a_n), \tag{2.18}$$

$$b_{n+1}/b_n = O(1). \tag{2.19}$$

Then, as $t \to \infty$,

$$(a_{M(t)} - t)/b_{M(t)} \to 0 \ \textit{a.s.}, \tag{2.20}$$

$$(a_{L(t)} - t)/b_{L(t)} \to 0 \ \textit{a.s.} \tag{2.21}$$

Moreover, if $\{a_n\}$ *is nondecreasing, then*

$$(a_{N(t)} - t)/\max(b_{M(t)},\ b_{L(t)}) \to 0 \ \textit{a.s.} \tag{2.22}$$

Proof. First note that, with $X_n = Z_n - Z_{n-1}$, $n \geqslant 1$, assumptions (2.16), (2.17), and (2.19) yield

$$\frac{X_n}{b_n} = \frac{Z_n - a_n}{b_n} - \frac{Z_{n-1} - a_{n-1}}{b_{n-1}}\frac{b_{n-1}}{b_n} + \frac{a_n - a_{n-1}}{b_n} \to 0, \qquad n \to \infty. \tag{2.23}$$

Now, by inequality (1.14)

$$\frac{a_{M(t)}-Z_{M(t)+1}}{b_{M(t)}} < \frac{a_{M(t)}-t}{b_{M(t)}} \leqslant \frac{a_{M(t)}-Z_{M(t)}}{b_{M(t)}}. \tag{2.24}$$

Because of $M(t) \to \infty$ a.s., $t \to \infty$, and (2.16), the right-hand side of (2.24) tends to 0 a.s. On the other hand, in view of (2.23) and (2.19),

$$\frac{a_{M(t)}-Z_{M(t)}}{b_{M(t)}} - \frac{a_{M(t)}-Z_{M(t)+1}}{b_{M(t)}} = \frac{X_{M(t)+1}}{b_{M(t)+1}} \frac{b_{M(t)+1}}{b_{M(t)}} \to 0 \text{ a.s.},$$

so that also the left-hand side of (2.24) tends to 0 a.s. This completes the proof of (2.20).

For the proof of (2.21) we use the same arguments.

Finally, (2.22) follows from (2.20) and (2.21), since

$$|a_{N(t)}-t| \leqslant \max(|a_{M(t)}-t|, |a_{L(t)}-t|)$$

because of $\{a_n, n \geqslant 1\}$ being monotone. ■

Now assume that $a(\cdot)$ has a continuous derivative $a'(\cdot)$ on $(t_0, +\infty)$ satisfying

$$a'(t) \asymp a'(s) \quad \text{if } t \asymp s, \tag{2.25}$$

i.e. $|a'(t)/a'(s)|$ is bounded away from 0 and ∞, if $|t/s|$ is bounded away from 0 and ∞, as $t, s \to \infty$. Moreover, let $\{b(t), t \geqslant 0\}$ be an extension of $\{b_n\}$ such that

$$b(t) \asymp b(s) \quad \text{if } t \asymp s. \tag{2.26}$$

COROLLARY 2.2. *Assume* (2.16) *together with* (2.6), (2.7), (2.17), (2.18), (2.25), *and* (2.26). *Then, as* $t \to \infty$,

$$\frac{a'(a^{-1}(t))}{b(a^{-1}(t))}(M(t)-a^{-1}(t)) \to 0 \ \textit{a.s.}, \tag{2.27}$$

$$\frac{a'(a^{-1}(t))}{b(a^{-1}(t))}(N(t)-a^{-1}(t)) \to 0 \ \textit{a.s.}, \tag{2.28}$$

$$\frac{a'(a^{-1}(t))}{b(a^{-1}(t))}(L(t)-a^{-1}(t)) \to 0 \ \textit{a.s.} \tag{2.29}$$

Proof. We only prove (2.27). The proofs of (2.28) and (2.29) are similar.

Observe that, in view of (2.16) and (2.18), as $n \to \infty$,

$$\frac{Z_n}{a_n} - 1 = \frac{Z_n - a_n}{b_n} \frac{b_n}{a_n} = o(1) \text{ a.s.} \tag{2.30}$$

Since conditions (2.17) and (2.18) also yield (2.2), assertion (2.30) implies (2.3) and, moreover, (2.8) under the given assumptions, i.e. $M(t) \sim a^{-1}(t)$ a.s., $t \to \infty$.

Now, by the mean value theorem,

$$a_{M(t)}-t = a(M(t))-a(a^{-1}(t)) = a'(\xi(t))(M(t)-a^{-1}(t)), \tag{2.31}$$

where $\xi(t) \sim a^{-1}(t)$ a.s., $t \to \infty$. On the other hand, by (2.20) and (2.26),

$$\frac{a_{M(t)}-t}{b(a^{-1}(t))} = \frac{a_{M(t)}-t}{b_{M(t)}} \frac{b(M(t))}{b(a^{-1}(t))} = o(1) \text{ a.s.}, \quad t \to \infty, \tag{2.32}$$

so that a combination of (2.25), (2.31), and (2.32) completes the proof of (2.27). ■

Corollary 2.3. *Assume that, for some $a > 0$ and $r > 1$, as $n \to \infty$,*

$$\frac{Z_n-na}{n^{1/r}} \to 0 \ \textit{a.s.} \tag{2.33}$$

Then, as $t \to \infty$,

$$\frac{M(t)-t/a}{t^{1/r}} \to 0 \ \textit{a.s.}, \tag{2.34}$$

$$\frac{N(t)-t/a}{t^{1/r}} \to 0 \ \textit{a.s.}, \tag{2.35}$$

$$\frac{L(t)-t/a}{t^{1/r}} \to 0 \ \textit{a.s.} \tag{2.36}$$

Proof. The functions $a(t) = ta$ and $b(t) = t^{1/r}$ satisfy the assumptions of Corollary 2.2 with $a^{-1}(t) = t/a$ and $a'(t) \equiv a$. ■

Remark 2.4. By the same technique as used to prove Corollary 2.2, one can also derive general LIL type results for renewal processes. Details will be omitted.

As in Theorem 2.2 the regularity assumptions of Corollary 2.2 can be considerably weakened if a different technique of proof is applied.

Theorem 2.4. *Assume* (2.16) *together with*

$$b(t) \uparrow \infty \quad \textit{as } t \to \infty, \tag{2.37}$$

$$a(t)/b(t) \uparrow 0 \quad \textit{as } t \to \infty, \tag{2.38}$$

$$a'(a^{-1}(t)) \asymp a'(a^{-1}(s)) \quad \textit{if } t \asymp s, \tag{2.39}$$

$$b(a^{-1}(t)) \asymp b(a^{-1}(s)) \quad \textit{if } t \asymp s, \tag{2.40}$$

where $a(t)$ is continuously differentiable on (t_0, ∞) with

$$a'(t) = o(b(t)) \quad \textit{as } t \to \infty. \tag{2.41}$$

Then, as $t \to \infty$, assertions (2.27)–(2.29) *retain.*

Proof. We prove only (2.28). The arguments for (2.27) and (2.29) are similar.

First, for any $\varepsilon > 0$, set $n = [a^{-1}(t)]$ and $m^{\pm} = \left[a^{-1}\left(t \pm \varepsilon(t)\right)\right]$, where $\varepsilon(t) = \varepsilon b\left(a^{-1}(t)\right)$. Note that

$$\varepsilon(t)/t \asymp \varepsilon b\left(a^{-1}(t)\right)/a\left(a^{-1}(t)\right) \asymp \varepsilon \quad \text{as } t \to \infty.$$

Then, as in (2.11),

$$N(t) - n = -N_1(t) - N_2(t) + N_3(t) + N_4(t),$$

and we have to show that the four summands are of order $o\left(b\left(a^{-1}(t)\right)/a'\left(a^{-1}(t)\right)\right)$ as $t \to \infty$.

For example, if $k \leqslant m^- \leqslant a^{-1}\left(t - \varepsilon(t)\right)$, then

$$\frac{t - a_k}{b_k} \geqslant \frac{t - a\left(a^{-1}\left(t - \varepsilon(t)\right)\right)}{b\left(a^{-1}(t)\right)} = \frac{\varepsilon(t)}{b\left(a^{-1}(t)\right)} = \varepsilon,$$

so that, in view of (2.41), we have

$$\frac{a'\left(a^{-1}(t)\right)}{b\left(a^{-1}(t)\right)} N_1(t) \to 0 \text{ a.s.}, \quad t \to \infty,$$

since only finitely many summands in $N_1(t)$ can be nonzero.

Similarly, for $k \geqslant m^+ + 1 > a^{-1}\left(t + \varepsilon(t)\right)$ we get

$$\frac{t - a_k}{b_k} \leqslant \frac{t - a\left(a^{-1}\left(t + \varepsilon(t)\right)\right)}{b\left(a^{-1}\left(t + \varepsilon(t)\right)\right)} = -\frac{\varepsilon(t)}{b\left(a^{-1}\left(t + \varepsilon(t)\right)\right)} \asymp -\varepsilon, \quad t \to \infty,$$

which, by the same reasoning, implies

$$\frac{a'\left(a^{-1}(t)\right)}{b\left(a^{-1}(t)\right)} N_4(t) \to 0 \text{ a.s.}, \quad t \to \infty.$$

It remains to prove that

$$\frac{a'\left(a^{-1}(t)\right)}{b\left(a^{-1}(t)\right)} \left(N_2(t) + N_3(t)\right) \to 0 \text{ a.s.}, \quad t \to \infty,$$

which, analogously to (2.14)–(2.15), follows from the expansion

$$a^{-1}\left(t \pm \varepsilon(t)\right) - a^{-1}(t) = \pm \frac{\varepsilon(t)}{a'\left(a^{-1}(\tau^{\pm})\right)} \asymp \pm \varepsilon \frac{b\left(a^{-1}(t)\right)}{a'\left(a^{-1}(t)\right)}.$$

Since, in view of (2.40),

$$\frac{a'\left(a^{-1}(t)\right)}{b\left(a^{-1}(t)\right)} \left(N(t) - [a^{-1}(t)]\right) = \frac{a'\left(a^{-1}(t)\right)}{b\left(a^{-1}(t)\right)} \left(N(t) - a^{-1}(t)\right) + o(1),$$

the proof of (2.28) is complete. ■

EXAMPLE 2.1. There are still situations in which the assumptions of Theorems 2.2 and 2.4 do not hold true, but yet a strong law of large numbers may be available. Consider, for instance, a sequence $\{Z_n, n \geqslant 1\}$ satisfying

$$Z_n/\log n \to 1 \text{ a.s.} \quad n \to \infty,$$

and assume a rate of convergence therein, e.g.

$$\limsup_{n\to\infty} \left|\frac{Z_n - \log n}{b(n)}\right| \leqslant B,$$

with some nonrandom constant $B > 0$. Let the function $\{b(t), t > 0\}$ be such that, for any $A > B$,

$$a_{\pm}(t) = \log t \pm Ab(t)$$

have inverse functions (say) $a_{\pm}^{-1}(t)$ satisfying

$$a_{\pm}^{-1}(t) = e^t \mp o(e^t) \quad \text{as } t \to \infty.$$

Then the SLLN for $\{N(t), t \geqslant 0\}$ retains, i.e.

$$\lim_{t\to\infty} \frac{N(t)}{e^t} = 1 \text{ a.s.}$$

The proof is similar to that of Theorem 2.4 by dividing $N(t) = \sum_{n=1}^{\infty} I\{Z_n \leqslant t\}$ into four subseries according to the conditions $n \leqslant a_{+}^{-1}(t)$, $a_{+}^{-1}(t) < n \leqslant e^t$, $e^t < n \leqslant a_{-}^{-1}(t)$, $a_{+}^{-1}(t) < n$. Details are omitted.

Note that the function $a(t) = \log t$ with $a^{-1}(t) = e^t$ violates conditions (2.7) and (2.39), so that neither Theorem 2.2 nor Theorem 2.4 is applicable in this situation. Nevertheless, an SLLN for the renewal process holds true.

3. EXAMPLES

In this section, we demonstrate the applicability of our general results from previous sections by a series of examples. Various situations of renewal processes are discussed related to i.i.d. schemes, to independent, but nonidentically distributed renewal times, and also to certain dependent sequences.

3.1. Renewal processes related to i.i.d. sequences. For the first four examples assume that $\{X_n, n \geqslant 1\}$ are i.i.d. random variables, and let $S_0 = 0$, $S_n = X_1 + \ldots + X_n$, $n \geqslant 1$.

EXAMPLE 3.1 (Linear renewal process). Let $EX_1 = a > 0$, and $a(t) = ta$ with $a^{-1}(t) = t/a$. Then by Kolmogorov's SLLN, for $Z_n = S_n$, as $n \to \infty$,

$$Z_n/na \to 1 \text{ a.s.}$$

Consequently, with $L(t)$, $M(t)$, $N(t)$ as in (1.6)–(1.8), Corollary 2.1 implies, as $t \to \infty$,

$$\frac{L(t)}{t/a} \to 1, \quad \frac{M(t)}{t/a} \to 1, \quad \frac{N(t)}{t/a} \to 1 \quad \text{a.s.} \tag{3.1}$$

For a convergence rate statement we refer to an extension of a classical result of Feller [2] obtained by Martikainen and Petrov [10]: If $b(t)$ is a positive, increasing, unbounded function such that

$$\sum_{k=n}^{\infty} \frac{1}{b^2(k)} = O\left(\frac{n}{b^2(n)}\right), \tag{3.2}$$

then the following assertions are equivalent:

$$\frac{S_n - na}{b(n)} \to 0 \text{ a.s.}, \quad n \to \infty, \tag{3.3}$$

$$\sum_{n=1}^{\infty} P(|X_1 - a| \geqslant b(n)) < \infty, \quad \lim_{n\to\infty} \frac{n}{b(n)} \int_{\{|x| < b(n)\}} x dF(x) = 0, \tag{3.4}$$

where $F(\cdot)$ is the distribution function of $X_1 - a$.

Combining the Martikainen–Petrov [10] result with Theorem 2.4 we have, under (3.4),

$$\lim_{t\to\infty} \frac{L(t) - t/a}{b(t)} = 0, \quad \lim_{t\to\infty} \frac{M(t) - t/a}{b(t)} = 0, \quad \lim_{t\to\infty} \frac{N(t) - t/a}{b(t)} = 0$$

almost surely.

One of the possible choices for $b(t)$ is the so-called Marcinkiewicz–Zygmund normalization $b(t) = t^{1/r}$, $1 < r < 2$: if $E|X_1|^r < \infty$ for some $1 < r < 2$, then

$$\lim_{t\to\infty} \frac{L(t) - t/a}{t^{1/r}} = 0, \quad \lim_{t\to\infty} \frac{M(t) - t/a}{t^{1/r}} = 0, \quad \lim_{t\to\infty} \frac{N(t) - t/a}{t^{1/r}} = 0$$

almost surely.

EXAMPLE 3.2 (Nonlinear renewal process; cf. Gut [5], pp. 133–138). Consider, as before, an i.i.d. sequence $\{X_n, n \geqslant 1\}$ with $EX_1 = a > 0$, but set now $Z_n = S_n/\alpha(n)$, where $\{\alpha(t), t > 0\}$ is a positive continuous function such that

$$t/\alpha(t) \uparrow \infty \quad \text{as } t \uparrow \infty.$$

For example, the first-passage time

$$M(t) + 1 = \inf\{n\colon Z_n > t\} = \inf\{n\colon S_n > t\alpha(n)\},$$

$\inf \varnothing = +\infty$, is of some statistical importance in sequential analysis and plays a key role in what is called nonlinear renewal theory (cf. e.g. Woodroofe

[15] and Siegmund [14]). Now, by Theorem 2.2, if $a(t) = ta/\alpha(t)$ with inverse function $a^{-1}(t)$ satisfying (2.7), then, as $t \to \infty$,

$$\lim_{t\to\infty} \frac{M(t)}{a^{-1}(t)} = 1, \quad \lim_{t\to\infty} \frac{N(t)}{a^{-1}(t)} = 1, \quad \lim_{t\to\infty} \frac{L(t)}{a^{-1}(t)} = 1 \quad \text{a.s.},$$

where $L(t)$, $M(t)$, $N(t)$ are as in (1.6)–(1.8).

If, additionally, $E|X_1|^r < \infty$ for some $1 < r < 2$, then with $b(n) = n^{1/r}/\alpha(n)$, as $n \to \infty$,

$$\frac{Z_n - a(n)}{b(n)} = \frac{S_n - na}{n^{1/r}} \to 0 \text{ a.s.},$$

so that by Theorem 2.4 we have, as $t \to \infty$,

$$\lim_{t\to\infty} \frac{M(t) - a^{-1}(t)}{(a^{-1}(t))^{1/r}} = \lim_{t\to\infty} \frac{N(t) - a^{-1}(t)}{(a^{-1}(t))^{1/r}} = \lim_{t\to\infty} \frac{L(t) - a^{-1}(t)}{(a^{-1}(t))^{1/r}} = 0 \text{ a.s.}$$

(cf. Gut [5], Theorem 5.5 in Chapter IV).

EXAMPLE 3.3 (Renewal process based on subsequences). Strong laws of large numbers and other convergence properties have also been extensively studied for subsequences of partial sums of an i.i.d. sequence. Corresponding properties of their renewal processes, which may be viewed as being related to certain nonlinear inspection schemes, can also be derived from the general results of Section 2. Consider e.g. a subsequence $\{a_n, n \geq 1\}$ of integers with an extension $\{a(t), t \geq 0\}$ and inverse $\{a^{-1}(t), t \geq 0\}$. Let $Z_n = S_{a_n}$, $n \geq 1$, and let the renewal processes $N(t)$, $M(t)$, and $L(t)$ be defined as in (1.6)–(1.8).

Assume a_n/n is increasing, so that $a(t)/t$ is also increasing. This assumption is sufficient for (2.7). Indeed, for any $\varepsilon > 0$,

$$\frac{a\big((1+\varepsilon)a^{-1}(t)\big)}{(1+\varepsilon)a^{-1}(t)} \geq \frac{a\big(a^{-1}(t)\big)}{a^{-1}(t)} = \frac{t}{a^{-1}(t)},$$

whence $a\big((1+\varepsilon)a^{-1}(t)\big) \geq (1+\varepsilon)t$ or $(1+\varepsilon)a^{-1}(t) \geq a^{-1}\big((1+\varepsilon)t\big)$. A similar approach allows one to get corresponding estimates with $1-\varepsilon$ instead of $1+\varepsilon$. This, in turn, implies (2.7), so that also (2.8)–(2.10) are satisfied.

In order to get a rate of convergence result, one has to assume conditions (2.38)–(2.41) in Theorem 2.4. Let us consider the case of Marcinkiewicz–Zygmund normalizations. Assuming $EX_1 > 0$ and $E|X_1|^r < \infty$ for some $1 < r < 2$, we obtain, as $n \to \infty$,

$$\frac{S_{a_n} - a_n EX_1}{a_n^{1/r}} \to 0 \text{ a.s.}$$

Choose $b(n) = a_n^{1/r}$. One would now expect a corresponding result for renewal processes, but this is not always true. For the sake of simplicity, consider

$a_n = [n^\nu]$ with $\nu > 1$. Then $a(t) \asymp t^\nu$, $a'(t) \asymp t^{\nu-1}$, $a^{-1}(t) \asymp t^{1/\nu}$, and $b(t) \asymp t^{\nu/r}$. Conditions (2.38)–(2.40) are obvious in this case, but condition (2.41) requires a restriction on r, i.e. $r < (\nu-1)/\nu$.

EXAMPLE 3.4 (Renewal processes when extremes are excluded). It is well known that, under certain assumptions, partial sums of i.i.d. random variables with infinite expectation can still satisfy a strong law of large numbers if the extremal terms are removed from the sums. Consider, for example, S_n as above, and put

$$Z_n = S_n - \max_{1 \leqslant k \leqslant n} |X_k|,$$

i.e. Z_n is the n-th partial sum with the maximal term being excluded. Mori [11] proved that

$$\lim_{n\to\infty} \frac{Z_n - nc_n}{n} = 0$$

almost surely for some nonrandom sequence $\{c_n, n \geqslant 1\}$ if and only if

$$\int_0^\infty x\bar{F}^2(x)\,dx < \infty, \quad \text{where } \bar{F}(x) = P(|X_1| > x). \tag{3.5}$$

Without loss of generality the constants c_n can be chosen as $c_n = EX_1 I\{|X_1| < n\tau\}$ with some positive constant τ, e.g. $\tau = 1$. Put $c(t) = EX_1 I\{|X_1| < t\}$ and $a(t) = tc(t)$. Then, under the conditions of Theorem 2.2, as $t \to \infty$,

$$\frac{L(t)}{a^{-1}(t)} \to 1, \quad \frac{M(t)}{a^{-1}(t)} \to 1, \quad \frac{N(t)}{a^{-1}(t)} \to 1 \quad \text{a.s.}$$

Moreover, under the assumptions of Theorem 2.4, with $b(t) = t$ we have, as $t \to \infty$,

$$\lim_{t\to\infty} \frac{a'(a^{-1}(t))}{a^{-1}(t)}(L(t) - a^{-1}(t)) = 0, \quad \lim_{t\to\infty} \frac{a'(a^{-1}(t))}{a^{-1}(t)}(M(t) - a^{-1}(t)) = 0,$$

$$\lim_{t\to\infty} \frac{a'(a^{-1}(t))}{a^{-1}(t)}(N(t) - a^{-1}(t)) = 0$$

almost surely, provided (3.5) holds.

For example, choose $F(x) = P(X_1 \leqslant x)$ as follows:

$$F(x) = \begin{cases} 1 - 1/(x\log x), & x \geqslant e, \\ 0, & x < e. \end{cases}$$

The expectation does not exist for this distribution, and therefore the renewal process based on sums has no linear asymptotic. On the other hand, the renewal process based on sums with excluded maximal term has a (nonlinear) asymptotic. Indeed,

$$c(t) = \log\log t + O(1), \qquad a(t) = t\log\log t + O(t),$$
$$a'(t) = \log\log t + o(1), \qquad a^{-1}(t) = t/\log\log t + O\big(t/(\log\log t)^2\big)$$

in this case, so that, as $t \to \infty$,

$$\frac{(\log\log t)^2}{t}\left(N(t) - \frac{t}{\log\log t}\right) \to 0 \text{ a.s.}$$

Similar results can be obtained for sums with r maximal terms excluded, if condition (3.5) is replaced by

$$\int_0^\infty x^r \bar{F}^{r+1}(x)\,dx < \infty. \tag{3.6}$$

EXAMPLE 3.5 (Extended renewal processes). Horváth [7] developed a strong approximation approach for extended renewal processes based on a sequence of d-dimensional i.i.d. random variables $\{\boldsymbol{X}_n, n \geqslant 1\}$ with $E\boldsymbol{X}_1 = \boldsymbol{a}$. Put $\boldsymbol{S}_0 = \boldsymbol{0}$, $\boldsymbol{S}_n = \boldsymbol{X}_1 + \ldots + \boldsymbol{X}_n$, $n \geqslant 1$, and let h: $\boldsymbol{R}^d \to \boldsymbol{R}^1$ be homogeneous of degree 1, continuously differentiable, and assume $h(\boldsymbol{a}) > 0$. For fixed $0 \leqslant p < 1$, set $Z_n = h(\boldsymbol{S}_n)/n^p$, and e.g.

$$M(t)+1 = \inf\{n\colon Z_n > t\} = \inf\{n\colon h(\boldsymbol{S}_n) > tn^p\},$$

$\inf\varnothing = +\infty$. Under $E|X|^r < \infty$ for some $r > 2$, Horváth [7] was able to derive a strong approximation of

$$M(t) - \big(t/h(\boldsymbol{a})\big)^{1/q}, \qquad q = 1-p, \tag{3.7}$$

by a suitable Wiener process which, in turn, results in a number of strong (and weak) limit theorems for $\{M(t), t \geqslant 0\}$.

Under weaker assumptions, we are still able to retain some SLLN or convergence rate type results by our methods of Section 2. For example, if only $E|\boldsymbol{X}_1| < \infty$, then similarly to Example 3.2, as $t \to \infty$,

$$\frac{L(t)}{\big(t/h(\boldsymbol{a})\big)^{1/q}} \to 1, \qquad \frac{M(t)}{\big(t/h(\boldsymbol{a})\big)^{1/q}} \to 1, \qquad \frac{N(t)}{\big(t/h(\boldsymbol{a})\big)^{1/q}} \to 1 \qquad \text{a.s.}$$

If $E|\boldsymbol{X}_1|^r > \infty$ for some $1 < r < 2$, then, as $t \to \infty$,

$$\frac{L(t) - \big(t/h(\boldsymbol{a})\big)^{1/q}}{t^{1/rq}} \to 0, \qquad \frac{M(t) - \big(t/h(\boldsymbol{a})\big)^{1/q}}{t^{1/rq}} \to 0,$$

$$\frac{N(t) - \big(t/h(\boldsymbol{a})\big)^{1/q}}{t^{1/rq}} \to 0 \qquad \text{a.s.}$$

3.2. Non-i.i.d. or dependent renewal times.

EXAMPLE 3.6 (Weighted i.i.d. summands). Consider an i.i.d. sequence $\{X_n, n \geqslant 1\}$ with $EX_1 = 1$ and a positive function $\{w(t), t > 0\}$. Let

$Z_n = \sum_{k=1}^{n} w(k) X_k$ and define renewal processes $\{L(t), t \geqslant 0\}$, $\{M(t), t \geqslant 0\}$, and $\{N(t), t \geqslant 0\}$ according to (1.8), (1.7), and (1.6), respectively.

Then, with $a(n) = w(1)+\ldots+w(n)$, $n \geqslant 1$, it is clear that $EZ_n = a(n)$. By the Kolmogorov SLLN for nonidentically distributed random variables,

$$\lim_{n\to\infty} \frac{S_n - a(n)}{b(n)} = 0 \text{ a.s.,}$$

provided $EX_1^2 < \infty$ and the function $b(t)$ is positive, nondecreasing, unbounded, and such that

$$\sum_{n=1}^{\infty} \frac{w^2(n)}{b^2(n)} < \infty.$$

Note that the above moment restriction may be weakened by applying a truncation procedure, but for the sake of simplicity, we keep the above assumption. Moreover, we choose $w(t) = t^\theta - (t-1)^\theta \sim \theta t^{\theta-1}$ for some $\theta > 0$. Then $a(n) = n^\theta$. We set $a(t) = t^\theta$ and note that $a^{-1}(t) = t^{1/\theta}$, $a'(t) = \theta t^{\theta-1}$. To satisfy the conditions of Theorem 2.4 we assume

$$\lim_{n\to\infty} \frac{b(n)}{n^\theta} = 0, \qquad \lim_{n\to\infty} \frac{b(n)}{n^{\theta-1}} = \infty.$$

Then, if

$$\sum_{n=1}^{\infty} \frac{n^{2(\theta-1)}}{b^2(n)} < \infty,$$

we obtain

$$\lim_{t\to\infty} \left(L(t) - t^{1/\theta}\right) \frac{t^{(\theta-1)/\theta}}{b(t^{1/\theta})} = 0, \qquad \lim_{t\to\infty} \left(M(t) - t^{1/\theta}\right) \frac{t^{(\theta-1)/\theta}}{b(t^{1/\theta})} = 0,$$

$$\lim_{t\to\infty} \left(N(t) - t^{1/\theta}\right) \frac{t^{(\theta-1)/\theta}}{b(t^{1/\theta})} = 0 \qquad \text{a.s.}$$

One of the possible choices of $b(t)$ is $b(t) = t^\delta$, with $\theta - 1 < \delta < \theta$, for which

$$\lim_{t\to\infty} \frac{L(t) - t^{1/\theta}}{t^{(\delta-\theta+1)/\theta}} = 0, \qquad \lim_{t\to\infty} \frac{M(t) - t^{1/\theta}}{t^{(\delta-\theta+1)/\theta}} = 0, \qquad \lim_{t\to\infty} \frac{N(t) - t^{1/\theta}}{t^{(\delta-\theta+1)/\theta}} = 0 \qquad \text{a.s.}$$

In a recent paper, Fazekas and Klesov [1] developed a general approach to the strong law of large numbers. Their key idea was to show that Hajek–Rényi type inequalities can be obtained from appropriate maximal inequalities for cumulative sums, and that the latter, in turn, imply the SLLN. By this method, no assumptions on the dependency structure of the summands are required, and a number of examples can be covered including sums of independent, but nonidentically distributed summands, martingale difference schemes, mixing sequences, mixingales, orthogonal sequences, sequences with superadditive moment structure and many others.

All of these examples may, under appropriate conditions, be converted into strong laws for their corresponding renewal processes. Just for the sake of demonstration, we consider three final examples.

EXAMPLE 3.7 (ϱ-mixing renewal times). Let $\{X_n, n \geqslant 1\}$ be a sequence of identically distributed random variables, ϱ-mixing, with $EX_1 = a > 0$ and $E|X_1|^r < \infty$ for some $1 \leqslant r < 2$. Then, with $a(t) = ta$ and $b(t) = t^{1/r}$, as $t \to \infty$,

$$\frac{L(t)-t/a}{t^{1/r}} \to 0, \quad \frac{M(t)-t/a}{t^{1/r}} \to 0, \quad \frac{N(t)-t/a}{t^{1/r}} \to 0 \quad \text{a.s.},$$

provided $\sum \varrho(2^n) < \infty$, where ϱ denotes the Kolmogorov–Rozanov mixing coefficient of $\{X_n, n \geqslant 1\}$.

The above mixing condition can even by weakened (cf. Shao [13] and Fazekas and Klesov [1], Theorem 5.1). Extensions to mixing sequences of nonidentically distributed random variables are also available (see Fazekas and Klesov [1], Theorem 5.2). Naturally, the case of m-dependent renewal times is included (confer also Janson [8] for further asymptotics in the latter case).

EXAMPLE 3.8 (Martingale difference schemes). Let $\{X_n, n \geqslant 1\}$ be a martingale difference sequence with respect to the filtration $\{\mathfrak{F}_n, n \geqslant 1\}$, where $\mathfrak{F}_n$ is generated by $X_1, \ldots, X_n$. Assume $EX_1 = a > 0$, $q > 1/2$, and let $\{b_n, n \geqslant 1\}$ be nondecreasing, unbounded and such that

$$\sum_{n=1}^{\infty} \frac{E|Z_n|^{2q} - E|Z_{n-1}|^{2q}}{b_n^{2q}} < \infty.$$

Then, with $a(t) = ta$ and $b(t)$ satisfying the assumptions of Theorem 2.4, we have, as $t \to \infty$,

$$\frac{L(t)-t/a}{b(t/a)} \to 0, \quad \frac{M(t)-t/a}{b(t/a)} \to 0, \quad \frac{N(t)-t/a}{b(t/a)} \to 0 \quad \text{a.s.} \tag{3.8}$$

(cf. Fazekas and Klesov [1], Theorem 3.1).

As a consequence, we obtain a Brunk–Prokhorov type strong law for renewal processes based on martingale difference schemes: Let $\{X_n, n \geqslant 1\}$ and $\{b_n, n \geqslant 1\}$ be as above, but assume either $q = 1$, or $q > 1$ and $n^{-\delta} b_n$ be nondecreasing for some $\delta > (q-1)/2q$. If

$$\sum_{n=1}^{\infty} \frac{E|X_n|^{2q}}{b_n^{2q}} n^{q-1} < \infty,$$

then (3.8) retains (cf. Fazekas and Klesov [1], Corollary 3.1).

EXAMPLE 3.9 (Banach space schemes). Let $\{X_n, n \geqslant 1\}$ be a sequence of independent, identically distributed random variables assuming values in a Banach space with norm $\|\cdot\|$, and put $Z_n = \|S_n\|$. If $E\|X_1\| < \infty$, then (2.1) holds with $a_n = n\mu$, provided $\mu = \|EX_1\| > 0$ (cf. e.g. Mourier [12]). Since (2.6) and (2.7) are obviously satisfied for such a sequence $\{a_n, n \geqslant 1\}$, Theorem 2.2 gives the asymptotic of the renewal process constructed from a random walk in a Banach space, i.e., as $t \to \infty$,

$$\frac{1}{t} \sum_{n=1}^{\infty} I\{\|S_n\| \leqslant t\} \to \frac{1}{\mu} \text{ a.s.}$$

Further applications of the above results to schemes of Banach space valued random variables will be published elsewhere.

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REFERENCES

[1] I. Fazekas and O. Klesov, *A general approach to the strong laws of large numbers*, Probab. Theory Appl. 45 (2000), pp. 568–583.

[2] W. Feller, *A limit theorem with infinite moments*, Amer. J. Math. 68 (1946), pp. 257–262.

[3] A. Frolov, A. Martikainen and J. Steinebach, *Limit theorems for maxima of sums and renewal processes*, Bericht Nr 70 (2000), 12 pp., Fachbereich Mathematik und Informatik, Philipps-Universität Marburg.

[4] J. Galambos, *Asymptotic Theory of Extreme Order Statistics*, Wiley, New York 1978.

[5] A. Gut, *Stopped Random Walks*, Springer, New York 1988.

[6] A. Gut, O. Klesov and J. Steinebach, *Equivalences in strong limit theorems for renewal counting processes*, Statist. Probab. Lett. 35 (1997), pp. 381–394.

[7] L. Horváth, *Strong approximation of extended renewal processes*, Ann. Probab. 12 (1984), pp. 1149–1166.

[8] S. Janson, *Renewal theory for m-dependent variables*, Ann. Probab. 11 (1983), pp. 558–568.

[9] O. Klesov and J. Steinebach, *Asymptotic behavior of renewal processes defined by random walks with multidimensional time*, Theory Probab. Math. Statist. 56 (1998), pp. 107–113.

[10] A. I. Martikainen and V. V. Petrov, *On a Feller theorem* (in Russian), Probab. Theory Appl. 25 (1980), pp. 194–197.

[11] T. Mori, *The strong law of large numbers when extreme terms are excluded from sums*, Z. Wahrsch. verw. Gebiete 36 (1976), pp. 189–194.

[12] E. Mourier, *Eléments aléatoires dans un espace de Banach*, Ann. Inst. H. Poincaré 13 (1952–1953), pp. 159–244.

[13] Q.-M. Shao, *Maximal inequalities for partial sums of ϱ-mixing sequences*, Ann. Probab. 23 (1995), pp. 948–965.

[14] D. Siegmund, *Sequential Analysis. Tests and Confidence Intervals*, Springer, New York 1985.
[15] M. Woodroofe, *Nonlinear Renewal Theory in Sequential Analysis*, CBS–NSF Regional Conf. Ser. Appl. Math., Vol. 39, SIAM, Philadelphia 1982.

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