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ONE-DIMENSIONAL SYMMETRIC STABLE FEYNMAN–KAC SEMIGROUPS

BY

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Abstract. We investigate here one-dimensional Feynman–Kac semigroups based on symmetric α-stable processes. We begin with establishing the properties of Green operators of intervals and halflines on functions from the Kato class. Then we provide a sufficient condition for gaugeability of the halfline $(-\infty, b)$ and evaluate the critical value β.

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1. INTRODUCTION

In the present paper we continue studying, initiated in Section 7 of [4], the one-dimensional Feynman–Kac semigroups based on symmetric α-stable stochastic Lévy processes. In connection with this topic and the conditional gauge theorem we also mention [3], [5] and [6].

Let us note that if $\alpha \geqslant 1 = d$, we deal with recurrent processes; thus many previously known objects take on different meanings or have different properties. Therefore, in Section 3, we check that the well-known characterization of functions in Kato class $\mathscr{J}^\alpha$ (see [10]) remains valid also for the recurrent case. Although the transient case ($\alpha < d$) is well known, we include it here for the sake of completeness. In Section 4 we then establish properties of Green potentials of intervals and halflines. In Section 5 we consider the gauge function $u(x, y) = E^x e_q(\tau_{(-\infty,y)})$ and investigate conditions assuring that $u(x, b) < \infty$ for $x < b$ (gaugeability of the sets $(-\infty, b)$). The procedure provides us with some estimates of the critical value β, i.e. the maximal value y_0 for which $u(x, y) < \infty$, for $x < y < y_0$. In the end of the paper we estimate β for $q = \mathbf{1}_{(-c,c)}$.

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Let us mention that regarding potential kernels we rely on [2]; for general facts about Markov processes the reader is referred to [1]. We also apply some (unpublished) results from [9].

2. PRELIMINARIES

We adopt here the notation and terminology from [4].

We begin with some elementary facts concerning the density function of the symmetric α-stable process. Let p_t be the density function of the symmetric α-stable process X_t in R^d, starting from 0. The (strict) α-stability of X_t yields that the distribution of $s^{-1/\alpha} X_{ts}$ is the same as that of X_t. We refer to this as to the *scaling property* of X_t. As a direct consequence we obtain

$$p_t(x) = s^{d/\alpha} p_{ts}(s^{1/\alpha} x). \tag{1}$$

Putting $s = t^{-1}$ in (1), we get

$$p_t(x) \leqslant t^{-d/\alpha} \sup_{x \in R^d} p_1(x) = Ct^{-d/\alpha}.$$

Using the inverse Fourier transform we obtain $C = \Gamma(d)\,\omega_d/(2\pi)^d$.

Moreover, it is well known that

$$p_1(x) \leqslant C_d |x|^{-\alpha-d}.$$

Thus, we have

$$p_t(x) \leqslant C_d\, t\, |x|^{-\alpha-d}.$$

Applying these estimates we obtain

LEMMA. *For all $t > 0$ the following holds:*

$$\int_0^t p_s(x)\,ds = |x|^{\alpha-d} \int_0^{t/|x|^\alpha} p_u(x/|x|)\,du \leqslant \frac{C_d t^2}{2} |x|^{-\alpha-d}. \tag{2}$$

For $d = 1 < \alpha$ we obtain

$$\int_0^t p_s(x)\,ds \leqslant \frac{\alpha}{\pi(\alpha-1)} t^{1-1/\alpha}. \tag{3}$$

We now put for $0 < \alpha < 2$ and $d \neq \alpha$:

$$K_\alpha(x) = \frac{\mathscr{A}(d, \alpha)}{|x|^{d-\alpha}}; \tag{4}$$

if $d = 1$ we get

$$K_\alpha(x) = \frac{|x|^{\alpha-1}}{2\Gamma(\alpha)\cos(\pi\alpha/2)}.$$

For $\alpha = 1 = d$ we put

$$K_1(x) = \frac{1}{\pi}\ln\frac{1}{|x|}. \tag{5}$$

Let us mention that if $\alpha \geqslant d = 1$, then the α-stable process is no longer transient, so its (free) Green function on the whole real line is no longer properly defined; in that case we refer to [2] for an appropriate interpretation of (4) and (5).

3. KATO CLASS $\mathscr{J}^\alpha$

DEFINITION. We say that a Borel function q on $\boldsymbol{R}^d$ belongs to the *Kato class* $\mathscr{J}^\alpha$ if

$$\lim_{\gamma \downarrow 0} \sup_{x \in \boldsymbol{R}^d} \int_{|x-y| \leqslant \gamma} |K_\alpha(x-y)\, q(y)|\, dy = 0; \tag{6}$$

we write $q \in \mathscr{J}^\alpha_{\text{loc}}$ if for every bounded Borel set B we have $\mathbf{1}_B q \in \mathscr{J}^\alpha$.

PROPERTIES OF THE CLASS $\mathscr{J}^\alpha$.

(i) *We have* $L^\infty(\boldsymbol{R}^d) \subseteq \mathscr{J}^\alpha$. *If* $f \in L^\infty(\boldsymbol{R}^d)$ *and* $q \in \mathscr{J}^\alpha$, *then* $fq \in \mathscr{J}^\alpha$.

(ii) *If* $q \in \mathscr{J}^\alpha$, *then* $\sup_{x \in \boldsymbol{R}^d} \int_{|x-y| \leqslant 1} |q(y)|\, dy < \infty$. *Hence, if* $q \in \mathscr{J}^\alpha_{\text{loc}}$, *then* $q \in L^1_{\text{loc}}(\boldsymbol{R}^d)$.

(iii) *If* $q \in \mathscr{J}^\alpha$, *then* $\sup_{x \in \boldsymbol{R}^d} \int_{|x-y| \geqslant 1} |x-y|^{-\alpha-d} |q(y)|\, dy < \infty$.

Proof. Condition (i) follows directly by the definition.

We verify (ii). For $0 < \alpha \leqslant d$ and $0 < \gamma \leqslant 1$ we obtain

$$\int_{|x-y| \leqslant \gamma} |K_\alpha(x-y)\, q(y)|\, dy \geqslant |K_\alpha(\gamma)| \int_{|x-y| \leqslant \gamma} |q(y)|\, dy.$$

There exists $N = N(d, \gamma)$ such that for every $x \in \boldsymbol{R}^d$ we can find $x_1, \ldots, x_N \in \boldsymbol{R}^d$ such that $\overline{B(x, 1)} \subseteq \bigcup_{i=1}^N B(x_i, \gamma)$. Thus, we obtain

$$\int_{|x-y| \leqslant 1} |q(y)|\, dy \leqslant |K_\alpha(\gamma)|^{-1} \sum_{i=1}^N \int_{|x_i-y| \leqslant \gamma} |K_\alpha(x_i-y)\, q(y)|\, dy.$$

For $d = 1 < \alpha < 2$ we choose $N = N(\gamma/4)$ such that for every $x \in \boldsymbol{R}^1$ there exist $x_1, \ldots, x_N \in \boldsymbol{R}^1$ such that $\overline{B(x, 1)} \subseteq \bigcup_{i=1}^N B(x_i, \gamma/4)$. Put $z_i = x_i + \gamma/2$. For y satisfying $|y - x_i| \leqslant \gamma/4$ we obtain $|y - z_i| \geqslant \gamma/2 - \gamma/4 = \gamma/4$. Thus, we get $1 \leqslant (4|y-z_i|/\gamma)^{\alpha-1}$ over the set $\{y; |y-x_i| \geqslant \gamma/4\}$. Hence, we have

$$\begin{aligned}\int_{|x-y| \leqslant 1} |q(y)|\, dy &\leqslant \sum_{i=1}^N (4/\gamma)^{\alpha-1} \int_{|x_i-y| \leqslant \gamma/4} |z_i-y|^{\alpha-1} |q(y)|\, dy \\ &\leqslant \sum_{i=1}^N (4/\gamma)^{\alpha-1} \int_{|z_i-y| \leqslant \gamma} |z_i-y|^{\alpha-1} |q(y)|\, dy.\end{aligned}$$

Remark. For $d=1<\alpha<2$ we have for $0<\gamma\leqslant 1$

$$\sup_{x\in R^d}\int_{|y-x|\leqslant\gamma}|y-x|^{\alpha-1}|q(y)|\,dy\leqslant\gamma^{\alpha-1}\sup_{x\in R^d}\int_{|x-y|\leqslant 1}|q(y)|\,dy.$$

Thus, $q\in\mathcal{J}^\alpha$ for $d=1<\alpha<2$ if and only if

$$\sup_{x\in R^d}\int_{|x-y|\leqslant 1}|q(y)|\,dy<\infty. \tag{7}$$

We now prove (iii). As in [9] we put

$$S_\gamma=\sup_{x\in R^d}\int_{|x-y|\geqslant\gamma}|x-y|^{-\alpha-d}|q(y)|\,dy.$$

There exists a constant $C(d,\gamma)$ such that $\{y;\ \gamma n\leqslant|x-y|\leqslant(n+1)\gamma\}$ can be covered by at most $C(d,\gamma)n^{d-1}$ balls with radius γ, for every n. Thus, we have

$$\begin{aligned}S_\gamma&\leqslant\sup_{x\in R^d}\Big[\sum_{n=1}^{\infty}\int_{n\gamma\leqslant|x-y|\leqslant(n+1)\gamma}|x-y|^{-\alpha-d}|q(y)|\,dy\Big]\\&\leqslant C(d,\gamma)\Big(\sum_{n=1}^{\infty}n^{-d-\alpha}n^{d-1}\Big)\sup_{x\in R^d}\int_{|x-y|\leqslant\gamma}|q(y)|\,dy<\infty.\end{aligned}$$

Now we obtain

THEOREM 1. *We have*

$$q\in\mathcal{J}^\alpha\quad\textit{if and only if}\quad\lim_{t\downarrow 0}\sup_{x\in R^d}\int_0^t P_s|q|(x)\,ds=0. \tag{8}$$

Proof. We first show that the condition on the right-hand side of (8) implies that $q\in\mathcal{J}^\alpha$. With the exception of the case $1=d=\alpha$ we have

$$\begin{aligned}\int_0^t P_s|q|(x)\,ds&=\int_{R^d}\Big(\int_0^t p_s(x-y)\,ds\Big)|q(y)|\,dy\\&=\mathcal{A}(d,\alpha)^{-1}\int_{R^d}|K_\alpha(x-y)|\int_0^{t/|x-y|^\alpha}p_u\Big(\frac{x-y}{|x-y|}\Big)du\,|q(y)|\,dy\\&\geqslant\mathcal{A}(d,\alpha)^{-1}\int_{|x-y|\leqslant t^{1/\alpha}}|K_\alpha(x-y)|\int_0^1 p_u\Big(\frac{x-y}{|x-y|}\Big)du\,|q(y)|\,dy\\&=C\mathcal{A}(d,\alpha)^{-1}\int_{|x-y|\leqslant t^{1/\alpha}}|K_\alpha(x-y)\,q(y)|\,dy,\end{aligned}$$

since

$$\int_0^1 p_u\Big(\frac{x-y}{|x-y|}\Big)du=C>0,$$

by the rotational invariance of p_u.

When $1 = d = \alpha$, by Fubini's theorem we obtain

$$\int_0^t P_s|q|(x)\,ds = \int_{R^1}\Big(\int_0^t p_s(x-y)\,ds\Big)|q|(y)\,dy$$

$$= \frac{1}{\pi}\int_{R^1}\int_0^t \frac{s\,ds}{s^2+(x-y)^2}|q(y)|\,dy = \frac{1}{2\pi}\int_{R^1}\ln\frac{t^2+(x-y)^2}{(x-y)^2}|q(y)|\,dy$$

$$\geqslant \frac{1}{2\pi}\int_{|x-y|\leqslant t^2}\ln\frac{1}{|x-y|}|q(y)|\,dy,$$

because

$$\frac{t^2+(x-y)^2}{(x-y)^2} \geqslant \frac{t^2}{(x-y)^2} \geqslant \frac{1}{|x-y|},$$

whenever $|x-y| \leqslant t^2$.

We now prove that the condition on the right-hand side of (8) holds whenever $q \in \mathscr{J}^\alpha$.

By the previous case we have

$$\int_0^t P_s|q|(x)\,ds = \int_{R^d}|x-y|^{\alpha-d}\int_0^{t/|x-y|^\alpha} p_u\Big(\frac{x-y}{|x-y|}\Big)\,du\,|q(y)|\,dy$$

$$= \int_{|x-y|\leqslant\gamma}\ldots + \int_{|x-y|>\gamma}\ldots = I_1+I_2.$$

We first consider the case $\alpha < d$. We then obtain

$$I_1 \leqslant \mathscr{A}(d,\alpha)^{-1}\int_{|x-y|\leqslant\gamma}|K_\alpha(x-y)\,q(y)|\int_0^\infty p_u\Big(\frac{x-y}{|x-y|}\Big)\,du\,dy$$

$$\leqslant \int_{|x-y|\leqslant\gamma}|K_\alpha(x-y)\,q(y)|\,dy,$$

because

$$\int_0^\infty p_u\Big(\frac{x-y}{|x-y|}\Big)\,du = K_\alpha\Big(\frac{x-y}{|x-y|}\Big) = \mathscr{A}(d,\alpha).$$

When $1 = d = \alpha$, we get

$$I_1 = \frac{1}{2\pi}\int_{|x-y|\leqslant\gamma}\ln\frac{t^2+(x-y)^2}{(x-y)^2}|q(y)|\,dy$$

$$\leqslant \frac{1}{2\pi}\int_{|x-y|\leqslant\gamma}\ln\frac{t^2+\gamma^2}{(x-y)^2}|q(y)|\,dy \leqslant \frac{1}{\pi}\int_{|x-y|\leqslant\gamma}\ln\frac{1}{|x-y|}|q(y)|\,dy$$

whenever $t^2+\gamma^2 < 1$.

When $\alpha > 1 = d$, we apply the estimate (3) to obtain

$$I_1 = \int\limits_{|x-y|\leqslant\gamma} \int\limits_0^t p_u(x-y)\,du\,dy \leqslant \frac{\alpha}{\pi(\alpha-1)} t^{1-1/\alpha} \int\limits_{|x-y|\leqslant\gamma} |q(y)|\,dy.$$

The Remark below the proof of the properties of the Kato class shows that this estimate is sufficient for our purposes.

What remains now is to estimate I_2. We have

$$I_2 = \int\limits_{|x-y|>\gamma} |x-y|^{\alpha-d} \int\limits_0^{t/|x-y|^\alpha} p_u\left(\frac{x-y}{|x-y|}\right) du\,|q(y)|\,dy$$
$$\leqslant \frac{Ct^2}{2} \int\limits_{|x-y|>\gamma} |x-y|^{\alpha-d}|x-y|^{-2\alpha}|q(y)|\,dy \leqslant \frac{Ct^2}{2} S_\gamma.$$

The above estimate completes the proof of the theorem.

4. GREEN OPERATORS

We now examine properties of the Green operator G_D when D is either an interval or halfline. We always assume that $\alpha\in(0, 2)$. We begin with the case of intervals. For this purpose, following [4], we state some useful estimates for the Green function of the interval. We put

$$I_\alpha(t) = \int\limits_0^t \frac{u^{\alpha/2-1}}{(u+1)^{1/2}}\,du, \qquad t \geqslant 0,$$

and

$$w(x, y) = \frac{(1-x^2)(1-y^2)}{(x-y)^2}, \qquad |x|, |y| \leqslant 1.$$

Let us recall [2] that the Green function $G(x, y)$ of the interval $(-1, 1)$ can be represented as

$$G(x, y) = \mathscr{B}_\alpha |x-y|^{\alpha-1} I_\alpha(w(x, y)), \qquad |x|, |y| \leqslant 1,$$

where $\mathscr{B}_\alpha = 2^{-\alpha}\Gamma(\alpha/2)^{-2}$.

The behavior of the function G is determined by the asymptotic properties of the integral $I_\alpha(t)$, which are summarized in the following lemma:

LEMMA 2. *There are constants $C_i = C_i(\alpha)$, $i = 1, 2$, such that for all $t > 0$*

(i) $$C_1 \leqslant I_\alpha(t)/[t^{\alpha/2} \wedge 1] \leqslant C_2 \quad \textit{if } \alpha < 1,$$

(ii) $$C_1 \leqslant I_\alpha(t)/[t^{\alpha/2} \wedge t^{(\alpha-1)/2}] \leqslant C_2 \quad \textit{if } \alpha > 1,$$

(iii) $$C_1 \leqslant I_\alpha(t)/t^{\alpha/2} \leqslant C_2 \quad \textit{if } t \leqslant 1,\ \alpha\in(0, 2),$$

and for $\alpha = 1$

(iv) $$C_1 \leqslant I_1(t)/\ln(t^{1/2}+1) \leqslant C_2.$$

A calculation allows for the choice of $C_2 = 2/(\alpha-1)$ in (ii) and $C_1 = 2^{1/2}/\alpha$, $C_2 = 2/\alpha$ in (iii).

Using Lemma 2 we describe the behavior of the function G.

COROLLARY 3. *For all* $x, y \in (-1, 1)$ $(x \neq y)$ *we have*

$$C_1 \leqslant G(x, y)/\big(\mathscr{B}_\alpha |x-y|^{\alpha-1}[w(x, y)^{\alpha/2} \wedge 1]\big) \leqslant C_2 \quad \text{if } \alpha < 1,$$
$$C_1 \leqslant G(x, y)/\big(\mathscr{B}_1 \ln(w(x, y)^{1/2}+1)\big) \leqslant C_2 \quad \text{if } \alpha = 1,$$
$$C_1 \leqslant G(x, y)/\big(\mathscr{B}_\alpha |x-y|^{\alpha-1}[w(x, y)^{\alpha/2} \wedge w(x, y)^{(\alpha-1)/2}]\big) \leqslant C_2 \quad \text{if } \alpha > 1.$$

We then obtain

THEOREM 4. *Let* $a, b \in \boldsymbol{R}^1$, $a < b$ *and* $D = (a, b)$. *Assume that* $q \in \mathscr{J}^\alpha$, $\alpha \geqslant 1$. *Then*

$$G_D q \in C_0(D).$$

Proof. We assume that $D = (-1, 1)$. First consider the case $\alpha > 1$. Then, by Lemma 2 and the remark regarding the choice of the constants, we have

$$G_D(x, y) \leqslant \frac{2\big((1-x^2)(1-y^2)\big)^{(\alpha-1)/2}}{(\alpha-1)\,2^\alpha \Gamma(\alpha/2)^2}.$$

Since D is bounded, we obtain $q \in L^1(D)$, so $G_D q(x)$ is a continuous and bounded function of x, continuously vanishing at ∂D.

When $\alpha = 1$, we obtain

$$G_D(x, y) = \frac{1}{\pi}\ln\frac{1-xy+\sqrt{(1-x^2)(1-y^2)}}{|x-y|} \leqslant \frac{1}{\pi}\ln\frac{3}{|x-y|}.$$

The proof of the theorem in this case $(\alpha = 1)$ is similar to the one for the (standard) case $\alpha < 1$; it is included here for the sake of completeness. We put $G_D^{(n)}(x, y) = G_D(x, y) \wedge n$. Then for fixed $\gamma < 1$ and $|x-y| > \gamma$ we obtain $G_D(x, y) \leqslant (1/\pi)\ln(3/\gamma)$. Therefore, if $n_0 > (1/\pi)\ln(3/\gamma)$, then for $n \geqslant n_0$ we have $|G_D(x, y) - G_D^{(n)}(x, y)| > 0$ only on the set $|x-y| \leqslant \gamma$. Thus, we obtain for $n \geqslant n_0$

$$|G_D q(x) - G_D^{(n)} q(x)| \leqslant \int_{|x-y| \leqslant \gamma} |G_D(x, y) - G_D^{(n)}(x, y)|\,|q(y)|\,dy$$
$$\leqslant \sup_{x \in \boldsymbol{R}^d} \int_{|x-y| \leqslant \gamma} \frac{1}{\pi}\ln\frac{3}{|x-y|}\,|q(y)|\,dy.$$

Since $q \in \mathscr{J}^\alpha$, the above inequality yields that $G_D^{(n)} q \to G_D q$ uniformly. Obviously, $G_D^{(n)} q \in C_0(D)$, which completes the proof.

Next, we consider the case of the halflines. As in the case of intervals, we establish first some elementary properties of the Green function of the set $D=(-\infty, b)\subset R^1$.

We put

$$J_\alpha(t)=\int_0^t \frac{u^{\alpha/2-1}}{(u+1)^{1-\alpha/2}}du, \quad t\geqslant 0,$$

and

$$v_b(x, y)=\frac{(b-x)\wedge(b-y)}{|x-y|}, \quad x\leqslant b,\ y\leqslant b.$$

Let us recall [8] that the Green function $G_D(x, y)$ of $(-\infty, b)$ can be represented as

$$G_D(x, y)=\frac{|x-y|^{\alpha-1}}{\Gamma(\alpha/2)^2}J_\alpha(v_b(x, y)), \quad x\leqslant b,\ y\leqslant b.$$

As before, we first summarize the asymptotic properties of the integral J_α, which determine the behavior of the function G_D.

LEMMA 5. *There are constants $C_i=C_i(\alpha)$, $i=1, 2$, such that for all $t>0$*

(i) $$C_1\leqslant J_\alpha(t)/[t^{\alpha/2}\wedge 1]\leqslant C_2 \quad \text{if } \alpha<1,$$

(ii) $$C_1\leqslant J_\alpha(t)/[t^{\alpha/2}\wedge t^{\alpha-1}]\leqslant C_2 \quad \text{if } \alpha>1,$$

(iii) $$C_1\leqslant J_\alpha(t)/t^{\alpha/2}\leqslant C_2 \quad \text{if } t\leqslant 1,\ \alpha\in(0, 2).$$

For $\alpha>1$ and $t\geqslant 1$

(iv) $$C_1\leqslant J_\alpha(t)/t^{\alpha-1}\leqslant C_2,$$

and for $\alpha=1$

(v) $$C_1\leqslant J_1(t)/\ln(t^{1/2}+1)\leqslant C_2.$$

The proof of Lemma 5 is omitted.

We note that the upper estimates in (iii) and (iv) hold for all $t>0$; in the latter case with the restriction $\alpha>1$. Following calculations, we can take $C_1=2^{\alpha/2}/\alpha$ and $C_2=2/\alpha$ in (iii), and $C_1=(2^{\alpha-1}-1)/(\alpha-1)$ and $C_2=1/(\alpha-1)$ in (iv).

The following corollary describes the asymptoting behavior of the function G_D for $D=(-\infty, b)$.

COROLLARY 6. *For all $x, y\in(-\infty, b)$ $(x\neq y)$ we have*

$$C_1\leqslant \Gamma(\alpha/2)^2 G_D(x, y)/(|x-y|^{\alpha-1}[v_b(x, y)^{\alpha/2}\wedge 1])\leqslant C_2 \quad \text{if } \alpha<1,$$

$$C_1\leqslant \pi G_D(x, y)/\ln(v_b(x, y)^{1/2}+1)\leqslant C_2 \quad \text{if } \alpha=1,$$

$$C_1\leqslant \Gamma(\alpha/2)^2 G_D(x, y)/(|x-y|^{\alpha-1}[v_b(x, y)^{\alpha/2}\wedge v_b(x, y)^{\alpha-1}])\leqslant C_2 \quad \text{if } \alpha>1.$$

Whenever $\alpha > 1$ we easily obtain from Lemma 5, the remark below and Corollary 6

$$2^{\alpha-1}-1 \leqslant \frac{(\alpha-1)\Gamma(\alpha/2)^2 G_D(x, y)}{((b-x)\wedge(b-y))^{\alpha-1}} \leqslant 1, \tag{9}$$

where the estimate from below holds true if $((b-x)\wedge(b-y))/|x-y| \geqslant 1$.

Note that X_t is transient for $\alpha < 1$, so we obtain in this case, as usual,

$$G_D(x, y) \leqslant K_\alpha(x-y). \tag{10}$$

For $\alpha = 1$ we obtain from Corollary 6

$$G_D(x, y) \leqslant C \ln\left(\sqrt{\frac{(b-x)\wedge(b-y)}{|x-y|}}+1\right). \tag{11}$$

We now state and prove a version of Theorem 4 for a halfline.

THEOREM 7. *Let $D = (-\infty, b)$ and let $q \in \mathcal{J}^\alpha$. Then $G_D|q|(x) < \infty$ whenever $(1+|y|)^{\alpha-1} q(y) \in L^1(D)$ for $\alpha < 1$, or if $q \in L^1(D)$ for $\alpha \geqslant 1$. If, additionally,*

(i) $$q \in L^1(R^1) \quad \text{if } \alpha < 1,$$

(ii) $$\ln(|y|^{1/2}+1)\, q(y) \in L^1(R^1) \quad \text{if } \alpha = 1,$$

(iii) $$(1+|y|)^{\alpha-1} q(y) \in L^1(R^1) \quad \text{if } \alpha > 1,$$

then $G_D q(x)$ is a continuous function of x and

$$\lim_{b\to-\infty} \sup_{x\leqslant b} G_{(-\infty,b)}|q|(x) = 0.$$

Proof. We prove the first part of the theorem for $\alpha < 1$. For a fixed γ, $0 < \gamma < 1$, we have

$$G_D q(x) = \int_{-\infty}^{b} G_D(x, y) q(y)\, dy$$

$$= \int_{y\leqslant b, |x-y|\leqslant\gamma} G_D(x, y) q(y)\, dy + \int_{y\leqslant b, |x-y|>\gamma} G_D(x, y) q(y)\, dy.$$

Since $\alpha < 1$, we have $G_D(x, y) \leqslant K_\alpha(x-y)$. This and the assumption $q \in \mathcal{J}^\alpha$ yield that for a given $\varepsilon > 0$ the supremum over $x \in R^1$ of the first term on the right-hand side of the above equality is less than ε whenever γ is small enough. On the other hand, if $|x-y| \geqslant \gamma$, then

$$\frac{|y|+1}{|x-y|} \leqslant \frac{|x-y|+|x|+1}{|x-y|} \leqslant 1+\frac{|x|+1}{|x-y|} \leqslant 1+\gamma^{-1}(|x|+1).$$

Therefore, the integrand in the second term is continuous in x, vanishes at b, and is bounded by

$$K_\alpha(x-y)|q(y)| \leqslant C[1+\gamma^{-1}(|x|+1)]^{1-\alpha}\frac{|q(y)|}{(1+|y|)^{1-\alpha}} \in L^1(D).$$

If we assume that $q \in L^1(R^1)$, then we obtain

$$\sup_{x \leqslant b} G_{(-\infty,b)}|q|(x) \leqslant \varepsilon + \mathscr{A}(1, \alpha)\gamma^{\alpha-1}\int_{-\infty}^{b}|q(y)|\,dy,$$

where ε and γ are as above. This completes the proof of the case $\alpha < 1$.

When $\alpha = 1$, we apply the inequality (11). Under the assumption $q \in L^1(D)$ we obtain, for $0 < \gamma < 1$,

$$\begin{aligned}\int_{-\infty}^{b} G_D(x, y)|q(y)|\,dy &\leqslant C\int_{-\infty}^{b}\ln\left(\sqrt{\frac{(b-x)\wedge(b-y)}{|x-y|}}+1\right)|q(y)|\,dy\\ &\leqslant C\int_{y\leqslant b, |x-y|\leqslant\gamma}\ln\frac{(b-x)^{1/2}+\gamma^{1/2}}{|x-y|^{1/2}}|q(y)|\,dy\\ &\quad + C\int_{y\leqslant b, |x-y|>\gamma}\ln\left(\sqrt{(b-x)/\gamma}+1\right)|q(y)|\,dy\\ &\leqslant C\int_{|x-y|\leqslant\gamma}\ln\frac{1}{|x-y|}|q(y)|\,dy + C\ln\left(\sqrt{(b-x)/\gamma}+1\right)\|q\|_{L^1(D)}.\end{aligned}$$

Taking into account $q \in \mathscr{J}^\alpha \cap L^1(D)$ we obtain the first conclusion in the case $\alpha = 1$. The continuity follows from similar arguments as in the first part of the proof.

When additionally $\ln(|y|^{1/2}+1)\,q(y) \in L^1(R^1)$, then, arguing as before, we obtain

$$G_{(-\infty,b)}|q|(x) \leqslant C\int_{|x-y|\leqslant\gamma}\ln\frac{1}{|x-y|}|q(y)|\,dy + C\int_{-\infty}^{b}\ln\left(\sqrt{(b-y)/\gamma}+1\right)|q(y)|\,dy.$$

Since for $b < 0$ we have $b-y < -y = |y|$, the proof of this part of the theorem is complete.

Finally, if $\alpha > 1 = d$, we apply the inequality (9). Assume first that $q \in L^1(D)$. We then obtain

$$|G_D q(x)| \leqslant \frac{(b-x)^{\alpha-1}}{(\alpha-1)\Gamma(\alpha/2)^2}\|q\|_{L^1(D)}.$$

When $(1+|y|)^{\alpha-1} q(y) \in L^1(R^1)$, then we obtain

$$G_{(-\infty,b)}|q|(x) \leqslant \frac{1}{(\alpha-1)\Gamma(\alpha/2)^2} \int_{-\infty}^{b} (b-y)^{\alpha-1}|q(y)|\,dy.$$

For $b<0$ we obtain, as before, $(b-y)^{\alpha-1} \leqslant |y|^{\alpha-1} < (1+|y|)^{\alpha-1}$ and this completes the proof of the last part of the theorem.

5. FUNDAMENTAL EXPECTATION

Let $q \in \mathcal{J}^\alpha$. We recall that

$$u(x, y) = E^x e_q(\tau_y), \quad \text{where } \tau_y = \tau_{(-\infty,y)}.$$

Obviously, $u(x, y) = 1$ for $x \geqslant y$. In [4] it is shown (see Section 7) that if $u(x, y) < \infty$ for a single $x < y$, then $u(\cdot, y)$ is a continuous regular q-harmonic function on $(-\infty, y)$. In particular, $u(w, y) < \infty$ for every $w < y$.

We put

$$\beta = \sup\{y \in R^1;\ u(x, y) < \infty \text{ for all } x < y\},$$

with the usual convention that $\sup\emptyset = -\infty$.

We now establish a condition under which $\beta > -\infty$.

THEOREM 8. *Under the assumptions* (i), (ii) *or* (iii), *respectively, from Theorem* 7, *we have*

$$\beta \geqslant \sup\{b \in R^1;\ \sup_{x\leqslant b} G_{(-\infty,b)}|q|(x) < 1\} > -\infty.$$

Proof. Let $q \not\equiv 0$. Define

$$M_n(x, b) = \frac{1}{n!} E^x \Big(\int_0^{\tau_b} |q(X_t)|\,dt\Big)^n \quad \text{and} \quad M_n(b) = \sup_{x\leqslant b} M_n(x, b).$$

Obviously,

$$M_1(b) = \sup_{x\leqslant b} G_{(-\infty,b)}|q|(x).$$

By the proof of the Khasminski lemma (see [7]) we obtain

$$M_n(b) \leqslant M_1(b)^n.$$

Observe that if $b_1 \leqslant b_2$, then $M_1(x, b_1) \leqslant M_1(x, b_2)$ for all $x \leqslant b_1$. Consequently,

$$M_1(b_1) = \sup_{x\leqslant b_1} M_1(x, b_1) \leqslant \sup_{x\leqslant b_1} M_1(x, b_2) \leqslant \sup_{x\leqslant b_2} M_1(x, b_2) = M_1(b_2).$$

Moreover, $M_1(b)$ is a continuous function of b. To prove this we show that $G_{(-\infty,b)}|q|(\cdot)$ is continuous, with respect to the supremum norm, as a function of b. For this purpose, assume that $b_1 < b_2$. Then for $b_1 \leqslant x \leqslant b_2$ we have

$$G_{(-\infty,b_2)}|q|(x) - G_{(-\infty,b_1)}|q|(x) = G_{(-\infty,b_2)}|q|(x).$$

We thus estimate first $G_{(-\infty,b_2)}|q|(x)$ for $b_1 \leqslant x \leqslant b_2$.

Assume that $\alpha > 1$. Using the estimate (9) we obtain

$$G_{(-\infty,b_2)}|q|(x) \leqslant \frac{(b_2-b_1)^{\alpha-1}}{(\alpha-1)\Gamma(\alpha/2)^2}\|q\|_{L^1(R^1)}.$$

If now $\alpha = 1$, then for a given $\varepsilon > 0$ there exists a γ, $0 < \gamma < 1$, such that

$$\sup_{x\in R^1} \frac{C}{2} \int_{|x-y|\leqslant\gamma} \ln\frac{1}{|x-y|}|q(y)|\,dy \leqslant \varepsilon,$$

where C is the constant from the estimate (11). We then obtain

$$\begin{aligned} G_{(-\infty,b_2)}|q|(x) &= \int_{-\infty}^{b_2} G_{(-\infty,b_2)}(x,y)|q(y)|\,dy \\ &\leqslant C\int_{-\infty}^{b_2} \ln\left(\sqrt{\frac{b_2-x}{|x-y|}}+1\right)|q|(y)\,dy \\ &\leqslant C\ln(\sqrt{b_2-b_1}+1) \int_{y\leqslant b_2, |x-y|\leqslant\gamma} |q(y)|\,dy \\ &\quad + \frac{C}{2}\int_{|x-y|\leqslant\gamma} \ln\frac{1}{|x-y|}|q(y)|\,dy \\ &\quad + C\ln(\sqrt{(b_2-b_1)/\gamma}+1) \int_{y\leqslant b_2, |x-y|>\gamma} |q(y)|\,dy \\ &\leqslant C\ln(\sqrt{(b_2-b_1)/\gamma}+1)\|q\|_{L^1(R^1)} + \varepsilon. \end{aligned}$$

We now consider the case $\alpha < 1$. As in the previous case, for a given $\varepsilon > 0$ we choose γ, $0 < \gamma < 1$, such that

$$\sup_{x\in R^1} \int_{|x-y|\leqslant\gamma} K_\alpha(x,y)|q(y)|\,dy \leqslant \varepsilon.$$

For $|x-y| > \gamma$ we estimate $G_{(-\infty,b_2)}$ by $2\alpha^{-1}\Gamma(\alpha/2)^{-2}(b_2-x)^{\alpha/2}|x-y|^{\alpha/2-1}$. Thus, we obtain

$$G_{(-\infty,b_2)}|q|(x) \leqslant \varepsilon + \frac{2(b_2-b_1)^{\alpha/2}}{\alpha\Gamma(\alpha/2)^2\gamma^{1-\alpha/2}}\|q\|_{L^1(R^1)}.$$

For $x \leqslant b_1$ we proceed as follows:

$$G_{(-\infty,b_2)}|q|(x) - G_{(-\infty,b_1)}|q|(x) = E^x\big[\tau_{b_1} < \tau_{b_2};\ \int_{\tau_{b_1}}^{\tau_{b_2}} |q(X_t)|\,dt\big]$$

$$= E^x\big[\tau_{b_1} < \tau_{b_2};\ \big(\int_0^{\tau_{b_2}} |q(X_t)|\,dt\big)\circ\theta_{\tau_{b_1}}\big] = E^x\big[\tau_{b_1} < \tau_{b_2};\ G_{(-\infty,b_2)}|q|(X_{\tau_{b_1}})\big]$$

$$= E^x\big[G_{(-\infty,b_2)}|q|(X_{\tau_{b_1}})\big] \leqslant \sup_{b_1 \leqslant x \leqslant b_2} G_{(-\infty,b_2)}|q|(x).$$

The result follows by application of the first part.

Define

(12) $$b_0 = \sup\{b \in R^1;\ M_1(b) < 1\}.$$

By continuity of M_1 and Theorem 7 we have $b_0 > -\infty$ and $M_1(b_0) = 1$ if $b_0 < \infty$. Also, for $b < b_0$ we obtain $M(b) < 1$. Therefore, for $b < b_0$ and all $x \leqslant b$ we have

$$u(x, b) = 1 + \sum_{n=1}^{\infty} M_n(x, b) \leqslant 1 + \sum_{n=1}^{\infty} M_n(b)$$

$$\leqslant 1 + \sum_{n=1}^{\infty} M_1(b)^n = \frac{1}{1 - M_1(b)} < \infty.$$

This proves that $\beta \geqslant b_0 > -\infty$. The proof is now complete.

We now establish additional properties of the function $u(x, y)$, under the assumptions of Theorem 7, unless stated otherwise. First, observe that by the final part of the above proof we get

$$\sup_{x \leqslant b} |u(x, b) - 1| \leqslant \frac{M_1(b)}{1 - M_1(b)}.$$

Since we have shown in Theorem 7 that $\lim_{b\to-\infty} M_1(b) = 0$, it follows that

$$\lim_{b\to-\infty} \sup_{x \leqslant b} |u(x, b) - 1| = 0.$$

We also note that if $\alpha < 1$, then $\lim_{b\to\infty} M_1(b) < \infty$. Indeed, since $q \in \mathscr{J}^\alpha \cap L^1(R^1)$, we obtain

$$\sup_{x\in R^1} \int_{|x-y|\leqslant 1} K_\alpha(x, y)\,|q(y)|\,dy \leqslant C_1.$$

Thus, by the estimate $G_{(-\infty,b)} \leqslant K_\alpha$, we obtain

$$G_{(-\infty,b)}|q|(x) \leqslant \int_{|x-y|\leqslant 1} K_\alpha(x, y)\,|q(y)|\,dy + \mathscr{A}(1, \alpha) \int_{|x-y|>1} |q(y)|\,dy$$

$$\leqslant C_1 + \mathscr{A}(1, \alpha)\,\|q(y)\|_{L^1(R^1)}.$$

On the other hand, for $\alpha \geqslant 1$ we obtain

$$\lim_{b\to\infty} M_1(b) = \infty.$$

We show this first for $\alpha > 1$. Observe that for $b > 0$ and $-b/2 < y < b/4$ we have $(b-y)/(y+b/2) > 1$. Hence using the lower estimate (9) we obtain

$$\begin{aligned} M_1(b) &\geqslant M_1(-b/2, b) \\ &\geqslant \frac{1}{\Gamma(\alpha/2)^2} \int_{-b/2}^{b/4} (y+b/2)^{\alpha-1} \int_0^{(b-y)/(y+b/2)} \frac{u^{\alpha/2-1}\,du}{(u+1)^{1-\alpha/2}} |q(y)|\,dy \\ &\geqslant \frac{2^{\alpha-1}-1}{(\alpha-1)\Gamma(\alpha/2)^2} \int_{-b/2}^{b/4} (y+b/2)^{\alpha-1} \left(\frac{b-y}{y+b/2}\right)^{\alpha-1} |q(y)|\,dy \\ &\geqslant \frac{2^{\alpha-1}-1}{(\alpha-1)\Gamma(\alpha/2)^2} \int_{-b/2}^{b/4} (b-y)^{\alpha-1} |q(y)|\,dy \\ &\geqslant (3/4\,b)^{\alpha-1} \frac{2^{\alpha-1}-1}{(\alpha-1)\Gamma(\alpha/2)^2} \int_{-b/2}^{b/4} |q(y)|\,dy. \end{aligned}$$

For $\alpha = 1$ we argue as follows:

$$\begin{aligned} G_{(-\infty,b)}|q|(x) &\geqslant C \int_{-\infty}^{x-1} \ln\left(\sqrt{b-x}+\sqrt{x-y}\right)|q(y)|\,dy - C\int_{-\infty}^{x-1} \ln|x-y|\,|q(y)|\,dy \\ &\geqslant C \int_{-\infty}^{x-1} \ln\left(\sqrt{b-x}+1\right)|q(y)|\,dy - C\int_{-\infty}^{x-1} \ln|x-y|\,|q(y)|\,dy. \end{aligned}$$

Now, in the last term, for fixed x we have

$$0 \leqslant \int_{-\infty}^{x-1} \ln|x-y|\,|q(y)|\,dy \leqslant C_1 < \infty,$$

while the previous term tends to ∞ as $b \to \infty$, whenever $\|q\|_{L^1(-\infty,x-1)} \neq 0$.

We now show that for fixed b we have $\lim_{x\to-\infty} G_{(-\infty,b)}\,q(x) = 0$. Note that this property has no Brownian motion counterpart.

Let $\alpha > 1$. By the upper estimate (9) we have

$$G_{(-\infty,b)}(x, y)\,|q(y)| \leqslant \frac{(b-y)^{\alpha-1}}{(\alpha-1)\Gamma(\alpha/2)^2}|q(y)| \in L^1(R^1),$$

by the assumption. At the same time, when $x \to -\infty$, then $b-x > b-y$ for fixed y, whenever $|x|$ is big enough. Therefore,

$$\frac{(b-x)\wedge(b-y)}{|x-y|} = \frac{b-y}{y-x} \to 0 \quad \text{as } x \to -\infty.$$

Using the same estimate we thus obtain, again when $|x|$ is sufficiently big:

$$G_{(-\infty,b)}(x, y) \leqslant \frac{2}{\alpha} \frac{(y-x)^{\alpha-1}}{\Gamma(\alpha/2)^2} \frac{(b-y)^{\alpha/2}}{(y-x)^{\alpha/2}} = \frac{2}{\alpha} \frac{(b-y)^{\alpha/2}}{(y-x)^{1-\alpha/2}} \to 0 \quad \text{as } x \to -\infty.$$

The application of the Lebesgue Dominated Convergence Theorem gives now the claim.

If $\alpha = 1$, then we show first that given $\varepsilon > 0$ we can choose $A < b$ such that

$$\sup_{x \leqslant b} \int_{-\infty}^{A} G_{(-\infty,b)}(x, y) |q(y)| \, dy \leqslant \varepsilon.$$

For this purpose, assume that $0 < \gamma \leqslant 1$ and estimate the above expression as follows:

$$\begin{aligned}
&\int_{-\infty}^{A} G_{(-\infty,b)}(x, y) |q(y)| \, dy \\
&\leqslant C \int_{-\infty}^{A} \ln\left(\sqrt{\frac{b-y}{|x-y|}} + 1\right) |q(y)| \, dy \\
&\leqslant C \int_{y \leqslant A, |x-y| \leqslant \gamma} \ln(\sqrt{b-y} + 1) |q(y)| \, dy \\
&\quad + C \int_{|x-y| \leqslant \gamma} \ln \frac{1}{|x-y|} |q(y)| \, dy + C \int_{y \leqslant A, |x-y| > \gamma} \ln(\sqrt{(b-y)/\gamma} + 1) |q(y)| \, dy \\
&\leqslant C \int_{|x-y| \leqslant \gamma} \ln \frac{1}{|x-y|} |q(y)| \, dy + C \int_{-\infty}^{A} \ln(\sqrt{(b-y)/\gamma} + 1) |q(y)| \, dy.
\end{aligned}$$

The assumption $q \in \mathscr{J}^{\alpha}$ and $\ln(|y|^{1/2} + 1) q(y) \in L^1(R^1)$ yield now the above claim.

The conclusion now follows from the fact that $G_{(-\infty,b)}(x, y) |q(y)|$ is dominated by the expression

$$C \ln\left(\sqrt{\frac{b-y}{|x-y|}} + 1\right) |q(y)|,$$

which converges to 0 as $x \to -\infty$ and is uniformly with respect to x integrable over the interval (A, b) for fixed but otherwise arbitrary $A < b$.

The proof of the case $\alpha < 1$ is even easier and is omitted.

We now show that if $b < b_0 = \sup\{b \in R^1;\, M_1(b) < 1\}$, then

$$\lim_{x \to -\infty} u(x, b) = 1.$$

We claim that

$$\lim_{x \to -\infty} E^x \Big(\int_0^{\tau_b} |q(X_t)| \, dt\Big)^n = 0, \qquad n = 1, 2, \ldots$$

Observe that since

$$E^x \int_0^{\tau_b} |q(X_t)|\, dt = G_{(-\infty,b)} |q| (x),$$

this has just been shown for $n = 1$. Next, if we define

$$f_n(x) = \frac{1}{n!} E^x \Big(\int_0^{\tau_b} |q(X_t)|\, dt\Big)^n,$$

then $f_{n+1} = G_{(-\infty,b)}(|q|\, f_n)$ for $n \geqslant 1$. However, $\sup_{x \leqslant b} |f_n(x)| \leqslant M_1(b)^n$, so qf_n satisfy all the assumptions imposed on q and this justifies the claim. Now, since $b < b_0$, we have $M_1(b) < 1$, so for a given $\varepsilon > 0$ we can find n_0 such that

$$\sum_{n \geqslant n_0} M_1(b)^n \leqslant \varepsilon/2.$$

If we now choose $x_0 < b$ such that for $x \leqslant x_0$

$$\sum_{n=0}^{n_0-1} |M_n(x, b)| \leqslant \varepsilon/2,$$

then for $x < x_0$ we obtain $|u(x, b) - 1| \leqslant \varepsilon$, and the conclusion follows.

We conclude our considerations by providing a condition opposite to those described in Theorem 8.

PROPOSITION 9. *Suppose that* $q \in \mathscr{J}^\alpha$ *and* $q \geqslant 0$. *If*

$$\int_{-\infty}^{b} \frac{q(y)}{(1+|y|)^{1-\alpha/2}}\, dy = \infty,$$

then $\beta = -\infty$. *If* $\alpha > 1$ *and*

$$\int_{-\infty}^{b} |y|^{\alpha-1} q(y)\, dy = \infty,$$

then either $\beta = -\infty$ *or* $\lim_{x \to -\infty} u(x, b) = \infty$.

Proof. By virtue of Jensen's inequality

$$u(x, b) \geqslant \exp\big(G_{(-\infty,b)}\, q(x)\big),$$

it is enough to analyse $G_{(-\infty,b)}\, q(x)$. Assume that the first condition holds true and suppose that $y < 2x - b = x - (b - x) < x$. Then we have $(b-x)/(x-y) < 1$. Thus, we obtain

$$G_{(-\infty,b)}\, q(x) \geqslant \frac{1}{\Gamma(\alpha/2)^2} \int_{-\infty}^{2x-b} (x-y)^{\alpha-1} \int_0^{(b-x)/(x-y)} \frac{u^{\alpha/2-1}\, du}{(u+1)^{1-\alpha/2}}\, q(y)\, dy$$

$$\geqslant \frac{2^{\alpha/2}}{\alpha \Gamma(\alpha/2)^2} \int_{-\infty}^{2x-b} (x-y)^{\alpha/2-1} (b-x)^{\alpha/2} q(y)\, dy.$$

Since obviously $x-y \leqslant b-y$, the last expression is not less than

$$\frac{2^{\alpha/2}}{\alpha\Gamma(\alpha/2)^2}(b-x)^{\alpha/2}\int_{-\infty}^{2x-b}\frac{q(y)}{(b-y)^{1-\alpha/2}}dy \geqslant \frac{2^{\alpha/2}}{\alpha\Gamma(\alpha/2)^2}\frac{(b-x)^{\alpha/2}}{b^{1-\alpha/2}}\int_{-\infty}^{2x-b}\frac{q(y)}{(1+|y|)^{1-\alpha/2}}dy$$

if only $b > 0$ and $y < 0$. Indeed, we then have $b-y < b-by$. This proves the first part of the proposition, since the last integral is infinite.

Let now $\alpha > 1$. Observe that for $2y > b+x$ we obtain $b-y < y-x$, so $(b-y)/(y-x) < 1$. Thus, we have

$$\begin{aligned} G_{(-\infty,b)}q(x) &\geqslant \frac{1}{\Gamma(\alpha/2)^2}\int_{(b+x)/2}^{b}(y-x)^{\alpha-1}\int_{0}^{(b-y)/(y-x)}\frac{u^{\alpha/2-1}\,du}{(u+1)^{1-\alpha/2}}q(y)\,dy \\ &\geqslant \frac{2^{\alpha/2}}{\alpha\Gamma(\alpha/2)^2}\int_{(b+x)/2}^{b}(y-x)^{\alpha-1}\left(\frac{b-y}{y-x}\right)^{\alpha/2}q(y)\,dy \\ &\geqslant \frac{2^{\alpha/2}}{\alpha\Gamma(\alpha/2)^2}\int_{(b+x)/2}^{b}(b-y)^{\alpha-1}q(y)\,dy \geqslant \frac{2^{\alpha/2}}{\alpha\Gamma(\alpha/2)^2}\int_{(b+x)/2}^{0}|y|^{\alpha-1}q(y)\,dy \end{aligned}$$

because for $y < 0$ and $b > 0$ we obtain $b-y > -y = |y|$. When $x \to -\infty$, the conclusion follows, completing the proof of the proposition.

We now indicate how to compute b_0, given by the formula (12) for $q = \mathbf{1}_{(-c,c)}$ with $c > 0$. We start with transforming the Green operator $G_{(-\infty,b)}$ into a form which is more suitable for computation. We always assume that $q \in \mathscr{J}^\alpha \cap L^1(R^1)$. Then Theorem 7 applies, so for every $x \leqslant b$ we obtain $G_{(-\infty,b)}|q|(x) < \infty$, and this justifies changing the order of integration in the following calculations:

$$\begin{aligned} G_{(-\infty,b)}q(x) &= \int_{-\infty}^{b}\frac{1}{\Gamma(\alpha/2)^2}\left\{\int_{0}^{(b-x)\wedge(b-y)}\frac{u^{\alpha/2-1}\,du}{(u+|x-y|)^{1-\alpha/2}}\right\}q(y)\,dy \\ &= \frac{1}{\Gamma(\alpha/2)^2}\int_{-\infty}^{x}\left\{\int_{0}^{b-x}\frac{du}{\big(u(u+x-y)\big)^{1-\alpha/2}}\right\}q(y)\,dy \\ &\quad+\frac{1}{\Gamma(\alpha/2)^2}\int_{x}^{b}\left\{\int_{0}^{b-y}\frac{du}{\big(u(u+y-x)\big)^{1-\alpha/2}}\right\}q(y)\,dy \\ &= \frac{1}{\Gamma(\alpha/2)^2}\int_{-\infty}^{x}\left\{\int_{x}^{b}\frac{du}{\big((u-x)(u-y)\big)^{1-\alpha/2}}\right\}q(y)\,dy \\ &\quad+\frac{1}{\Gamma(\alpha/2)^2}\int_{x}^{b}\left\{\int_{y}^{b}\frac{du}{\big((u-x)(u-y)\big)^{1-\alpha/2}}\right\}q(y)\,dy \\ &= \frac{1}{\Gamma(\alpha/2)^2}\int_{x}^{b}\left\{\int_{-\infty}^{x}\frac{q(y)\,dy}{\big((u-x)(u-y)\big)^{1-\alpha/2}}\right\}du \end{aligned}$$

$$+\frac{1}{\Gamma(\alpha/2)^2}\int_x^b\left\{\int_x^u\frac{q(y)\,dy}{((u-x)(u-y))^{1-\alpha/2}}\right\}du$$
$$=\frac{1}{\Gamma(\alpha/2)^2}\int_x^b\frac{du}{(u-x)^{1-\alpha/2}}\left\{\int_{-\infty}^u\frac{q(y)\,dy}{(u-y)^{1-\alpha/2}}\right\}.$$

We also define for $0 \leqslant u < 1$

$$P_1^\alpha(u)=\int_0^u\frac{t^{\alpha/2-1}\,dt}{(1-t)^{\alpha+1}} \quad\text{and}\quad P_2^\alpha(u)=\int_0^u\frac{t^{\alpha/2}\,dt}{(1-t)^{\alpha+1}}. \tag{13}$$

We are now able to examine the above-mentioned example. As usual, we assume here that $\alpha\in(0, 2)$.

EXAMPLE. *Let* $q = \mathbb{1}_{(-c,c)}$, $c > 0$. *For* $b > -c$ *we obtain*

$$\sup_{x\leqslant b} G_{(-\infty,b)}q(x) = \max_{-c<x<c} G_{(-\infty,b)}q(x).$$

For $-c < b \leqslant c$ *the greatest value of* $G_{(-\infty,b)}q(x)$ *is attained at the point* x_0 *determined by the unique solution* u_0 *of the equation*

$$\alpha(1-u)^\alpha P_1^\alpha(u) = u^{\alpha/2-1}, \quad 0 < u < 1, \tag{14}$$

with $u = (b-x)/(b+c)$. *The corresponding maximal value* $G_{(-\infty,b)}q(x_0)$ *is given as*

$$\sup_{x\leqslant b} G_{(-\infty,b)}q(x) = \frac{(b+c)^\alpha}{2\Gamma(1+\alpha/2)^2}u_0^{\alpha/2-1}. \tag{15}$$

For $b \geqslant c$ *the corresponding value* u_1 *is the unique solution of the equation*

$$(1-u)^{\alpha-1}P_1^\alpha(u)+(u-d)^{\alpha-1}P_2^\alpha(d/u) = \frac{u^{\alpha/2-1}}{\alpha}\left(\frac{1}{1-u}+\frac{d^{\alpha/2+1}}{u-d}\right), \tag{16}$$

$0 < u < 1$, *where* $d = (b-c)/(b+c)$ *and the maximal value of the Green operator is given by*

$$\frac{2(b+c)^\alpha}{\alpha\Gamma(\alpha/2)^2}\left[(1-d)(1-u_1)^{\alpha-1}P_1^\alpha(u_1)-\frac{u_1^{\alpha/2-1}}{\alpha}\left(\frac{u_1-d}{1-u_1}+d^{\alpha/2+1}\right)\right]. \tag{17}$$

We first note that since $G_{(-\infty,b)}(x, y) = 0$ for $y \geqslant b$, we obtain

$$G_{(-\infty,b)}\mathbb{1}_{(-c,c)}(x) = \int_{-\infty}^b G_{(-\infty,b)}(x, y)\mathbb{1}_{(-c,c)}(y)\,dy = 0$$

whenever $b \leqslant -c$. We therefore assume throughout the remainder that $b > -c$.

By the form of the Green operator we obtain

$$(18)\qquad G_{(-\infty,b)}\,q(x) = \frac{1}{\Gamma(\alpha/2)^2}\int_x^b \frac{du}{(u-x)^{1-\alpha/2}}\left\{\int_{-\infty}^{u}\frac{\mathbf{1}_{(-c,c)}(y)\,dy}{(u-y)^{1-\alpha/2}}\right\}$$

$$= \frac{1}{\Gamma(\alpha/2)^2}\int_{x\vee(-c)}^{b} \frac{du}{(u-x)^{1-\alpha/2}}\left\{\int_{-c}^{u\wedge c}\frac{dy}{(u-y)^{1-\alpha/2}}\right\}$$

$$= \frac{c^{\alpha}}{\Gamma(\alpha/2)^2}\int_{x'\vee(-1)}^{b'} \frac{du}{(u-x')^{1-\alpha/2}}\left\{\int_{-1}^{u\wedge 1}\frac{dy}{(u-y)^{1-\alpha/2}}\right\},$$

where $x' = x/c$ and $b' = b/c$.

Assume first that $x \leqslant -c$. By (18) we then obtain

$$(19)\qquad G_{(-\infty,b)}\,\mathbf{1}_{(-c,c)}(x) = \frac{c^{\alpha}}{\Gamma(\alpha/2)^2}\int_{-1}^{b'} \frac{du}{(u-x')^{1-\alpha/2}}\left\{\int_{-1}^{u\wedge 1}\frac{dy}{(u-y)^{1-\alpha/2}}\right\}$$

$$= \frac{2c^{\alpha}}{\alpha\Gamma(\alpha/2)^2}\int_{-1}^{b'}\frac{(u+1)^{\alpha/2}\,du}{(u-x')^{1-\alpha/2}}.$$

Since the above expression is a nondecreasing function of x, we obtain

$$G_{(-\infty,b)}\,\mathbf{1}_{(-c,c)}(x) \leqslant G_{(-\infty,b)}\,\mathbf{1}_{(-c,c)}(-c), \qquad x \leqslant -c.$$

Direct calculations provide the value of the last quantity for $|b| \leqslant c$ as $2(b+c)^{\alpha}/\alpha^2\,\Gamma(\alpha/2)^2$ or

$$\frac{2}{\alpha\Gamma(\alpha/2)^2}\left[\frac{(b+c)^{\alpha}}{\alpha} - (2c)^{\alpha}\,P_2^{\alpha}\left(\frac{b-c}{b+c}\right)\right]$$

whenever $b \geqslant c$. We now write (19) in a form which is more suitable for computations. We introduce a new variable t defined by the formula $u+1 = (-1-x)/(t-1)$. We then obtain

$$u-x = t(-1-x)/(t-1) \quad\text{ and }\quad du = (1+x)\,dt/(t-1)^2$$

and, after substituting, we get

$$G_{(-\infty,b)}\,q(x) = \frac{2(b+c)^{\alpha}}{\alpha\Gamma(a/2)}(u-1)^{\alpha}\int_u^{\infty}\frac{t^{\alpha/2-1}}{(t-1)^{\alpha+1}}\,dt,$$

where $u = (b-x)/(b+c)$. Substituting once again $v = 1/t$, $t-1 = (1-v)/v$, $dt = -dv/v^2$ we finally obtain

$$(20)\qquad G_{(-\infty,b)}\,q(x) = \frac{2(b+c)^{\alpha}}{\alpha\Gamma(\alpha/2)^2}(u-1)^{\alpha}\,P_2^{\alpha}(1/u).$$

Assume now that $c \leqslant x \leqslant b$. By (18) we obtain

$$G_{(-\infty,b)}\,\mathbf{1}_{(-c,c)}(x) = \frac{c^{\alpha}}{\Gamma(\alpha/2)^2}\int_{x'}^{b'}\frac{du}{(u-x')^{1-\alpha/2}}\left\{\int_{-1}^{1}\frac{dy}{(u-y)^{1-\alpha/2}}\right\}.$$

Computing the inner integral and using integration by parts we further obtain

$$\frac{c^\alpha}{\Gamma(\alpha/2)^2}\int_{x'}^{b'}\frac{(u+1)^{\alpha/2}-(u-1)^{\alpha/2}}{(u-x')^{1-\alpha/2}}\,du$$

$$=\frac{4(b-x)^{\alpha/2}}{\alpha^2\Gamma(\alpha/2)^2}[(b+c)^{\alpha/2}-(b-c)^{\alpha/2}]$$

$$+\frac{2}{\alpha\Gamma(\alpha/2)^2}\int_x^b (u-x)^{\alpha/2}\left[\frac{1}{(u-c)^{1-\alpha/2}}-\frac{1}{(u+c)^{1-\alpha/2}}\right]du.$$

Since the above expression is a nonincreasing function of x, we obtain

$$G_{(-\infty,b)}\mathbb{1}_{(-c,c)}(x)\leqslant G_{(-\infty,b)}\mathbb{1}_{(-c,c)}(c),\qquad c\leqslant x<b.$$

Direct calculations again provide the last value as

$$\frac{2}{\alpha^2\Gamma(\alpha/2)^2}\left[2(b^2-c^2)^{\alpha/2}-(b-c)^\alpha-\alpha(2c)^\alpha P_2^\alpha\left(\frac{b-c}{b+c}\right)\right].$$

It is easy to see that $G_{(-\infty,b)}q(-c)>G_{(-\infty,b)}q(c)$.

We now consider the case $-c<x<c$. We may and do assume here that $c=1$. The expression (18) takes then, up to constants, the form

$$\int_x^b\frac{du}{(u-x)^{1-\alpha/2}}\int_{-1}^u\frac{dy}{(u-y)^{1-\alpha/2}}=\frac{2}{\alpha}\int_x^b\frac{(u+1)^{\alpha/2}}{(u-x)^{1-\alpha/2}}\,du. \tag{21}$$

Integrating by parts the right-hand side, we obtain

$$\left(\frac{2}{\alpha}\right)^2(b+1)^{\alpha/2}(b-x)^{\alpha/2}-\frac{2}{\alpha}\int_x^b\frac{(u-x)^{\alpha/2}\,du}{(u+1)^{1-\alpha/2}}.$$

Taking the derivative of the above expression with respect to x we obtain

$$-\frac{2}{\alpha}(b+1)^{\alpha/2}(b-x)^{\alpha/2-1}+\int_x^b\frac{du}{\big((u+1)(u-x)\big)^{1-\alpha/2}}. \tag{22}$$

Observe that for $\alpha\leqslant 1$ the integral above tends to ∞ when $x\downarrow -1$, so the derivative also tends to ∞. For $\alpha>1$ it follows that it converges to

$$(b+1)^{\alpha-1}\big(1/(\alpha-1)-1/\alpha\big)>0.$$

On the other hand, when $x\uparrow b$, the integral above tends to 0 while the first term converges to $-\infty$. Consequently, the derivative tends to $-\infty$. Therefore, there exists $x_0\in(-1,b)$ where the expression (21) attains the maximum. Integrating by parts once again in (22) we see that x_0 is a solution of the equation

$$\frac{2-\alpha}{\alpha}\int_x^b\frac{(u-x)^{\alpha/2}}{(u+1)^{2-\alpha/2}}\,du=\frac{2(1+x)}{\alpha\big((b+1)(b-x)\big)^{1-\alpha/2}}. \tag{23}$$

Since the left-hand side of (23) is decreasing and the right-hand side is increasing in x, the solution x_0 is unique.

As before, we again transform the expression (21). We introduce a new variable t defined by the formula $u+1 = (1+x)/(1-t)$. We then obtain

$$u-x = t(1+x)/(1-t) \quad \text{and} \quad du = (1+x)\,dt/(1-t)^2.$$

Taking into account the appropriate constants we obtain from (21)

$$G_{(-\infty,b)}\,q(x) = \frac{2(b+c)^\alpha}{\alpha\Gamma(\alpha/2)^2}(1-u)^\alpha P_1^\alpha(u), \tag{24}$$

where, as before, $u = (b-x)/(b+c)$, $0 < u < 1$.

The maximal value of the expression (24) is determined now by the root of its derivative:

$$\alpha(1-u)^{\alpha-1} P_1^\alpha(u) = (1-u)^\alpha \frac{u^{\alpha/2-1}}{(1-u)^{\alpha+1}},$$

and this equation is equivalent to (14). Taking this into account we obtain (15).

We now consider the case when $b \geqslant c$ and $-c < x < c$. Again, we assume for time being that $c = 1$. By the formula (18) we then obtain

$$\begin{aligned}\Gamma(\alpha/2)^2\, G_{(-\infty,b)}\,q(x) &= \int_x^1 \frac{du}{(u-x)^{1-\alpha/2}} \int_{-1}^{u} \frac{dy}{(u-y)^{1-\alpha/2}} + \int_1^b \frac{du}{(u-x)^{1-\alpha/2}} \int_{-1}^{1} \frac{dy}{(u-y)^{1-\alpha/2}} \\ &= \frac{2}{\alpha}\int_x^b \frac{(u+1)^{\alpha/2}}{(u-x)^{1-\alpha/2}}\,du - \frac{2}{\alpha}\int_1^b \frac{(u-1)^{\alpha/2}}{(u-x)^{1-\alpha/2}}\,du.\end{aligned}$$

We introduce a new variable in the first integral in the expression above by the formula $u+1 = (1+x)/(1-t)$. We then obtain

$$u-x = t(1+x)/(1-t), \qquad du = (1+x)\,dt/(1-t)^2$$

and

$$\frac{2}{\alpha}\int_x^b \frac{(u+1)^{\alpha/2}}{(u-x)^{1-\alpha/2}}\,du = \frac{2}{\alpha}(1+x)^\alpha P_1^\alpha\left(\frac{b-x}{b+1}\right).$$

Analogously, substituting $u-x = (1-x)/(1-t)$ in the second integral we obtain

$$u-1 = t(1-x)/(1-t), \qquad du = (1-x)\,dt/(1-t)^2$$

and

$$\frac{2}{\alpha}\int_1^b \frac{(u-1)^{\alpha/2}}{(u-x)^{1-\alpha/2}}\,du = \frac{2}{\alpha}(1-x)^\alpha P_2^\alpha\left(\frac{b-1}{b-x}\right).$$

Denoting $(b-x)/(b+c)$ by u and $(b-c)/(b+c)$ by d and taking into account constants, we transform the expression $\Gamma(\alpha/2)^2 G_{(-\infty,b)} q(x)$ into the following form:

$$\frac{2}{\alpha}(b+c)^\alpha [(1-u)^\alpha P_1^\alpha(u)-(u-d)^\alpha P_2^\alpha(d/u)]. \tag{25}$$

On the other hand, by integration by parts we obtain

$$P_1^\alpha(u)+P_2^\alpha(u) = \frac{2}{\alpha}\frac{u^{\alpha/2}}{(1-u)^\alpha},$$

which allows us to transform (25) into the following:

$$\frac{2}{\alpha}(b+c)^\alpha\left[(1-u)^\alpha P_1^\alpha(u)+(u-d)^\alpha P_1^\alpha(d/u)-\frac{2}{\alpha}u^{\alpha/2}d^{\alpha/2}\right]. \tag{26}$$

The derivative of (26) again leads to the equation

$$(1-u)^{\alpha-1}P_1^\alpha(u)-(u-d)^{\alpha-1}P_1^\alpha(d/u) = \frac{u^{\alpha/2-1}}{\alpha}\left[d^{\alpha/2}+\frac{d^{\alpha/2}u}{u-d}-\frac{1}{1-u}\right]. \tag{27}$$

It is not difficult to observe that (27) is equivalent to (16). The justification of the existence of the unique solution u_1 satisfying (27) is similar to that of (14) and is omitted. Taking into account (26) and (27) we obtain (17).

Although we do not need the expressions for $G_{(-\infty,b)} q(x)$ for $x < -c$ or for $c < x < b$, we provide them for the sake of completeness. We begin with the case $x < -c$:

$$G_{(-\infty,b)} q(x) = \frac{2(b+c)^\alpha}{\alpha\Gamma(\alpha/2)^2}[(u-1)^\alpha P_2^\alpha(1/u)-(u-d)^\alpha P_2^\alpha(d/u)], \tag{28}$$

where $u = (b-x)/(b+c) > 1$. For $c < x < b$ we obtain, analogously,

$$G_{(-\infty,b)} q(x) = \frac{2(b+c)^\alpha}{\alpha\Gamma(\alpha/2)^2}[(1-u)^\alpha P_1^\alpha(u)-(d-u)^\alpha P_1^\alpha(u/d)] \tag{29}$$

with $u = (b-x)/(b+c) < d$.

We now provide a more detailed analysis for the case $\alpha = 1$. Note that in this case we compute directly the integrals (13) as follows:

$$P_1^1(u) = \frac{\sqrt{u}}{1-u}+\frac{1}{2}\ln\frac{1+\sqrt{u}}{1-\sqrt{u}}, \tag{30}$$

$$P_2^1(u) = \frac{\sqrt{u}}{1-u}-\frac{1}{2}\ln\frac{1+\sqrt{u}}{1-\sqrt{u}}. \tag{31}$$

Substituting (30) into (14) we obtain the equation

$$\sqrt{u}(1-u)\left(\frac{\sqrt{u}}{1-u}+\frac{1}{2}\ln\frac{1+\sqrt{u}}{1-\sqrt{u}}\right) = 1$$

or, equivalently,

$$\frac{\sqrt{u}}{2}\ln\frac{1+\sqrt{u}}{1-\sqrt{u}} = 1, \quad 0 < u < 1. \tag{32}$$

Substituting $\exp 2t = (1+\sqrt{u})/(1-\sqrt{u})$, $0 < t < \infty$, we obtain $\sqrt{u} = \tanh t$, and so (32) takes the form

$$t \tanh t = 1, \quad 0 < t < \infty. \tag{33}$$

Let t_0 be the unique root of the equation (33). Then we obtain

$$\sup_{x \leqslant b} G_{(-\infty,b)} q(x) = 2\pi^{-1}(b+c)t_0.$$

Assume that $c > \pi/4t_0$. We get $b_0 = \pi/2t_0 - c$.

Let now $b \geqslant c$ and $-c < x < c$. From (16) we have

$$P_1^1(u) + P_2^1(d/u) = \frac{1}{\sqrt{u}}\left(\frac{1}{1-u} + \frac{d^{3/2}}{u-d}\right).$$

By (30) and (31) we then obtain equivalently

$$\ln\left(\frac{1+\sqrt{u}}{1-\sqrt{u}}\frac{\sqrt{u}-\sqrt{d}}{\sqrt{u}+\sqrt{d}}\right) = 2\frac{1-\sqrt{d}}{\sqrt{u}}. \tag{34}$$

If we introduce a new variable by the formula

$$\exp 2t = \frac{1+\sqrt{u}}{1-\sqrt{u}}\frac{\sqrt{u}-\sqrt{d}}{\sqrt{u}+\sqrt{d}}, \quad 0 < t < \infty,$$

then we obtain

$$\tanh t = \frac{u-\sqrt{d}}{\sqrt{u}(1-\sqrt{d})}, \tag{35}$$

so the equation (34) is being transformed into

$$\sqrt{u}\, t = 1-\sqrt{d}. \tag{36}$$

The equations (35) and (36) provide then the following equivalent version of (16):

$$\tanh t = \frac{1}{t} - \frac{\sqrt{d}}{(1-\sqrt{d})^2}t. \tag{37}$$

Let t_1 be the unique root of (37) and u_1 the corresponding value of the original variable. According to (17) we then obtain

$$\sup_{x \leqslant b} G_{(-\infty,b)} q(x) = \frac{2(b+c)}{\pi}\left[\frac{1-d}{2}\ln\frac{1+\sqrt{u_1}}{1-\sqrt{u_1}} + \frac{d}{\sqrt{u_1}}(1-\sqrt{d})\right]. \tag{38}$$

If we put $\exp 2w_1 = (1+\sqrt{u_1})/(1-\sqrt{u_1})$, then $\tanh w_1 = \sqrt{u_1}$ and (38) becomes

$$\sup_{x \leqslant b} G_{(-\infty,b)} q(x) = \frac{2(b+c)}{\pi}[(1-d)w_1 + dt_1]. \tag{39}$$

Remark. In the Brownian motion case the Green function equals $G_{(-\infty,b)}(x, y) = (b-x) \wedge (b-y)$, $x, y \leqslant b$, and the formula (18) is also valid. In this case the greatest value of $G_{(-\infty,b)} q(x)$ is attained at $x = -c$, for q as before, and is equal to $(b+c)^2/2$ if $-c < b < c$ and to $2bc$ if $c < b$. Thus, $b_0 = \sqrt{2}-c$ if $c > \sqrt{2}/2$ and $b_0 = 1/(2c)$ if $c \leqslant \sqrt{2}/2$. However, even in that case $b_0 < \beta$ (see the Example in Section 9 of [7]) and to determine β the more advanced methods are required.

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REFERENCES

[1] R. M. Blumenthal and R. K. Getoor, *Markov Processes and Their Potential Theory*, Pure Appl. Math., Academic Press, New York 1968.

[2] R. M. Blumenthal, R. K. Getoor and D. B. Ray, *On the distribution of first hits for the symmetric stable processes*, Trans. Amer. Math. Soc. 99 (1961), pp. 540–554.

[3] K. Bogdan and T. Byczkowski, *Potential theory for the α-stable Schrödinger operator on bounded Lipschitz domains*, Studia Math. 133 (1999), pp. 53–92.

[4] K. Bogdan and T. Byczkowski, *Potential theory of Schrödinger operator based on fractional Laplacian*, Probab. Math. Statist. 20 (2000), pp. 293–335.

[5] Z.-Q. Chen and R. Song, *Intrinsic ultracontractivity and conditional gauge for symmetric stable processes*, J. Funct. Anal. 150 (1997), pp. 204–239.

[6] Z.-Q. Chen and R. Song, *Intrinsic ultracontractivity, conditional lifetimes and conditional gauge for symmetric stable processes on rough domains*, Illinois J. Math. 44 (2000), pp. 138–160.

[7] K. L. Chung and Z. Zhao, *From Brownian Motion to Schrödinger's Equation*, Springer, New York 1995.

[8] D. B. Ray, *Stable processes with an absorbing barrier*, Trans. Amer. Math. Soc. 89 (1958), pp. 16–24.

[9] K. Samotij, *Non-Brownian Schrödinger operators*, preprint, 1997, pp. 1–35.

[10] Z. Zhao, *A probabilistic principle and generalized Schrödinger perturbation*, J. Funct. Anal. 101 (1991), pp. 162–176.

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