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ON THE DUGUÉ PROBLEM WITH A SOLUTION IN THE SET OF SIGNED MEASURES

BY

MAREK T. MALINOWSKI* AND JOLANTA K. MISIEWICZ* (ZIELONA GÓRA)

Abstract. There are two methods of obtaining symmetric probability measure on a base of an arbitrary probability measure μ corresponding to the random variable X. The first relies on considering distribution of $Y = X - X'$, where X' is an independent copy of X. In the language of measures we have then $\mathscr{L}(Y) = \mu * \mu^-$, where $\mu^-(A) = \mu(-A)$. In the second method we consider the mean of two measures μ and μ^-. In the paper we want to present some known and new results on characterizing such measures μ for which both methods coincide, i.e. measures for which

$$\tfrac{1}{2}(\mu + \mu^-) = \mu * \mu^-.$$

In the literature one can find also the following generalization of this question: for fixed $p \in (0, 1]$ what is the characterization of such pairs of distributions μ and ν for which

$$p\mu + (1-p)\nu = \mu * \nu?$$

This problem was posed by Dugué in 1939 and it was extensively studied since then. However, the full characterization has not been found yet. In the paper we show some constructions of the Dugué question with the properties of simple fractions classes of characteristic functions. We give also a collection of new solutions and an example of three measures μ, ν and η such that

$$p\mu + q\nu + r\eta = \mu * \nu * \eta.$$

In the last section we give also some solutions in the set of signed σ-finite measures. The authors would like to express their gratitude to Professor D. Szynal for his interesting questions and discussions.

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* Institute of Mathematics, University of Zielona Góra.

1. INTRODUCTION

In this paper we study the following problem: For fixed probability distribution μ and fixed constant $p \in (0, 1]$ we want to find whether or not there exists any signed, maybe σ-finite measure ν such that

$$p\mu + (1-p)\nu = \mu * \nu. \tag{1}$$

If φ and ψ are the Fourier transforms of the measures μ and ν, respectively, then the equation (1) can be rewritten in the following form:

$$p\varphi(t) + (1-p)\psi(t) = \varphi(t)\psi(t). \tag{1'}$$

This problem, in a slightly different form, was posed by D. Dugué in 1939 (see [1], [2]). It was studied later by Szynal and Wolińska (see [8], [10], [11]). Also recently there appeared a paper of Krakowiak (see [3]). The beginning is very simple: in probability there are two methods of symmetrization of the given distribution μ; one is based on taking the mean of the original measure and its symmetric image; the other leads to the distribution of $X - X'$, where X and X' are independent with the same distribution μ. Dugué's question was to characterize such distributions μ for which these two methods coincide, i.e. to characterize such distributions μ for which

$$\tfrac{1}{2}(\mu + \mu^-) = \mu * \mu^-, \tag{2}$$

where $\mu^-(A) = \mu(-A)$ for every Borel set A. If φ is the characteristic function of μ, we can equivalently say that we want to characterize such characteristic functions φ for which

$$\mathrm{Re}\varphi = |\varphi|^2. \tag{2'}$$

After some generalizations (convex linear combination instead of mean and any probability distribution ν instead of μ^-) the Dugué problem was to characterize pairs of probability distributions (μ, ν) such that for given $p \in (0, 1]$ the following equation is satisfied:

$$p\mu + (1-p)\nu = \mu * \nu.$$

In this paper we denote by Φ the set of all characteristic functions. For every $p \in (0, 1]$ and $q = 1-p$, we define two operators T_0^p, T_1^p: $\Phi \to \Phi$ by the following formulas:

$$T_0^p(\varphi) = \frac{p}{1-q\varphi}, \quad T_1^p(\varphi) = \frac{p\varphi}{1-q\varphi}.$$

Calculating the Fourier transform ψ of the measure ν from equation (1′) we obtain

$$\psi = \frac{p\varphi}{\varphi - q} \overset{\text{def}}{=} G^p(\varphi).$$

For every $p \in (0, 1]$ the function $G^p(\varphi)$ is well defined on the set $\varphi(t) \neq q$. Notice that in general $G^p(\varphi)$ does not have to be a characteristic function.

The following lemma will be very useful in this paper:

LEMMA 1. *For every $p \in (0, 1]$ and every probability measure μ with $\hat{\mu}(t) = \varphi(t)$ the functions $T_0^p(\varphi)$ and $T_1^p(\varphi)$ are the characteristic functions of the following measures:*

$$T_0^p(\mu) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} pq^k \mu^{*k}, \qquad T_1^p(\mu) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} pq^{k-1} \mu^{*k}.$$

Proof. It is enough to calculate the characteristic functions for the measures $T_0^p(\mu)$ and $T_1^p(\mu)$. ∎

2. WHEN DO TWO METHODS OF SYMMETRIZATION COINCIDE?

We consider the equations (2) and (2′). In [2] Dugué noticed that the characteristic function $\varphi(t) = 1/(1-it/a)$ of an exponential distribution $\Gamma(1, a)$ satisfies the equation $(\varphi+\bar{\varphi})/2 = \varphi\bar{\varphi}$. This means that for $\Gamma(1, a)$ two types of stochastic symmetrization coincide.

PROPOSITION 1. *If a probability measure μ satisfies the equation (2), then there exists a function $u\colon \boldsymbol{R} \to \boldsymbol{R}$ such that the characteristic function $\hat{\mu}(t)$ of the measure μ is of the form*

$$\hat{\mu}(t) = \tfrac{1}{2}+\tfrac{1}{2}\exp\{iu(t)\}.$$

The function $u(t)$ is continuous, uniquely determined and $u(-t) = -u(t)$.

Proof. We can write $\hat{\mu}(t) = x(t)+iy(t)$, where $x, y\colon \boldsymbol{R} \to \boldsymbol{R}$. Then using the equation (2′) we have $(2x(t)-1)^2+(2y(t))^2 = 1$. Now it is enough to define $2x(t)-1 = \sin u(t)$ and $2y(t) = \cos u(t)$. The properties of the function $u(t)$ are simple implications of general properties of characteristic functions. ∎

EXAMPLE. It is known that the following probability measures satisfy equation (2):

1) $\mu_1 = \frac{1}{2}\delta_0+\frac{1}{2}\delta_a$, $a \in \boldsymbol{R}$;

2) $\mu_2(dx) = ae^{-ax}\mathbf{1}_{(0,\infty)}(x)\,dx$, $a > 0$;

3) $\mu_3 = (\frac{1}{2}\delta_0+\frac{1}{2}\delta_a) * T_0^p(\delta_a)$, where $p \in (0, 1)$, $a \in \boldsymbol{R}$.

Let $\hat{\mu}_i(t) = \frac{1}{2}+\frac{1}{2}\exp\{u_i(t)\}$. Evidently, $\hat{\mu}_1(t) = \frac{1}{2}+\frac{1}{2}e^{ita}$, so $u_1(t) = at$. For the function $\hat{\mu}_2$ we have

$$\frac{1}{2}+\frac{1}{2}\exp\{iu_2(t)\} = \frac{1}{1-it/a}.$$

Taking the derivative of both the sides of this equation we obtain

$$u_2'(t)\exp\{iu_2(t)\} = \frac{2}{a(1-it/a)^2},$$

which leads us to the equation

$$u_2'(t) = \frac{2}{a\big(1+(t/a)^2\big)}.$$

Hence $u_2(t) = 2\arctan(t/a)+C$, and since $u_2(0) = 0$, we finally have

$$u_2(t) = 2\arctan(t/a).$$

For the measure μ_3 and its characteristic function we have the equation

$$\exp\{iu_3(t)\} = \frac{e^{ita}-q}{1-qe^{ita}}.$$

The complex logarithm gives

$$u_3(t) = \operatorname{Arg}\frac{\cos(at)-q+i\sin(at)}{1-q\cos(at)-iq\sin(at)}.$$

3. ON THE GENERAL DUGUÉ PROBLEM

For the first time the equation $p\varphi_1(t)+q\varphi_2(t) = \varphi_1(t)\varphi_2(t)$, $p+q = 1$, $p > 0$, $q > 0$, was considered by Kubik (see [4]). He gave two examples of pairs of measures for which the condition (1) is satisfied: a pair of discrete measures

$$\mu = q\delta_0+p\delta_{-a}, \qquad \nu = p\delta_0+q\delta_a, \qquad a\in \boldsymbol{R};$$

and a pair of exponential distributions

$$\mu(dx) = a\exp\{ax\}\mathbb{1}_{(-\infty,0)}(x)\,dx, \qquad \nu(dx) = \frac{ap}{q}\exp\left\{-\frac{ap}{q}x\right\}\mathbb{1}_{(0,\infty)}(x)\,dx.$$

Rossberg in [7] proved (see Theorem 3, p. 213) that only such a pair of exponential distributions can have all the following properties: $\operatorname{supp}(\mu) = (-\infty, 0)$, $\operatorname{supp}(\nu) = (0, \infty)$, ν is no lattice distribution, and $p\mu+q\nu = \mu*\nu$ for some $p = 1-q\in(0, 1)$.

It was proved (see [3], [11]) that if the pair (μ, ν) of probability measures satisfies (1), then both the measures are discrete or absolutely continuous or singular. In [8], [10], [11] one can find many examples of characteristic functions for which (1′) holds. Moreover, Wolińska in [10] and [11] gave some recursive formulas for a sequence of pairs of characteristic functions satisfying (1′) when we have at least one such pair.

For every $p\in(0, 1]$ we define the set $\mathscr{G}^p \subseteq \Phi$ by the following:

$$\mathscr{G}^p = \{\varphi\in\Phi\colon G^p(\varphi) \text{ is a characteristic function}\}.$$

The set $\mathscr{G} \subset \Phi$ is defined as

$$\mathscr{G} = \bigcap_{p\in(0,1]} \mathscr{G}^p.$$

It means that a characteristic function φ belongs to $\mathscr{G}$ if and only if $G^p(\varphi)$ is a characteristic function for every $p\in(0, 1]$. Krakowiak (see [3]) proved that for every $\varphi\in\Phi$ one of the following conditions holds:

1) $\varphi\notin\mathscr{G}^p$ for every $p\in(0, 1)$;

2) $\varphi\in\mathscr{G}$;

3) $\exists p_0\in(0, 1)$: $\varphi\in\mathscr{G}^p$ for every $p\in(0, p_0]$, and $\varphi\notin\mathscr{G}^p$ for $p > p_0$.

PROPOSITION 2. *If* $\varphi\in\mathscr{G}^p$ *for some* $p\in(0, 1)$, *then*:

(i) $|\varphi-(1+p)^{-1}| \geqslant p/(1+p)$;

(ii) $\varphi\in\mathscr{G}^u$ *for every* $u\in(0, p]$;

(iii) $\psi = G^p(\varphi)\in\mathscr{G}^u$ *for every* $u\in(0, 1-p]$.

Proof. Another proof of this proposition can be found in [3].

(i) We want to show that for $\varphi\in\mathscr{G}^p$ the function φ is taking values in the set

$$D_p = \left\{x+iy\colon \left(x-\frac{1}{1+p}\right)^2+y^2 \geqslant \frac{p^2}{(1+p)^2},\ x^2+y^2 \leqslant 1\right\}.$$

Let $\varphi(t) = x(t)+iy(t)$. Of course, $|\varphi| \leqslant 1$. Since $G^p(\varphi)\in\Phi$, we have $|G^p(\varphi)| \leqslant 1$. Thus

$$\left|\frac{p(x+iy)}{x+iy-q}\right| \leqslant 1.$$

It is easy to see that the set of $x+iy$ having these two properties is equal to D_p.

(ii) We have $\varphi\in\mathscr{G}^p$, i.e. $\psi = G^p(\varphi)\in\Phi$. Hence the function $T_1^r(\psi)\in\Phi$ for every $r\in(0, 1]$. On the other hand, we have

$$T_1^r(\psi) = \frac{r\dfrac{p\varphi}{\varphi-q}}{1-(1-r)\dfrac{p\varphi}{\varphi-q}} = \frac{rp\varphi}{\varphi(1-(1-r)p)-q}$$

$$= \frac{\dfrac{rp}{1-p+rp}\varphi}{\varphi-\dfrac{1-p}{1-p+rp}} = G^{rp/(1-p+rp)}(\varphi).$$

This proves that $G^u(\varphi)\in\Phi$ for every $u = rp/(1-p+rp)$, $r\in(0, 1]$, which was to be shown.

(iii) Let $v = 1-u$. It is easy to see that

$$G^u(\psi) = \frac{u\dfrac{p\varphi}{\varphi-q}}{\dfrac{p\varphi}{\varphi-q}-v} = \frac{up\varphi}{vq-(v-p)\varphi} = \frac{\dfrac{up}{vq}\varphi}{1-\left(\dfrac{v-p}{vq}\right)\varphi} = T_1^{(up)/(vq)}(\varphi).$$

There must be $0<(up)/(vq)\leqslant 1$ and $v-p\geqslant 0$. Thus $0<u\leqslant 1-p$. ∎

Corollary 1. *If $\varphi = \bar{\varphi}$ and $\varphi\in\mathcal{G}^p$ for some $p\in(0, 1)$, then $\varphi(t)\equiv 1$, since the real part of the set D_p is equal to $[-1, (1-p)/(1+p)]\cup\{1\}$ and the function φ is continuous.*

Proposition 3. *The characteristic functions of the following distributions belong to $\mathcal{G}$:*

(i) *Dirac measure $\mu = \delta_a$, where $a\in\mathbf{R}$;*

(ii) *exponential distribution $\mu(dx) = ae^{-ax}\mathbf{1}_{(0,\infty)}(x)\,dx$, $a>0$;*

(iii) *geometric distribution $\mu = s\sum_{k=1}^{\infty}u^{k-1}\delta_k$, $s, u\in(0, 1)$, $s+u=1$.*

Proof. To prove these facts it is enough to notice that $\nu = G^p(\mu)$ is a probability measure for every $p\in(0, 1]$.

(i)
$$G^p(e^{ita}) = \frac{pe^{ita}}{e^{ita}-q} = \frac{p}{1-qe^{-ita}} = T_0^p(e^{-ita}).$$

Thus $G^p(e^{ita})\in\Phi$ because $T_0^p(e^{-ita})$ is the characteristic function of the distribution

$$\nu = T_0^p(\delta_{-a}) = p\sum_{k=0}^{\infty}q^k\delta_{-ka}.$$

(ii)
$$\hat{\mu}(t) = \frac{a}{a-it}.$$

Hence

$$G^p(\hat{\mu}) = \frac{\dfrac{pa}{a-it}}{\dfrac{a}{a-it}-q} = \frac{pa}{pa+itq} = \frac{\dfrac{pa}{q}}{\dfrac{pa}{q}+it}.$$

It is a characteristic function of the exponential distribution with a density function

$$\nu(dx) = \frac{pa}{q}\exp\left\{\frac{pa}{q}x\right\}\mathbf{1}_{(-\infty,0)}(x)\,dx.$$

(iii) $$\hat{\mu}(t) = \frac{se^{it}}{1-ue^{it}} \quad \text{and} \quad G^p(\hat{\mu}) = \frac{pse^{it}}{(s+qu)e^{it}-q} = \frac{\dfrac{ps}{s+qu}}{1-\dfrac{q}{s+qu}e^{-it}}.$$

So it is the characteristic function of the probability measure

$$\nu = T_0^{(ps)/(s+qu)}(\delta_{-1}) = \frac{ps}{s+qu}\sum_{k=0}^{\infty}\left(\frac{q}{s+qu}\right)^k \delta_{-k}. \blacksquare$$

4. THE DUGUÉ PROBLEM AND SIMPLE FRACTIONS

In [5] the authors considered some special classes of probability measures and their characteristic functions. The main reason for studying these classes was that the convolution of the measures from a fixed class is equivalent to the linear combination of these measures. However, the coefficients of linear combination do not have to be positive. This concept seems to have a lot in common with the Dugué problem.

Following the construction given in [5] for every fixed probability measure μ with the characteristic function φ we define

$$h(t) = \begin{cases} 1/\varphi(t)-1 & \text{for } \varphi(t) \neq 0, \\ \infty & \text{for } \varphi(t) = 0. \end{cases}$$

We denote by $\Phi(\varphi)$ the following class of characteristic functions:

$$\Phi(\varphi) = \left\{\varphi_a(t) \stackrel{\text{def}}{=} \frac{a}{a+h(t)}:\ a \in \boldsymbol{R}\backslash\{0\},\ \varphi_a \in \Phi\right\}.$$

$\Phi(\varphi)$ is the set of all characteristic functions which are the simple fractions of the variable $h(t)$. It turns out that the crucial role in the Dugué problem plays the following set:

$$T(\varphi) = \{a \in \boldsymbol{R}\backslash\{0\}:\ \varphi_a \in \Phi(\varphi)\}.$$

It is easy to see that for the exponential distribution $\mu(dx) = e^{-x}\mathbf{1}_{(0,\infty)}(x)\,dx$ we obtain $\varphi(t) = 1/(1-it)$ and $\Phi(\varphi)$ is the class of all exponential distributions (also those which are supported on the negative half-line), and $T(\varphi) = \boldsymbol{R}\backslash\{0\}$. For more interesting examples of such classes we refer to [5].

It was shown in Proposition 1 of [5] that for every $p \in (0, 1]$ and every characteristic function φ we have $pT(\varphi) \subset T(\varphi)$. In particular, we infer then that $(0, 1] \subset T(\varphi)$ for every characteristic function φ. The following theorem shows the connection between $T(\varphi)$ and the Dugué problem:

THEOREM 1. *Let μ be a probability measure with the characteristic function φ. There exists $p \in (0, 1)$ such that $G^p(\mu)$ is a probability measure if and only if*

$[-p/(1-p), 0) \subset T(\varphi)$. *Moreover, if* $G^p(\mu)$ *is a probability measure, then its characteristic function* $G^p(\varphi)$ *belongs to* $\Phi(\varphi)$.

Proof. For the proof we calculate:

$$G^p(\varphi)(t) = \frac{p\varphi(t)}{\varphi(t)-1+p} = \frac{-\dfrac{p}{1-p}}{\dfrac{1}{\varphi(t)}-1-\dfrac{p}{1-p}} = \varphi_{-p/(1-p)}(t).$$

Now it is enough to notice that if $G^p(\varphi)$ is a characteristic function, then for every $r \in (0, p)$ $G^r(\varphi)$ is a characteristic function, and consequently $-r/(1-r) \in T(\varphi)$ for every $r \in (0, p]$. The opposite implications also hold. ■

As a trivial consequence of Theorem 1 we obtain

COROLLARY 2. *If* μ *is a probability measure with the characteristic function* φ, *then* $\varphi \in \mathscr{G}$ *if and only if* $(-\infty, 0) \subset T(\varphi)$.

THEOREM 2. *If* $T(\varphi) = \boldsymbol{R}\setminus\{0\}$, $p \in (0, 1)$, $p+q = 1$, *and*

$$qa_1/a_2 \notin (-(1+\sqrt{p})^2, -(1-\sqrt{p})^2),$$

then there exist $b_1, b_2 \neq 0$ *such that*

$$p\varphi_{a_1}\varphi_{a_2} + q\varphi_{b_1}\varphi_{b_2} = \varphi_{a_1}\varphi_{a_2}\varphi_{b_1}\varphi_{b_2}.$$

THEOREM 3. *If* $a_1, \ldots, a_n, b_1, \ldots, b_n \in T(\varphi)$, $n \in N$, *then for every* $p \in (0, 1)$, $p+q = 1$, *we have*

$$p\prod_{k=1}^{n}\varphi_{a_k} + q\prod_{k=1}^{n}\varphi_{b_k} = \prod_{k=1}^{n}\varphi_{a_k}\prod_{k=1}^{n}\varphi_{b_k}$$

if and only if for $n = 1$

$$pa_1 + qb_1 = 0,$$

and for $n \geqslant 2$

$$p\textstyle\prod_{i=1}^{n} a_i + q\prod_{i=1}^{n} b_i = 0,$$

$$\textstyle\sum_{\Pi(k,n)} a_{\pi(1,k)} \ldots a_{\pi(k,k)} = \sum_{\Pi(k,n)} b_{\pi(1,k)} \ldots b_{\pi(k,k)}, \qquad k = 1, \ldots, n-1,$$

where $\Pi(k, n)$ *is the set of all choices* $\pi = \{\pi(1, k), \ldots, \pi(k, k)\}$ *of* k *different elements from the set* $\{1, 2, \ldots, n\}$.

Proof. The proof is only a matter of laborious calculations. In the paper we present it for $n = 2$. The equation

$$p\varphi_{a_1}\varphi_{a_2} + q\varphi_{b_1}\varphi_{b_2} = \varphi_{a_1}\varphi_{a_2}\varphi_{b_1}\varphi_{b_2}$$

implies that

$$pa_1a_2(b_1+h)(b_2+h) + qb_1b_2(a_1+h)(a_2+h) = a_1a_2b_1b_2,$$

which shall hold for every value of $h = h(t)$. ■

EXAMPLE. Let $n = 2$. Then

$$\hat{\mu}(t) = \frac{-6}{(6+h(t))(-1+h(t))}, \quad \hat{\nu}(t) = \frac{-6}{(3+h(t))(2+h(t))}$$

satisfy the equation $\frac{1}{2}\hat{\mu}(t)+\frac{1}{2}\hat{\nu}(t) = \hat{\mu}(t)\hat{\nu}(t)$.

Prof. D. Szynal asked whether or not there exist three measures μ, ν and η such that for some $p, q, r \in (0, 1)$, $p+q+r = 1$, the following condition holds:

$$p\mu+q\nu+r\eta = \mu * \nu * \eta.$$

Now we are able to give a positive answer to this question:

PROPOSITION 4. *If μ is a probability distribution with the characteristic function φ such that $a, b, c, d \in T(\varphi)$, then for $p+q+r = 1$ the following equation is satisfied:*

$$p\varphi_a \varphi_b + q\varphi_c + r\varphi_d = \varphi_a \varphi_b \varphi_c \varphi_d,$$

if and only if

$$qc+rd = 0,$$
$$pab+rd(c-d) = 0,$$
$$a+b = c+d,$$
$$(r-q)^2 > (r+q)^3.$$

EXAMPLE. If a probability measure μ with the characteristic function φ is such that $[-15, 46] \subset T(\varphi)$, then choosing

$$\hat{\mu}(t) = \varphi_{30}(t)\varphi_1(t), \quad \hat{\nu}(t) = \varphi_{46}(t), \quad \hat{\eta}(t) = \varphi_{-15}(t)$$

we obtain

$$\frac{1403}{1464}\hat{\mu}(t)+\frac{15}{1464}\hat{\nu}(t)+\frac{46}{1464}\hat{\eta}(t) = \hat{\mu}(t)\hat{\nu}(t)\hat{\eta}(t).$$

5. SOLUTIONS IN THE SET OF SIGNED MEASURES

In this section we consider the following problem: for an arbitrary but fixed probability measure μ we want to find any signed σ-finite measure ν for which the equality (1) holds, i.e.

$$p\mu+(1-p)\nu = \mu * \nu.$$

It turns out that in some cases this problem has some interesting solution. The measure ν for which this equality holds will be denoted by $G^p(\mu)$.

EXAMPLE. Let π be a Poisson distribution with the parameter $\lambda > 0$, i.e.

$$\pi = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \delta_k.$$

Then, for $p \in (0, 1)$ and the probability measure $\mu = T_0^{1-p}(\pi)$, $G^p(\mu)$ is a signed, finite measure such that

$$G^p(\mu) = e^{\lambda} \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \delta_k.$$

THEOREM 4. *Let η be a probability measure on $\mathbf{R}$, $r \in (0, 1)$, $r+s = 1$, and let $\mu = r\delta_0 + s\eta$.*

1. *If $s < p/2$, $p+q = 1$, then $G^p(\mu)$ is a signed measure defined by*

$$G^p(\mu) = \frac{pr}{p-s}\delta_0 + \sum_{k=1}^{\infty} \left(\frac{pq}{p-s}\right)\left(-\frac{s}{p-s}\right)^k \eta^{*k}.$$

2. *If $s = p/2$, then $G^p(\mu)$ is a signed measure of the form*

$$G^p(\mu) = (2-p)\delta_0 + 2(1-p) \sum_{k=1}^{\infty} (-1)^k \eta^{*k}.$$

Proof. Ad 1. Notice first that the function $G^p(\hat{\mu})$ can be written in the following way:

$$G^p(\hat{\mu}) = p + q \frac{\dfrac{p}{p-s}}{1 + \dfrac{s}{p-s}\hat{\eta}}.$$

Under our assumptions $p > 2s$, so $s/(p-s) < 1$ and we can write

$$G^p(\hat{\mu}) = p + \frac{pq}{p-s} \sum_{k=0}^{\infty} \left(-\frac{s}{p-s}\right)^k \hat{\eta}^k = \frac{pr}{p-s} + \frac{pq}{p-s} \sum_{k=1}^{\infty} \left(-\frac{s}{p-s}\right)^k \hat{\eta}^k,$$

since the corresponding series converges. It is easy to see now that $G^p(\hat{\mu})$ is the Fourier transform of the measure $G^p(\mu)$ defined in the theorem.

Ad 2. For $p = 2s$ we can write

$$G^p(\hat{\mu}) = \frac{p\left(1 - \dfrac{p}{2} + \dfrac{p}{2}\hat{\eta}\right)}{\dfrac{p}{2} + \dfrac{p}{2}\hat{\eta}} = \left(1 - \frac{p}{2} + \frac{p}{2}\hat{\eta}\right) \frac{1}{\dfrac{1}{2}(1+\hat{\eta})}.$$

Notice that for every $t \in \boldsymbol{R}$ such that $|\hat{\eta}(t)| < 1$ we have $|\frac{1}{2}(1-\hat{\eta})| < 1$, so

$$\frac{1}{\frac{1}{2}(1+\hat{\eta})} = \frac{1}{1-\frac{1}{2}(1-\hat{\eta})} = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n \sum_{k=0}^{n}\binom{n}{k}(-\hat{\eta})^k$$

$$= \sum_{k=0}^{\infty}(-\hat{\eta})^k\left(\frac{1}{2}\right)^k \sum_{n=0}^{\infty}\binom{n+k}{n}\left(\frac{1}{2}\right)^n \overset{(*)}{=} \sum_{k=0}^{\infty}(-\hat{\eta})^k\left(\frac{1}{2}\right)^k 2^{k+1} = 2\sum_{k=0}^{\infty}(-\hat{\eta})^k.$$

The equality $(*)$ follows from the formula

$$\sum_{n=0}^{\infty}\binom{n+k}{n}x^n = (1-x)^{-k-1} \qquad \text{for } x \in (0, 1)$$

(see formula 5.2.11.3 in [6]) and the series converges unconditionally for $|\hat{\eta}| < 1$. Finally, we obtain

$$G^p(\hat{\mu}) = \left(1-\frac{p}{2}+\frac{p}{2}\hat{\eta}\right)\left(2\sum_{k=0}^{\infty}(-\hat{\eta})^k\right) = (2-p)+2(1-p)\sum_{k=1}^{\infty}(-\hat{\eta})^k. \blacksquare$$

Example. Assume that η has an exponential distribution $\Gamma(1, \alpha)$ and $p > 2s$. Then η^{*k} has the distribution $\Gamma(k, \alpha)$ and

$$G^p(\mu) = \frac{pr}{p-s}\delta_0 + \frac{pq}{p-s}\sum_{k=1}^{\infty}\left(-\frac{s}{p-s}\right)^k \eta^{*k}.$$

$G^p(\mu)$ has an atom at zero of the weight $pr/(p-s)$ and outside zero it is absolutely continuous with respect to the Lebesgue measure with the density

$$f(x) = -\frac{\alpha pqs}{(p-s)^2}\exp\left\{-\alpha x\frac{p}{p-s}\right\}\mathbf{1}_{(0,\infty)}(x).$$

Since $p > 2s$, the absolutely continuous part of the measure $G^p(\mu)$ is negative and finite with the support $[0, \infty)$. In the case $p = 2s$, $G^p(\mu)$ has an atom at zero of the weight $2r$ and the absolutely continuous part of this measure has the density function

$$f(x) = -2\alpha q\exp\{-2\alpha x\}\mathbf{1}_{(0,\infty)}(x).$$

Example. Let η be a Poisson distribution with a parameter $\lambda > 0$ and $p > 2s$. Then the support of the measure $G^p(\mu)$ is $N \cup \{0\}$, and

$$G^p(\mu)(\{n\}) = \begin{cases} \dfrac{pq}{p-s}\dfrac{\lambda^n}{n!}\sum_{k=1}^{\infty}\left(-\dfrac{se^{-\lambda}}{p-s}\right)^k k^n & \text{for } n \geqslant 1, \\ \dfrac{p(r+se^{-\lambda})}{p-s+se^{-\lambda}} & \text{for } n = 0. \end{cases}$$

For $p = 2s$ we have

$$G^p(\mu)(\{n\}) = \begin{cases} 2q\frac{\lambda^n}{n!}\sum_{k=1}^{\infty}(-1)^k e^{-\lambda k} k^n & \text{for } n \geqslant 1, \\ \frac{2(r+se^{-\lambda})}{1+e^{-\lambda}} & \text{for } n = 0. \end{cases}$$

EXAMPLE. Let η be a geometric distribution, i.e. $\eta = \sum_{n=1}^{\infty} rs^{n-1}\delta_n$. Then

$$\eta^{*k} = \sum_{n=k}^{\infty}\binom{n-1}{n-k} r^k s^{n-k}\delta_n, \quad k\in N.$$

For $p > 2s$ we have

$$G^p(\mu) = \frac{pr}{p-s}\delta_0 + \frac{pq}{p-s}\sum_{n=1}^{\infty} s^n\delta_n \sum_{k=1}^{n}(-1)^k\binom{n-1}{n-k}\left(\frac{r}{p-s}\right)^k$$
$$= \frac{pr}{p-s}\delta_0 - \frac{pqsr}{(p-s)^2}\sum_{n=1}^{\infty}\left(\frac{-sq}{p-s}\right)^{n-1}\delta_n,$$

and for $p = 2s$

$$G^p(\mu) = 2r\delta_0 - 2qr\sum_{n=1}^{\infty}(-q)^{n-1}\delta_n.$$

As a special case of Theorem 4 we obtain the following:

THEOREM 5. *Let* $p\in(0, 1)$ *and* $\mu = r\delta_0+(1-r)\delta_a$; $r, s > 0$, $r+s = 1$. *If* $r\in[0, 1-p]$, *then*

$$G^p(\mu) = rT_1^{p/(1-r)}(\delta_{-a})+(1-r)\,T_0^{p/(1-r)}(\delta_{-a})$$

is a probability measure. For $r > 1-p$, $G^p(\mu)$ *is a signed measure with non-trivial negative part, and*

$$G^p(\mu) = \begin{cases} p\delta_0+\frac{pq}{1-p-r}\sum_{k=1}^{\infty}\left(\frac{1-p-r}{1-r}\right)^k\delta_{-ak}, & r\in(1-p, 1-p/2), \\ \frac{pr}{p+r-1}\delta_0+\frac{pq}{p+r-1}\sum_{k=1}^{\infty}\left(-\frac{1-r}{p+r-1}\right)^k\delta_{ak}, & r\in(1-p/2, 1), \\ (2-p)\delta_0+2(1-p)\sum_{k=1}^{\infty}(-1)^k\delta_{ak}, & r = 1-p/2. \end{cases}$$

Proof. For $p\in(0, 1)$ and $r\in[0, 1-p]$ we have $p/(1-r)\in(0, 1]$, so

$$G^p(\hat{\mu}) = r\frac{\frac{p}{1-r}e^{-ita}}{1-\left(1-\frac{p}{1-r}\right)e^{-ita}}+(1-r)\frac{\frac{p}{1-r}}{1-\left(1-\frac{p}{1-r}\right)e^{-ita}}$$
$$= rT_1^{p/(1-r)}(e^{-ita})+(1-r)\,T_0^{p/(1-r)}(e^{-ita}),$$

which was to be shown.

Let now $r = 1-p+\varepsilon$, where $\varepsilon \in (0, p/2)$ and $\hat{\mu}(t) = 1-p+\varepsilon+(p-\varepsilon)e^{ita}$. Then

$$G^p(\hat{\mu}) = (p-\varepsilon)\frac{\dfrac{p}{p-\varepsilon}}{1-\left(-\dfrac{\varepsilon}{p-\varepsilon}\right)e^{-ita}} + (1-p+\varepsilon)\frac{\dfrac{p}{p-\varepsilon}e^{-ita}}{1-\left(-\dfrac{\varepsilon}{p-\varepsilon}\right)e^{-ita}}$$

$$= p\sum_{k=0}^{\infty}\left(-\frac{\varepsilon}{p-\varepsilon}\right)^k e^{-itka} - \frac{pr}{\varepsilon}\sum_{k=1}^{\infty}\left(-\frac{\varepsilon}{p-\varepsilon}\right)^k e^{-itka}$$

$$= p - \frac{pq}{\varepsilon}\sum_{k=1}^{\infty}\left(-\frac{\varepsilon}{p-\varepsilon}\right)^k e^{-itka},$$

since under our assumptions $\varepsilon/(p-\varepsilon) \in (0, 1)$ and the corresponding series are unconditionally convergent. The other two cases are simple consequences of Theorem 4. ■

REFERENCES

[1] D. Dugué, *Sur les fonctions méromorphes transformées de Fourier de fonctions*, C. R. Acad. Sci. 208 (1939), p. 1547.

[2] D. Dugué, *Arithmétique des lois de probabilités*, Mém. Sci. Math. 137, Gauthier-Villars, Paris 1957.

[3] W. Krakowiak, *Remarks about the Dugué problem*, preprint.

[4] L. Kubik, *Sur un problème de M. D. Dugué*, Comment. Math. (Prace Matem.) 13 (1969), pp. 1–2.

[5] J. K. Misiewicz and R. Cooke, *Simple fractions and linear decomposition of some convolutions of measures*, Discussiones Mathematicae Probability and Statistics 21 (2001), pp. 149–157.

[6] A. J. Prudnikov, Yu. A. Brychkov and O. I. Marychev, *Integrals and Series* (in Russian), Nauka, Moscow 1981.

[7] H. J. Rossberg, *Characterization of the exponential and Pareto distributions by means of some properties of the distribution which the differences and quotients of order statistics are subject to*, Math. Operationsforsch. Statist. 3 (3) (1972), pp. 207–216.

[8] D. Szynal and A. Wolińska, *On classes of couples of characteristic functions satisfying the condition of Dugué*, Comment. Math. (Prace Matem.) 23 (1983), pp. 325–328.

[9] S. Vallander, I. Ibragimov and N. Lindtrop, *On limiting distributions for moduli of sequential differences of independent variables*, Teor. Voroyatnost. i Primenen. 14 (4) (1969), pp. 693–707.

[10] A. Wolińska, *On a problem of Dugué*, Lecture Notes in Math. 982, Springer, Berlin 1982, pp. 244–253.

[11] A. Wolińska-Wełcz, *On a solution of the Dugué problem*, Probab. Math. Statist. 7 (1986), pp. 169–185.

Institute of Mathematics
University of Zielona Góra
ul. Szafrana
65-246 Zielona Góra, Poland
E-mail: m.malinowski@im.uz.zgora.pl
j.misiewicz@im.uz.zgora.pl

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