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SOME REMARKS ON $S\alpha S$, β-SUBSTABLE RANDOM VECTORS

BY

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Abstract. An $S\alpha S$ random vector X is *β-substable*, $\alpha < \beta \leqslant 2$, if $X \stackrel{d}{=} Y\Theta^{1/\beta}$ for some symmetric β-stable random vector Y, $\Theta \geqslant 0$ a random variable with the Laplace transform $\exp\{-t^{\alpha/\beta}\}$, Y and Θ are independent. We say that an $S\alpha S$ random vector is *maximal* if it is not β-substable for any $\beta > \alpha$.

In the paper we show that the canonical spectral measure for every $S\alpha S$, β-substable random vector X, $\beta > \alpha$, is equivalent to the Lebesgue measure on S_{n-1}. We show also that every such vector admits the representation $X = Y+Z$, where Y is an $S\alpha S$ sub-Gaussian random vector, Z is a maximal $S\alpha S$ random vector, Y and Z are independent. The last representation is not unique.

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Let us remind first the well-known definitions of symmetric α-stable random variables, random vectors and stochastic processes, $\alpha \in (0, 2]$. The random variable X is *symmetric α-stable* if there exists a positive constant A such that

$$E\exp\{itX\} = \exp\{-A|t|^{\alpha}\}.$$

A random vector $X = (X_1, \ldots, X_n)$ is *symmetric α-stable* if for every $\xi = (\xi_1, \ldots, \xi_n)$ the random variable $\langle \xi, X\rangle = \sum_{k=1}^{n} \xi_k X_k$ is symmetric α-stable. This is equivalent to the following condition:

$$\forall \xi = (\xi_1, \ldots, \xi_n)\ \exists c(\xi) > 0\ \ \langle \xi, X\rangle \stackrel{d}{=} c(\xi) X_1.$$

It is well known that if X is an $S\alpha S$ random vector on $\boldsymbol{R}^n$, then there exists a finite measure ν on $\boldsymbol{R}^n$ such that

$$(*)\qquad E\exp\{i\langle \xi, X\rangle\} = \exp\{-\int\limits_{\boldsymbol{R}^n}\!\!\ldots\!\int |\langle \xi, x\rangle|^{\alpha}\,\nu(dx)\}.$$

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The measure ν is called the *spectral measure* for an $S\alpha S$ random vector X. If ν is concentrated on the unit sphere $S_{n-1} \subset \boldsymbol{R}^n$, then it is called the *canonical spectral measure* for X. The canonical spectral measure for a given $S\alpha S$ vector X is uniquely determined.

An $S\alpha S$ random vector X is *β-substable*, $\alpha < \beta \leqslant 2$, if there exists a symmetric β-stable random vector Y such that

$$X \stackrel{d}{=} Y\Theta^{1/\beta},$$

where $\Theta \geqslant 0$ is an α/β-stable random variable with the Laplace transform $\exp\{-t^{\alpha/\beta}\}$, Y and Θ are independent.

DEFINITION 1. An $S\alpha S$ random vector X is *maximal* if for every $\beta \geqslant \alpha$ and every $S\beta S$ random vector Y, and every Θ independent of Y the equality $X \stackrel{d}{=} Y\Theta$ implies that $\alpha = \beta$ and $\Theta = \text{const}$.

A stochastic process $\{X_t\colon t \in T\}$ is *symmetric α-stable* if all its finite-dimensional distributions are symmetric α-stable, i.e., if for every $n \in N$ and every choice of $t_1, \ldots, t_n \in T$ the random vector $(X_{t_1}, \ldots, X_{t_n})$ is symmetric α-stable.

For more information on stable random vectors, processes and distributions see [2]. Almost all $S\alpha S$ random vectors and stochastic processes studied in literature are maximal; and even more, almost all of them have pure atomic spectral measure. In [1] one can find some results on characterizing maximal $S\alpha S$ random vectors in the language of geometry of reproducing kernel spaces, however, except some trivial cases, these results are given only for infinite-dimensional $S\alpha S$ random vectors. The following, surprisingly simple theorem characterizes maximal symmetric α-stable random vectors on $\boldsymbol{R}^n$:

THEOREM 1. *Assume that a random vector $X = (X_1, \ldots, X_n)$ is symmetric α-stable and β-substable for some $\beta \in (\alpha, 2]$. Then the canonical spectral measure ν for the vector X has a continuous density function $f(u)$ with respect to the Lebesgue measure on the unit sphere $S_{n-1} \subset \boldsymbol{R}^n$, and $f(u) > 0$ for every $u \in S_{n-1}$.*

Proof. From the assumptions we infer that there exists a symmetric β-stable random vector $Y = (Y_1, \ldots, Y_n)$ such that $X \stackrel{d}{=} Y\Theta^{1/\beta}$, where $\Theta > 0$ independent of Y is α/β-stable with a Laplace transform $\exp\{-t^{\alpha/\beta}\}$. Assume that

$$E\exp\{it\langle\xi, Y\rangle\} = \exp\{-c(\xi)^\beta |t|^\beta\}.$$

This means that for every ξ we have

$$\langle\xi, Y\rangle \stackrel{d}{=} c(\xi)Y_0, \quad \text{where } E\exp\{itY_0\} = \exp\{-|t|^\beta\}.$$

In particular,

$$E|\langle\xi, Y\rangle|^\alpha = c(\xi)^\alpha E|Y_0|^\alpha.$$

Since $\alpha < \beta$, we have $c^{-1} = \boldsymbol{E}|Y_0|^\alpha < \infty$ and $c(\xi)^\alpha = c\boldsymbol{E}|\langle \xi, Y\rangle|^\alpha$. Calculating now the characteristic function for the vector X we obtain

$$\begin{aligned} \boldsymbol{E}\exp\{i\langle \xi, X\rangle\} &= \boldsymbol{E}\exp\{i\langle \xi, Y\Theta^{1/\beta}\rangle\} \\ &= \boldsymbol{E}\exp\{-c(\xi)^\beta \Theta\} = \exp\{-c(\xi)^\alpha\} \\ &= \exp\{-c\boldsymbol{E}|\langle \xi, Y\rangle|^\alpha\} \\ &= \exp\{-\int_{\boldsymbol{R}^n}\!\!\dots\!\int |\langle \xi, x\rangle|^\alpha c f_\beta(x)\, dx\}, \end{aligned}$$

where $f_\beta(x)$ denotes the density function of the $S\beta S$ random vector Y. This means that the function $cf_\beta(x)$ is the density of a spectral measure for the random vector X.

To get the canonical spectral measure ν_0 for the $S\alpha S$ random vector X from this spectral measure it is enough to make the spherical substitution $x = ru$ and integrate out the radial part. Consequently, for every Borel set $A \subset S_{n-1}$ we obtain

$$\nu_0(A) = \int_A\!\!\dots\!\int \underbrace{\int_0^\infty c f_\beta(ru)\, r^{n-1+\alpha}\, dr}_{g(u)}\, w(du),$$

where w is the Lebesgue measure on S_{n-1}. Since f_β is uniformly continuous on $\boldsymbol{R}^n$ and $f_\beta > 0$ everywhere, $g(u)$ is a continuous function and $g(u) > 0$ everywhere. The uniqueness of the canonical spectral measure implies that the function $g(u)$ is the density of the measure ν_0, which completes the proof. ∎

COROLLARY 1. *Every random vector with a pure atomic spectral measure is maximal. In fact, for maximality of the SαS random vector it is enough that its spectral measure μ is zero on a set in S_{n-1} of positive Lebesgue measure.*

COROLLARY 2. *Let $(E, \mathscr{B}, \mu)$ be a σ-finite measure space and let $Y = \{Y(B);\ B \in \mathscr{B},\ \mu(B) < \infty\}$ be an independently scattered SαS random measure on $(E, \mathscr{B})$ controlled by the measure μ. We say that a stochastic process $X = \{X_t;\ t \in T\}$ is a set-indexed SαS-process if there exists a map S from T to $\mathscr{B}$ such that*

$$X_t = Y(S_t).$$

Every set-indexed SαS-process is maximal.

Proof. Notice that any finite-dimensional marginal distribution of a set--indexed $S\alpha S$-process has a pure point spectrum. For example, the 3-dimensional marginal characteristic function is

$$\begin{aligned}E\exp\{i(z_1 X_{t_1}+z_2 X_{t_2}+z_3 X_{t_3})\} &= E\exp\{i(z_1 Y(S_1)+z_2 Y(S_2)+z_3 Y(S_3))\}\\ &= \exp\{|z_1|^\alpha \mu(S_1\cap S_2^c\cap S_3^c)+|z_2|^\alpha \mu(S_1^c\cap S_2\cap S_3^c)\\ &\quad +|z_3|^\alpha \mu(S_1^c\cap S_2^c\cap S_3)+|z_2+z_3|^\alpha \mu(S_1^c\cap S_2\cap S_3)\\ &\quad +|z_3+z_1|^\alpha \mu(S_1\cap S_2^c\cap S_3)+|z_1+z_2|^\alpha \mu(S_1\cap S_2\cap S_3^c)\\ &\quad +|z_1+z_2+z_3|^\alpha \mu(S_1\cap S_2\cap S_3)\}. \blacksquare\end{aligned}$$

Some of important $S\alpha S$-processes are set-indexed processes: for example, multiparameter Lévy motion, multiparameter additive processes, generally linearly additive processes, a class of self-similar $S\alpha S$-processes (see, e.g., [3]–[6]). Moreover, all these processes have very interesting properties, called *determinisms*.

COROLLARY 3. *If an $S\alpha S$ random vector X is not maximal, i.e., if X is β-substable for some $\beta > \alpha$, then there exist a symmetric Gaussian random vector Z and a maximal $S\alpha S$ random vector Y such that*

$$X \stackrel{d}{=} Z\Theta^{1/2}+Y,$$

where $\Theta \geqslant 0$ has the Laplace transform $\exp\{-t^{\alpha/2}\}$, *Z, Y and Θ are independent.*

Proof. Since every continuous function attains its extremes on very compact set, we have

$$A = \inf\{g(u):\ u\in S_{n-1}\} > 0,$$

where $g(u)$ is the density of the canonical spectral measure for X obtained in Theorem 1. Now it is easy to see that $X \stackrel{d}{=} Z\Theta^{1/2}+Y$ for the Gaussian random vector Z with the characteristic function $\exp\{-A^{1/\alpha}\sum_{k=1}^{n}\xi_k^2\}$, and the $S\beta S$ random vector Y with the spectral measure given by the density function $f(u) = g(u)-A$. $\blacksquare$

Remark 1. The representation obtained in Corollary 3 is not unique. In fact, for every $S\alpha S$ β-substable random vector X and every symmetric Gaussian random vector Z taking values in the same space R^n there exist a constant $c > 0$ and a maximal $S\alpha S$ random vector Y such that

$$X \stackrel{d}{=} cZ\Theta^{1/2}+Y,$$

where Θ as in Corollary 3, Y, Z and Θ are independent.

Proof. The representation (*) for the characteristic function of an $S\alpha S$ random vector holds for every $\alpha\in(0, 2]$ including the Gaussian case. However, for $\alpha = 2$ we do not have uniqueness for the spectral measure ν. In fact, ν can always be taken here from the class of pure atomic measures on S_{n-1}, but such a representation is not useful for our construction. We will use the measure ν_A constructed as follows:

Let $\nu = \nu_I$ be the uniform distribution on the unit sphere $S_{n-1} \subset \boldsymbol{R}^n$, and let $U = (U_1, \ldots, U_n)$ be the random vector with the distribution ν. Then we have

$$\exp\{-\int_{S_{n-1}}\ldots\int |\langle \xi, u\rangle|^2 c_n \nu(du)\} = \exp\{-\tfrac{1}{2}\langle \xi, \xi\rangle\},$$

where $c_n^{-1} = 2EU_1^2$. Now let Σ be the covariance matrix for the random vector Z and let $\Sigma = AA^T$. We denote by ν_1 the distribution of the random vector AU. Then

$$\begin{aligned}\exp\{-\int_{R^n}\ldots\int \langle \xi, x\rangle^2 c_n \nu_1(dx)\} &= \exp\{-\int_{S_{n-1}}\ldots\int \langle \xi, Au\rangle^2 c_n \nu(du)\} \\ &= \exp\{-\int_{S_{n-1}}\ldots\int |\langle A^T\xi, u\rangle|^2 c_n \nu(du)\} \\ &= \exp\{-\tfrac{1}{2}\langle A^T\xi, A^T\xi\rangle\} = \exp\{-\tfrac{1}{2}\langle \xi, \Sigma\xi\rangle\},\end{aligned}$$

which is the characteristic function for the Gaussian vector Z. It is easy to see now that for a suitable constant $a > 0$

$$\exp\{-\int_{R^n}\ldots\int |\langle \xi, x\rangle|^\alpha c_n \nu_1(dx)\} = \exp\{-a(\langle \xi, \Sigma\xi\rangle)^{\alpha/2}\},$$

which is a characteristic function of the sub-Gaussian vector $Z\Theta^{1/2}$. We define now the measure ν_A as the projection (in the sense described in the proof of Theorem 1) of the measure ν_1 to the sphere S_{n-1} and we obtain

$$\int_{R^n}\ldots\int |\langle \xi, x\rangle|^\alpha c_n \nu_1(dx) = \int_{S_{n-1}}\ldots\int |\langle \xi, u\rangle|^\alpha \nu_A(du).$$

Since ν_1 is absolutely continuous with respect to the Lebesgue measure, ν_A has the same property and $\nu_A(du) = f_A(u)\,\omega(du)$ for some continuous positive function f_A. If $g(u)$ is the density of the spectral measure for X, then there exists $c_0 > 0$ such that

$$c_0 = \sup\{c > 0\colon\ g(u) - cf_A(u) \geqslant 0\}.$$

Now it is enough to define the maximal $S\alpha S$ random vector X by its canonical spectral measure absolutely continuous with respect to the Lebesgue measure with density $h(u) = g(u) - c_0 f_A(u)$ and put $c = c_0^{1/\alpha}$. ∎

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