PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 24, Fasc. 1 (2004), pp. 97–119

GENERALIZED t-TRANSFORMATIONS OF PROBABILITY MEASURES AND DEFORMED CONVOLUTIONS

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Abstract. In this paper, the generalized (two-parameterized) t-transformations on probability measures are introduced, in which the t-transformation of Bożejko and Wysoczański can be obtained as the special case, and the associated deformed convolutions are also investigated. We see that the generalized t-deformed free convolution can be realized as the conditionally free convolution of Bożejko, Leinert, and Speicher. We also study another special case of the generalized t-deformed free convolution, which is called the *τ-free convolution*, that gives an interpolation between the free and the Fermi convolutions.

AMS Subject Classification: Primary 46L53, 46L54; Secondary 60E10.

Key words and phrases: Convolution, conditionally free, moment-cumulant formula.

0. INTRODUCTION

Let V be an invertible map on the set of probability measures on $\boldsymbol{R}$ with finite moments of all orders. For two probability measures μ_1 and μ_2, and a given convolution $\oplus$ (for which the classical convolution, the Voiculescu's free convolution, and other convolutions may serve), one can have the following deformation of the convolution $\oplus$ associated with the invertible map V: We define the *deformed convolution* of the probability measures μ_1 and μ_2 by the relation

$$\mu = V^{-1}\big(V(\mu_1)\oplus V(\mu_2)\big),$$

that is, the *convoluted measure* μ is defined as the V-inversion of the convolution of the transformed measures $V(\mu_1)$ and $V(\mu_2)$.

Bożejko and Wysoczański introduced the invertible map U_t on probability measures for $t > 0$, which is called the *t-transformation*, and considered the associated deformations of the classical (t-classical) and of the free (t-free) con-

* The first-named author was partially sponsored with KBN grant no. 2PO3A00723 and RTN HPRN-CT-2002-00279.

volutions in the above manner in [4] and [5]. The central limits and the Poisson type limits with respect to the t-classical and to the t-free convolutions were also investigated. Furthermore, the model of the t-classical and the t-free Gaussian random variables were constructed on the t-deformed symmetric (boson) Fock space and on the t-deformed full Fock space, respectively.

In this paper we shall introduce a generalization of the t-transformation which is called the $\boldsymbol{t}$ (*"bold t")-transformation* or the *generalized t-transformation*. The definition is still based on the reciprocals of the Cauchy transforms as for the t-transformation, but we shall impose two parameters, the diagonal graph of which will give the original t-transformation.

The central limit measures with respect to the $\boldsymbol{t}$-deformed classical and the $\boldsymbol{t}$-deformed free convolutions are the same as the ones for the original t-deformations, but the Poisson limits depend on the two parameters. In Section 2, we calculate the $\boldsymbol{t}$-deformed classical Poisson limit and give the orthogonal polynomials that belong to its limit probability measure.

The subsequent sections are devoted to the study of the deformed free case. In Section 3, we see that the $\boldsymbol{t}$-deformed free convolution can be obtained as the conditionally free convolution of Bożejko et al. in [3], which enables us to apply the results on the conditionally free convolution to our $\boldsymbol{t}$-deformed free convolution. Using the combinatorial results in [3], we give the moment-cumulant formula for the $\boldsymbol{t}$-deformed free convolution, which requires a finer set partition statistic on non-crossing partitions than the number of inner blocks (see [5]) for the t-free convolution.

The Poisson limit with respect to the $\boldsymbol{t}$-deformed free convolution is calculated in Section 4 and we also give its limit measure explicitly with the orthogonal polynomials. In the last section, we shall restrict ourselves to the special case of parameters other than the usual t-free case, which yields the *τ-free convolution.* This new family of deformed free convolutions gives an interpolation between the free and the Fermi (see [7]) convolutions. We also construct the model of the τ-free Poisson process on the deformed full Fock space that is the same as the t-deformed full Fock space introduced in [5]. It is required to consider the gauge operator on the τ-free Fock space other than the creation and annihilation operators in order to give the model of the τ-free Poisson process. It would be notable that our model has exactly the same form as for the free Poisson process on the full Fock space constructed in [9].

1. GENERALIZED $\boldsymbol{t}$-TRANSFORMATIONS AND CONVOLUTIONS

We shall introduce the transformation of probability measures on $\boldsymbol{R}$, which is a certain generalization of the t-transformation investigated in [4] (see also [5]). Although the definition itself does not require that the probability measure has finite moments, we will work in this paper on the class of probability measures $\mathscr{P}^{\infty}(\boldsymbol{R})$ of finite moments of all orders because we would like to discuss on the continued fractions (Stieltjes expansion) and on moment-cumulant formulae.

The Cauchy transform G_μ of the probability measure μ is defined for $z \in \boldsymbol{C}^+ = \{z \in \boldsymbol{C}\colon \Im z > 0\}$ by

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{z-x}.$$

By the Nevanlinna theorem (see, for instance, [1]), we know that a function $F(z)$ is the reciprocal of the Cauchy transform of a probability measure if and only if there exists a positive measure ϱ and a real number α such that for every $\Im z > 0$

$$F(z) = \alpha + z + \int_{-\infty}^{\infty} \frac{1+xz}{x-z}\, d\varrho(x).$$

Now we shall define the transformation on $\mathscr{P}^\infty(\boldsymbol{R})$, which acts essentially on the pair (α, ϱ).

DEFINITION 1.1. Let μ be a probability measure in $\mathscr{P}^\infty(\boldsymbol{R})$ and we write the reciprocal of the Cauchy transform of μ as

$$\frac{1}{G_\mu(z)} = \alpha + z + \int_{-\infty}^{\infty} \frac{1+xz}{x-z}\, d\varrho(x).$$

For a real number a and a positive real number b, we consider a pair of numbers $\boldsymbol{t} = (a, b)$ and define the $\boldsymbol{t}(=(a, b))$*-transformation* $\tilde{U}^{(\boldsymbol{t})}$ by

$$\tilde{U}^{(\boldsymbol{t})}(\mu) = \mu^{(\boldsymbol{t})},$$

where the probability measure $\mu^{(\boldsymbol{t})}$ is determined by the formula

$$\frac{1}{G_{\mu^{(\boldsymbol{t})}}(z)} = a\alpha + z + b\int_{-\infty}^{\infty} \frac{1+xz}{x-z}\, d\varrho(x) = a\alpha + z + \int_{-\infty}^{\infty} \frac{1+xz}{x-z}\, d(b\varrho)(x). \tag{1.1}$$

Here $G_{\mu^{(\boldsymbol{t})}}(z)$ is the Cauchy transform of the probability measure $\mu^{(\boldsymbol{t})}$. Namely, the $\boldsymbol{t}(=(a, b))$-transformation induces the map on the pairs of a constant and a positive measure such that $(\alpha, \varrho) \mapsto (a\alpha, b\varrho)$.

Remark 1.2. The following formula is a direct consequence of the definition:

$$\frac{1}{G_{\mu^{(\boldsymbol{t})}}(z)} = \frac{b}{G_\mu(z)} + (1-b)z + (b-a)E(\mu), \tag{1.2}$$

where $E(\mu)$ denotes the mean (the first moment) of the probability measure μ.

In the case of $a = b = t$, the transform $\tilde{U}^{(\boldsymbol{t})}$ is reduced to the t-transformation U_t in [4]. Thus we also call the "bold t" transformation $\tilde{U}^{(\boldsymbol{t})}$ the *generalized t-transformation.*

Here we shall describe the change of moments on our $t(=(a, b))$-transformation.

LEMMA 1.3. *For $n \in N$, we write the nth moments of probability measures μ and $\mu^{(t)}$ as*

$$m_n = \int_{\mathbf{R}} x^n \, d\mu(x) \quad \text{and} \quad \tilde{m}_n = \int_{\mathbf{R}} x^n \, d\mu^{(t)}(x),$$

respectively. Then we have

$$m_n = b^{-1}\tilde{m}_n + (b^{-1}-1)\sum_{k=1}^{n-1} m_k \tilde{m}_{n-k} + (1-a/b)\, m_1 \sum_{k=0}^{n-1} m_k \tilde{m}_{n-k-1},$$

$$\tilde{m}_n = b m_n + (b-1)\sum_{k=1}^{n-1} m_k \tilde{m}_{n-k} + (a-b)\, m_1 \sum_{k=0}^{n-1} m_k \tilde{m}_{n-k-1}.$$

Proof. For a probability measure ν, we write

$$M_\nu(z) = \sum_{n=0}^{\infty} m_n(\nu)\, z^n,$$

the moment series of the probability measure ν, which is related to the Cauchy transform $G_\nu(z)$ by

$$G_\nu(z) = z^{-1} M_\nu(z^{-1}).$$

By the relation (1.2), we have

$$\frac{1}{z^{-1} M_{\mu^{(t)}}(z^{-1})} = \frac{b}{z^{-1} M_\mu(z^{-1})} + (1-b)\, z + (b-a)\, E(\mu),$$

which yields the equation

$$M_\mu(z) = b M_{\mu^{(t)}}(z) + \big((1-b) + (b-a)\, m_1 z\big) M_{\mu^{(t)}}(z)\, M_\mu(z).$$

Using the Leibnitz formula for differentiation at $z = 0$, we obtain the relation

$$\frac{M_\mu^{(n)}(0)}{n!} = b\frac{M_{\mu^{(t)}}^{(n)}(0)}{n!} + (1-b)\sum_{k=0}^{n} \frac{M_\mu^{(k)}(0)}{k!}\frac{M_{\mu^{(t)}}^{(n-k)}(0)}{(n-k)!}$$
$$+ (b-a)\, m_1 \sum_{k=0}^{n-1} \frac{M_\mu^{(k)}(0)}{k!}\frac{M_{\mu^{(t)}}^{(n-1-k)}(0)}{(n-1-k)!},$$

which implies that

$$m_n = b\tilde{m}_n + (1-b)\sum_{k=0}^{n} m_k \tilde{m}_{n-k} + (b-a)\, m_1 \sum_{k=1}^{n-1} m_k \tilde{m}_{n-k-1}$$
$$= b\tilde{m}_n + (1-b)\, m_n \tilde{m}_0 + (1-b)\, m_0 \tilde{m}_n$$
$$+ (1-b)\sum_{k=1}^{n-1} m_k \tilde{m}_{n-k} + (b-a)\, m_1 \sum_{k=0}^{n-1} m_k \tilde{m}_{n-k-1}.$$

Since $m_0 = \tilde{m}_0 = 1$, we have the desired formulae. ∎

PROPOSITION 1.4. *The following properties of the* $\mathbf{t}$*-transformation are satisfied*:

(1) *The* $\mathbf{t}$*-transformation is multiplicative, that is,*

$$\tilde{U}^{(\mathbf{t}_2)}\big(\tilde{U}^{(\mathbf{t}_1)}(\mu)\big) = \tilde{U}^{(\mathbf{t}_1\cdot\mathbf{t}_2)}(\mu),$$

where we use the notation $\mathbf{t}_1\cdot\mathbf{t}_2 = (a_1, b_1)\cdot(a_2, b_2) = (a_1 a_2, b_1 b_2)$.

(2) *Dilation of a measure* D_λ *commutes with* $\tilde{U}^{(\mathbf{t})}$, *that is,*

$$D_\lambda\big(\tilde{U}^{(\mathbf{t})}(\mu)\big) = \tilde{U}^{(\mathbf{t})}\big(D_\lambda(\mu)\big).$$

(3) *For a pair* $\mathbf{t} = (a, b)$, *where* $a \neq 0$ *and* $b > 0$, *we write* $\mathbf{t}^{-1} = (a^{-1}, b^{-1})$. *Then* $\tilde{U}^{(\mathbf{t})}$ *and* $\tilde{U}^{(\mathbf{t}^{-1})}$ *are inverses of each other.*

(4) *For any real number* α *and* $\mathbf{t} = (a, b)$, $\tilde{U}^{(\mathbf{t})}(\delta_\alpha) = \delta_{a\alpha}$, *where* δ_x *denotes the Dirac unit mass at* x.

Proof. Concerning the first property (1), it follows from the definition that

$$\frac{1}{G_{\tilde{U}^{(\mathbf{t}_2)}(\tilde{U}^{(\mathbf{t}_1)}(\mu))}(z)} = \frac{b_2}{G_{\tilde{U}^{(\mathbf{t}_1)}(\mu)}} + (1-b_2)z + (b_2-a_2)E\big(\tilde{U}^{(\mathbf{t}_1)}(\mu)\big)$$

$$= b_2\left(\frac{b_1}{G_\mu(z)} + (1-b_1)z + (b_1-a_1)E(\mu)\right) + (1-b_2)z + (b_2-a_2)E\big(\tilde{U}^{(\mathbf{t}_1)}(\mu)\big)$$

$$= \frac{b_1 b_2}{G_\mu(z)} + \big(b_2(1-b_1) + (1-b_2)\big)z + \big(b_2(b_1-a_1) + (b_2-a_2)a_1\big)E(\mu),$$

where we have used the relation

$$E\big(\tilde{U}^{(\mathbf{t}_1)}(\mu)\big) = a_1 E(\mu).$$

Thus we obtain

$$\frac{1}{G_{\tilde{U}^{(\mathbf{t}_2)}(\tilde{U}^{(\mathbf{t}_1)}(\mu))}(z)} = \frac{b_1 b_2}{G_\mu(z)} + (1-b_1 b_2)z + (b_1 b_2 - a_1 a_2)E(\mu)$$

$$= \frac{1}{G_{\tilde{U}^{(\mathbf{t}_1\cdot\mathbf{t}_2)}(\mu)}(z)}.$$

Concerning the second property (2), we should note that, for a probability measure ν and the dilation D_λ, we have

$$D_\lambda(\nu)(B) = \nu(\lambda^{-1}B) \quad \text{for any Borel set } B,$$

which implies the following formula on the Cauchy transforms:

$$G_{D_\lambda(\nu)}(z) = \int_{-\infty}^{\infty} \frac{1}{z-x}\,dD_\lambda(\nu)(x) = \int_{-\infty}^{\infty} \frac{1}{z-\lambda x}\,d\nu(x) = \frac{1}{\lambda}G_\nu\left(\frac{z}{\lambda}\right).$$

Hence we obtain

$$\frac{1}{G_{\tilde{U}^{(t)}(D_\lambda(\mu))}(z)} = \frac{b}{G_{D_\lambda(\mu)}(z)} + (1-b)z + (b-a)E(D_\lambda(\mu))$$
$$= \frac{\lambda b}{G_\mu(z/\lambda)} + (1-b)z + \lambda(b-a)E(\mu),$$

because we know the relation

$$E(D_\lambda(\nu)) = \lambda E(\nu).$$

On the other hand, it follows that

$$\frac{1}{G_{D_\lambda(\tilde{U}^{(t)}(\mu))}(z)} = \frac{\lambda}{G_{\tilde{U}^{(t)}(\mu)}(z/\lambda)}$$
$$= \lambda\left(\frac{b}{G_\mu(z/\lambda)} + (1-b)\frac{z}{\lambda} + (b-a)E(\mu)\right) = \frac{\lambda b}{G_\mu(z/\lambda)} + (1-b)z + \lambda(b-a)E(\mu).$$

Thus the dilatation D_λ commutes with the map $\tilde{U}^{(t)}$.

Now the properties (3) and (4) are obvious. ■

Remark 1.5. The $t(=(a, b))$-transformation of a probability measure can be seen in terms of continued fractions, that is, it just multiplies the coefficients α_1 and β_1 (the Jacobi parameters of the first level for the original probability measure) by a and b, respectively:

$$G_{\mu^{(t)}}(z) = \cfrac{1}{z-a\alpha_1-\cfrac{b\beta_1}{z-\alpha_2-\cfrac{\beta_2}{z-\alpha_3-\cfrac{\beta_3}{z-\alpha_4-\cfrac{\beta_4}{\ddots}}}}}.$$

EXAMPLE 1.6. We shall compute the $t(=(a, b))$-transformation of the Bernoulli measure. Let μ_B be the Bernoulli measure with success probability p, that is,

$$\mu_B = (1-p)\delta_0 + p\delta_1.$$

Since we know that

$$G_{\mu_B}(z) = \frac{1-p}{z} + \frac{p}{z-1} = \frac{z-(1-p)}{z(z-1)},$$

we can obtain, by the definition,

$$G_{\tilde{U}^{(t)}(\mu_B)}(z) = \left(\frac{bz(z-1)}{z-(1-p)} + (1-b)z + (b-a)p\right)^{-1}$$
$$= \frac{z-(1-p)}{z^2-(ap+1-p)z-(b-a)p(1-p)}.$$

If we put

$$R_1(z) = z-(1-p),$$
$$R_2(z) = z^2-(ap+1-p)z-(b-a)p(1-p),$$

then one can decompose $R_1(z)/R_2(z)$ into the partial fractions of the form

$$\frac{R_1(z)}{R_2(z)} = \frac{P}{z-A}+\frac{Q}{z-B},$$

where A and B are the zeros of the polynomial $R_2(z)$ which are given by

$$A = \frac{1-p+ap-\gamma}{2}, \quad B = \frac{1-p+ap+\gamma}{2}$$

with

$$\gamma = \sqrt{(1-p+ap)^2+4(b-a)p(1-p)},$$

and the coefficients P and Q are given by

$$P = \frac{B-ap}{\gamma}, \quad Q = \frac{ap-A}{\gamma}.$$

This means that the measure $\tilde{U}^{(t)}(\mu_B)$ is again a two-point measure:

$$\tilde{U}^{(t)}(\mu_B) = P\delta_A+Q\delta_B.$$

As we have seen in Proposition 1.4, for $a \neq 0$ and $b > 0$, the $\boldsymbol{t}(=(a, b))$-transformation $\tilde{U}^{(t)}$ is invertible, thus one can define the deformed convolution associated with the map $\tilde{U}^{(t)}$ in the same manner as for the t-deformation in [4] (see also [5]).

DEFINITION 1.7. Let μ_1 and μ_2 be two probability measures in $\mathscr{P}^\infty(\boldsymbol{R})$ and let a be a non-zero real number and b be a positive number. The $\boldsymbol{t}(=(a, b))$-*deformed convolution* $\mu_1 \oplus_{(t)} \mu_2$ can be defined as

$$\mu_1 \oplus_{(t)} \mu_2 = (\tilde{U}^{(t)})^{-1}(\tilde{U}^{(t)}(\mu_1) \oplus \tilde{U}^{(t)}(\mu_2)),$$

where $\oplus$ on the right-hand side is an original convolution, for instance, the classical convolution or the Voiculescu's free convolution. The map $(\tilde{U}^{(t)})^{-1}$ is the inverse of $\tilde{U}^{(t)}$, which is given by (3) in Proposition 1.4.

Remark 1.8. If a given convolution is associative, then its $\boldsymbol{t}$-deformation is also associative. It is not so difficult to see that the central limit measure with respect to the $\boldsymbol{t}(=(a, b))$-deformed convolution does not depend on a but it is the same as for the t-deformation in [4] and [5], where we should put $t = b$. The Poisson limit measure, however, depend both on a and on b as we will see later.

2. THE t-DEFORMED CLASSICAL POISSON LAW

In this section we shall study the Poisson limit theorem for t-deformed classical convolution. The t-free case will be discussed later in another section.

DEFINITION 2.1. For a number $0 \leqslant \lambda \leqslant 1$, we consider the sequence of measures

$$\mu_N = \left(1-\frac{\lambda}{N}\right)\delta_0 + \frac{\lambda}{N}\delta_1.$$

We define the *t-deformed classical Poisson measure* of parameter λ, c-$\mathrm{Po}_\lambda^{(t)}$, by the weak limit

$$\text{c-Po}_\lambda^{(t)} = \lim_{N\to\infty} \underbrace{\mu_N *_{(t)} \mu_N *_{(t)} \dots *_{(t)} \mu_N}_{N},$$

where $*$ means the classical convolution. In this section, we shall abbreviate c-$\mathrm{Po}_\lambda^{(t)}$ as p_λ.

Let us remind from Example 1.6 that the t-transformed measure of μ_N is given by

$$\tilde{U}^{(t)}(\mu_N) = P_N \delta_{A_N} + Q_N \delta_{B_N}, \tag{2.1}$$

where

$$A_N = \frac{1-\lambda N^{-1} + a\lambda N^{-1} - \gamma_N}{2}, \qquad B_N = \frac{1-\lambda N^{-1} + a\lambda N^{-1} + \gamma_N}{2},$$

$$P_N = \frac{B_N - a\lambda N^{-1}}{\gamma_N}, \qquad Q_N = \frac{a\lambda N^{-1} - A_N}{\gamma_N}$$

with putting

$$\gamma_N = \sqrt{\left(1-\frac{\lambda}{N}+a\frac{\lambda}{N}\right)^2 + 4(b-a)\frac{\lambda}{N}\left(1-\frac{\lambda}{N}\right)}.$$

In order to calculate the limit measure, we shall use the Fourier transform. We put

$$\varrho_N = \underbrace{\mu_N *_{(t)} \dots *_{(t)} \mu_N}_{N}.$$

By the definition it follows that

$$\mathscr{F}[\tilde{U}^{(t)}(\varrho_N)](x) = \left(\mathscr{F}[\tilde{U}^{(t)}(\mu_N)](x)\right)^N,$$

where $\mathscr{F}$ denotes the Fourier transform. We infer from the equation (2.1) that

$$\mathscr{F}[\tilde{U}^{(t)}(\mu_N)](x) = P_N \exp(ixA_N) + Q_N \exp(ixB_N)$$

and consider the Taylor expansion

$$\mathscr{F}[\tilde{U}^{(t)}(\mu_N)](x) = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}(P_N A_N^n + Q_N B_N^n).$$

Here we would like to study the limit, as N tends to infinity, of the Nth power of the Fourier transform and only terms containing N^{-1} in power 0 or 1 will contribute to the limit. By small calculation we obtain

$$A_N B_N = (a-b)\frac{\lambda}{N}\left(1-\frac{\lambda}{N}\right) = O\left(\frac{1}{N}\right).$$

Using this estimation, we get the following lemma:

LEMMA 2.2. *For each positive integer n, we obtain*

$$A_N^n + B_N^n = 1 + O\left(\frac{1}{N}\right).$$

Proof. For $n = 1$, we have

$$A_N + B_N = 1 - \frac{\lambda}{N} + a\frac{\lambda}{N} = 1 + O\left(\frac{1}{N}\right)$$

and, for $n \geqslant 2$,

$$1 + O\left(\frac{1}{N}\right) = (A_N + B_N)^n = A_N^n + B_N^n + A_N B_N \sum_{k=1}^{n-1} \binom{n}{k} A_N^k B_N^{n-k}$$
$$= A_N^n + B_N^n + O\left(\frac{1}{N}\right). \quad ■$$

The following lemma shows what contribution of each term of the series should be taken into account:

LEMMA 2.3. *For each positive integer n, we obtain*

$$P_n A_N^n + Q_N B_N^n = \begin{cases} a\lambda N^{-1} & \text{if } n = 1, \\ b\lambda N^{-1} + O(N^{-2}) & \text{if } n \geqslant 2. \end{cases}$$

Proof. Using the definitions and the relation $\gamma_N = B_N - A_N$, we may write, for $n \geqslant 3$, that

$$P_N A_N^n + Q_N B_N^n$$
$$= \frac{1}{\gamma_N}\left(\left(B_N - a\frac{\lambda}{N}\right)A_N^n + \left(a\frac{\lambda}{N} - A_N\right)B_N^n\right)$$
$$= a\frac{\lambda}{N}\left(\frac{B_N^n - A_N^n}{B_N - A_N}\right) - \left(\frac{A_N B_N^n - B_N A_N^n}{B_N - A_N}\right)$$

$$= a\frac{\lambda}{N}(B_N^{n-1}+B_N^{n-2}A_N+\ldots+A_N^{n-1})-B_N A_N(B_N^{n-2}+B_N^{n-3}A_N+\ldots+A_N^{n-2})$$

$$= a\frac{\lambda}{N}(B_N^{n-1}+A_N^{n-1})-(a-b)\frac{\lambda}{N}\left(1-\frac{\lambda}{N}\right)(B_N^{n-2}+A_N^{n-2})+O\left(\frac{1}{N^2}\right)$$

$$= a\frac{\lambda}{N}(-A_N B_N^{n-2}-A_N^{n-2}B_N)+b\frac{\lambda}{N}(B_N^{n-2}+A_N^{n-2})+O\left(\frac{1}{N^2}\right)$$

$$= b\frac{\lambda}{N}(B_N^{n-2}+A_N^{n-2})+O\left(\frac{1}{N^2}\right),$$

where, for the second last equality, we have used the estimations

$$A_N-1 = -B_N+O\left(\frac{1}{N}\right) \quad \text{and} \quad B_N-1 = -A_N+O\left(\frac{1}{N}\right).$$

For $n = 2$, we can easily calculate

$$P_N A_N^2+Q_N B_N^2 = \frac{B_N-a\lambda N^{-1}}{\gamma_N}A_N^2+\frac{a\lambda N^{-1}-A_N}{\gamma_N}B_N^2$$

$$= \frac{A_N^2 B_N N-a\lambda A_N^2+a\lambda B_N^2-B_N^2 A_N N}{N(B_N-A_N)}$$

$$= -A_N B_N+\frac{a\lambda(A_N+B_N)}{N}$$

$$= -(a-b)\frac{\lambda}{N}\left(1-\frac{\lambda}{N}\right)+\frac{a\lambda}{N}\left(1-\frac{\lambda}{N}+a\frac{\lambda}{N}\right)$$

$$= b\frac{\lambda}{N}+O\left(\frac{1}{N^2}\right).$$

For $n = 1$, we get

$$P_N A_N+Q_N B_N = a\lambda\frac{B_N-A_N}{N\gamma_N} = a\frac{\lambda}{N}. \quad \blacksquare$$

THEOREM 2.4. *The Fourier transform of the $\mathbf{t}(=(a, b))$-transformed measure of the $\mathbf{t}(=(a, b))$-deformed classical Poisson measure of parameter λ is given by the formula*

$$\mathcal{F}[\tilde{U}^{(\mathbf{t})}(p_\lambda)](x) = \exp\left(b\lambda(e^{ix}-1)\right)\exp\left(i(a-b)\lambda x\right).$$

Proof. It follows from the above lemmas that

$$\mathscr{F}[\tilde{U}^{(t)}(\mu_N)](x) = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}(P_N A_N^n + Q_N B_N^n)$$
$$= 1 + \frac{a\lambda}{N}(ix) + \sum_{n=2}^{\infty} \frac{(ix)^n}{n!}\left(\frac{b\lambda}{N} O\left(\frac{1}{N^2}\right)\right)$$
$$= 1 + \frac{a\lambda}{N}(ix) + \frac{b\lambda}{N}\sum_{n=2}^{\infty} \frac{(ix)^n}{n!} + O\left(\frac{1}{N^2}\right)$$
$$= 1 - \frac{b\lambda}{N} + (a-b)\frac{\lambda}{N}(ix) + \frac{\lambda\alpha}{N}\sum_{n=0}^{\infty} \frac{(ix)^n}{n!} + O\left(\frac{1}{N^2}\right)$$
$$= 1 + \left((a-b)ix - b + be^{ix}\right)\frac{\lambda}{N} + O\left(\frac{1}{N^2}\right).$$

Hence we have

$$\lim_{N\to\infty} \mathscr{F}[\tilde{U}^{(t)}(\varrho_N)](x) = \lim_{N\to\infty} \left(\mathscr{F}[\tilde{U}^{(t)}(\mu_N)](x)\right)^N$$
$$= \lim_{N\to\infty}\left(1 + \frac{a\lambda}{N}(ix) + \sum_{n=2}^{\infty}\frac{(ix)^n}{n!}\left(\frac{b\lambda}{N} + O\left(\frac{1}{N^2}\right)\right)\right)^N$$
$$= \lim_{N\to\infty}\left(1 + \left((a-b)ix - b + be^{ix}\right)\frac{\lambda}{N} + O\left(\frac{1}{N^2}\right)\right)^N$$
$$= \exp\left(\left((a-b)ix - b + be^{ix}\right)\lambda\right) = \exp\left(b\lambda(e^{ix}-1)\right)\exp\left(i(a-b)\lambda x\right). \blacksquare$$

COROLLARY 2.5. *The $t(=(a, b))$-transformed measure of the $t(=(a, b))$-deformed classical Poisson measure of parameter λ is given by*

$$\tilde{U}^{(t)}(p_\lambda) = e^{-b\lambda}\sum_{k=0}^{\infty}\frac{(b\lambda)^k}{k!}\delta_{k+(a-b)\lambda}. \tag{2.2}$$

We shall investigate the orthogonal polynomials for the probability measure p_λ. First we recall that the Charlier polynomials belong to the classical Poisson measure which is, of course, the discrete measure,

$$e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\delta_k. \tag{2.3}$$

Namely, the Charlier polynomials have the orthogonal relation

$$\sum_{x=0}^{\infty}\frac{\lambda^x}{x!}C_m(x, \lambda)\,C_n(x, \lambda) = \lambda^{-n}e^{\lambda}n!\,\delta_{mn}$$

and satisfy the recurrence relation

$$C_{-1}(X, \lambda) = 0, \qquad C_0(X, \lambda) = 1,$$
$$(\lambda + n - X)\,C_n(X, \lambda) = \lambda C_{n+1}(X, \lambda) + nC_{n-1}(X, \lambda) \qquad \text{for } n \geqslant 0.$$

We can also consider the monic polynomials

$$P_n(X, \lambda) = (-\lambda)^n C_n(X, \lambda),$$

which satisfy the recurrence relation

$$P_{-1}(X, \lambda) = 0, \quad P_0(X, \lambda) = 1,$$

$$(X - \lambda - n) P_n(X, \lambda) = P_{n+1}(X, \lambda) + n\lambda P_{n-1}(X, \lambda) \quad \text{for } n \geqslant 0.$$

By comparing the measures in (2.2) and (2.3), it can be seen that the measure $\widetilde{U}^{(t)}(p_\lambda)$ can be obtained as the right shift by $(a-b)\lambda$ of the classical Poisson measure of parameter $b\lambda$, so the monic orthogonal polynomials $\{\widetilde{P}_n(X)\}$ for the measure $\widetilde{U}^{(t)}(p_\lambda)$ are given by

$$\widetilde{P}_n(X) = P_n\big(X - (a-b)\lambda, b\lambda\big),$$

the recurrence relation of which becomes

$$\widetilde{P}_0(X) = 1, \quad \widetilde{P}_1(X) = X - b\lambda,$$

$$\widetilde{P}_{n+1}(X) = (X - a\lambda - n)\widetilde{P}_n(X) - nb\lambda \widetilde{P}_{n-1}(X) \quad \text{for } n \geqslant 0,$$

that is, the Jacobi parameters, for $n \geqslant 1$, can be given by

$$\alpha_n = a\lambda + n - 1, \quad \beta_n = nb\lambda.$$

Hence we can obtain the Cauchy transform of the probability measure $\widetilde{U}^{(t)}(p_\lambda)$ in the continued fraction (Stieltjes expansion):

$$G_{\widetilde{U}^{(t)}(p_\lambda)}(z) = \cfrac{1}{z - \alpha\lambda - \cfrac{b\lambda}{z - a\lambda - 1 - \cfrac{2b\lambda}{z - a\lambda - 2 - \cfrac{3b\lambda}{z - a\lambda - 3 - \cfrac{4b\lambda}{\ddots}}}}}.$$

Since p_λ can be obtained as the $t^{-1} = (a^{-1}, b^{-1})$-transformed measure of $\widetilde{U}^{(t)}(p_\lambda)$, it follows, with the help of Remark 1.5, that the Cauchy transform of the probability measure p_λ can be given in the continued fraction

$$G_{p_\lambda}(z) = \cfrac{1}{z - \lambda - \cfrac{\lambda}{z - a\lambda - 1 - \cfrac{2b\lambda}{z - a\lambda - 2 - \cfrac{3b\lambda}{z - a\lambda - 3 - \cfrac{4b\lambda}{\ddots}}}}}.$$

Thus we have the following orthogonal polynomials:

THEOREM 2.6. *The orthogonal polynomials* $\{P_n^{(p_\lambda)}(X)\}$ *for the* **t**-*deformed classical Poisson measure* p_λ *are given by the following recurrence relations*:

$$P_0^{(p_\lambda)}(X) = 1, \quad P_1^{(p_\lambda)}(X) = X-\lambda,$$

$$P_2^{(p_\lambda)}(X) = \big(X-(a\lambda+1)\big)P_1^{(p_\lambda)}(X)-\lambda P_0^{(p_\lambda)}(X),$$

$$P_{n+1}^{(p_\lambda)}(X) = \big(X-(a\lambda+n)\big)P_n^{(p_\lambda)}(X)-nb\lambda P_{n-1}^{(p_\lambda)}(X) \quad \text{for } n \geqslant 2.$$

3. REMARKS ON THE t-DEFORMED FREE CONVOLUTION

For a given map V on probability measures on $\boldsymbol{R}$, one can define a deformed free convolution μ_0 of the probability measures μ_1 and μ_2 associated with the map V using the following formula of the conditionally free convolution in [3] (see also [2]):

$$(\mu_0, V(\mu_1) \boxplus V(\mu_2)) = (\mu_1, V(\mu_1)) \boxplus (\mu_2, V(\mu_2));$$

namely, we use the transformed probability measures as the conditional part.

Now we shall take the **t**-transformation $\tilde{U}^{(\mathbf{t})}$ as the map V and consider the conditionally free convolution as above. Then we shall see that the $\tilde{U}^{(\mathbf{t})}$-free convolution coincides with the **t**-deformed free convolution introduced in Definition 1.7, which allows us to apply the results on the conditionally free convolution to our **t**-free case.

PROPOSITION 3.1. *Let* μ_1 *and* μ_2 *be probability measures in* $\mathscr{P}^\infty(\boldsymbol{R})$. *For a given* $\mathbf{t}\,(=(a, b))$, *the* **t**-*deformed free convolution* $\mu_1 \boxplus_{(\mathbf{t})} \mu_2$ *satisfies the relation*

$$(\mu_1 \boxplus_{(\mathbf{t})} \mu_2, \tilde{U}^{(\mathbf{t})}(\mu_1) \boxplus \tilde{U}^{(\mathbf{t})}(\mu_2)) = (\mu_1, \tilde{U}^{(\mathbf{t})}(\mu_1)) \boxplus (\mu_2, \tilde{U}^{(\mathbf{t})}(\mu_2)),$$

where $\boxplus$ *on the right-hand side denotes the conditionally free convolution for pairs of probability measures in* [3].

Proof. We denote the **t**-deformed free convolution of μ_1 and μ_2 by μ_0, that is:

$$\mu_0^{(\mathbf{t})} = \mu_1^{(\mathbf{t})} \boxplus \mu_2^{(\mathbf{t})},$$

where $\mu_i^{(\mathbf{t})} = \tilde{U}^{(\mathbf{t})}(\mu_i)$. Since $\mu_0^{(\mathbf{t})}$ is the free convolution of $\mu_1^{(\mathbf{t})}$ and $\mu_2^{(\mathbf{t})}$, and the Voiculescu $\mathscr{R}$-transform of $\mu_i^{(\mathbf{t})}$ is given by

$$\mathscr{R}_{\mu_i^{(\mathbf{t})}}(z) = G_{\mu_i^{(\mathbf{t})}}^{\langle -1 \rangle}(z) - z^{-1},$$

we have the following relation:

$$G_{\mu_0^{(t)}}^{\langle -1\rangle}(z)-z^{-1}=\left(G_{\mu_1^{(t)}}^{\langle -1\rangle}(z)-z^{-1}\right)+\left(G_{\mu_2^{(t)}}^{\langle -1\rangle}(z)-z^{-1}\right), \tag{3.1}$$

where $G_{\mu_i^{(t)}}^{\langle -1\rangle}$ is the inverse of the function $G_{\mu_i^{(t)}}$ with respect to the composition.

On the other hand, the relation (1.2) in Remark 1.2 can be reformulated as

$$z-\frac{1}{G_{\mu_i^{(t)}}(z)}=b\left(z-\frac{1}{G_{\mu_i}(z)}\right)+(a-b)\,E(\mu_i),$$

and, by substituting $G_{\mu_i^{(t)}}^{\langle -1\rangle}(z)$ into z, we obtain the relation

$$G_{\mu_i^{(t)}}^{\langle -1\rangle}(z)-\frac{1}{z}=b\left(G_{\mu_i^{(t)}}^{\langle -1\rangle}(z)-\frac{1}{G_{\mu_i}\left(G_{\mu_i^{(t)}}^{\langle -1\rangle}(z)\right)}\right)+\left(1-\frac{b}{a}\right)E(\mu_i^{(t)}), \tag{3.2}$$

where we have used the fact that $E(\mu_i^{(t)})=aE(\mu_i)$. Combining the relations (3.1) and (3.2), we obtain

$$\begin{aligned}
&G_{\mu_0^{(t)}}^{\langle -1\rangle}(z)-\frac{1}{G_{\mu_0}\left(G_{\mu_0^{(t)}}^{\langle -1\rangle}(z)\right)}+\left(\frac{1}{b}-\frac{1}{a}\right)E(\mu_0^{(t)})\\
&=\left(G_{\mu_1^{(t)}}^{\langle -1\rangle}(z)-\frac{1}{G_{\mu_1}\left(G_{\mu_1^{(t)}}^{\langle -1\rangle}(z)\right)}\right)+\left(G_{\mu_2^{(t)}}^{\langle -1\rangle}(z)-\frac{1}{G_{\mu_2}\left(G_{\mu_2^{(t)}}^{\langle -1\rangle}(z)\right)}\right)\\
&\qquad+\left(\frac{1}{b}-\frac{1}{a}\right)\left(E(\mu_1^{(t)})+E(\mu_2^{(t)})\right).
\end{aligned}$$

Since the mean $E(\mu_i^{(t)})$ is the first cumulant of $\mu_i^{(t)}$ with respect to the free convolution, namely

$$E(\mu_0^{(t)})=E(\mu_1^{(t)})+E(\mu_2^{(t)}),$$

we obtain

$$\begin{aligned}
&G_{\mu_0^{(t)}}^{\langle -1\rangle}(z)-\frac{1}{G_{\mu_0}\left(G_{\mu_0^{(t)}}^{\langle -1\rangle}(z)\right)}\\
&=\left(G_{\mu_1^{(t)}}^{\langle -1\rangle}(z)-\frac{1}{G_{\mu_1}\left(G_{\mu_1^{(t)}}^{\langle -1\rangle}(z)\right)}\right)+\left(G_{\mu_2^{(t)}}^{\langle -1\rangle}(z)-\frac{1}{G_{\mu_2}\left(G_{\mu_2^{(t)}}^{\langle -1\rangle}(z)\right)}\right).
\end{aligned} \tag{3.3}$$

The above relation ensures that the probability measure μ_0 should be given by the $\tilde{U}^{(t)}$-free convolution of μ_1 and μ_2. Actually, for a probability measure μ, the cumulant series $\mathscr{R}_\mu^{(t)}$ for the $\tilde{U}^{(t)}$-free convolution can be determined by the relation (see [3])

$$\frac{1}{G_\mu(z)}=z-\mathscr{R}_\mu^{(t)}(G_\nu(z)), \tag{3.4}$$

where $\nu = \mu^{(t)}$, that is,

$$\mathscr{R}_{\mu}^{(t)}(z) = G_{\mu^{(t)}}^{\langle -1\rangle}(z) - \frac{1}{G_{\mu}(G_{\mu^{(t)}}^{\langle -1\rangle}(z))}.$$

Thus the relation (3.3) implies

$$\mathscr{R}_{\mu_0}^{(t)}(z) = \mathscr{R}_{\mu_1}^{(t)}(z) + \mathscr{R}_{\mu_2}^{(t)}(z). \quad \blacksquare$$

Now we shall discuss the moment-cumulant formula for the t-deformed free convolution. We write the cumulant series for the t-deformed free convolution of a probability measure μ as

$$\mathscr{R}_{\mu}^{(t)}(z) = \sum_{n\geqslant 1} R_n^{(t)}(\mu) z^{n-1},$$

and the Voiculescu $\mathscr{R}$-transform of the t-transformed measure $\nu = \mu^{(t)}$ in the form

$$\mathscr{R}_{\nu}(z) = \sum_{n\geqslant 1} r_n(\nu) z^{n-1}.$$

With the help of the combinatorial investigations on the conditionally free convolutions in [3] (see also [4]), we have the following relation among the moment sequence $\{m_n(\mu)\}_{n\geqslant 1}$ and the cumulant sequences $\{R_n^{(t)}(\mu)\}_{n\geqslant 1}$ and $\{r_n(\nu)\}_{n\geqslant 1}$:

$$m_n(\mu) = \sum_{\pi\in \mathrm{NC}(n)} \Big(\prod_{\substack{B_l\in\pi \\ B_l:\text{inner}}} r_{|B_l|}(\nu)\Big)\Big(\prod_{\substack{B_k\in\pi \\ B_k:\text{outer}}} R_{|B_k|}^{(t)}(\mu)\Big), \tag{3.5}$$

where $\mathrm{NC}(n)$ denotes the set of non-crossing partitions of n elements. Here a block B_l of a non-crossing partition is called *inner* if it is contained in some other block. A block B_k which is not inner is called *outer*.

As we have observed in the proof of Proposition 3.1, the relation (1.2) for the t-transformation can be reformulated as

$$z - \frac{1}{G_{\mu^{(t)}}(z)} = b\left(z - \frac{1}{G_{\mu}(z)}\right) + (a-b)E(\mu).$$

By substituting $G_{\mu^{(t)}}^{\langle -1\rangle}(z)$ for z, we can derive the relation between $\mathscr{R}_{\mu}^{(t)}(z)$ and $\mathscr{R}_{\nu}(z)$:

$$\mathscr{R}_{\nu}(z) = b\mathscr{R}_{\mu}^{(t)}(z) + (a-b)E(\mu).$$

Since the mean of μ might be regarded as the first cumulant for the t-deformed free convolution, namely $E(\mu) = R_1^{(t)}(\mu)$, we obtain

$$\sum_{n\geqslant 1} r_n(\nu) z^{n-1} = b \sum_{n\geqslant 1} R_n^{(t)}(\mu) z^{n-1} + (a-b) R_1^{(t)}(\mu),$$

which implies

$$r_1(\nu) = aR_1^{(t)}(\mu),$$
(3.6)
$$r_n(\nu) = bR_n^{(t)}(\mu) \quad \text{for } n \geq 2.$$

In order to give the moment-cumulant formula for the $\boldsymbol{t}$-free convolution, we shall introduce the statistics on non-crossing partitions: $\text{ins}(\pi)$ defined as the *number of inner singletons*, and $\text{nsi}(\pi)$ determined as the *number of non-singletonic inner blocks*. Putting the relations (3.6) into the formula (3.5), we obtain the following:

THEOREM 3.2. *The moment-cumulant formula for the $\boldsymbol{t}(=(a, b))$-deformed free convolution can be given by*

$$m_n(\mu) = \sum_{\pi \in \mathrm{NC}(n)} a^{\mathrm{ins}(\pi)} b^{\mathrm{nsi}(\pi)} \prod_{B \in \pi} R_{|B|}^{(t)}(\mu).$$

As we have mentioned before, in the case of $a = b = t$, the above formula will be reduced to, of course, the one for the t-free convolution because $(\text{ins}(\pi) + \text{nsi}(\pi))$ is nothing else but the number of inner blocks of the non-crossing partition π.

4. THE $\boldsymbol{t}$-DEFORMED FREE POISSON LAW

In this section we shall study the Poisson limit theorem for $\boldsymbol{t}$-deformed free convolution.

As we have done in Section 2, it is natural to define the *$\boldsymbol{t}$-deformed free Poisson measure* of parameter λ, $\text{f-Po}_\lambda^{(t)}$, by the weak limit

$$\text{f-Po}_\lambda^{(t)} = \lim_{N \to \infty} \underbrace{\mu_N \boxplus_{(t)} \mu_N \boxplus_{(t)} \ldots \boxplus_{(t)} \mu_N}_{N},$$

where

$$\mu_N = \left(1 - \frac{\lambda}{N}\right)\delta_0 + \frac{\lambda}{N}\delta_1$$

and, of course, $\boxplus$ means Voiculescu's free convolution. In this section, let us simply denote $\text{f-Po}_\lambda^{(t)}$ by p_λ.

We write the $\boldsymbol{t}(=a, b))$-transformed measure of μ_N in the form

$$\tilde{U}^{(t)}(\mu_N) = P_N \delta_{A_N} + Q_N \delta_{B_N},$$

where A_N, B_N, P_N, and Q_N are the same as in (2.1). The Voiculescu $\mathscr{R}$-transform of $\tilde{U}^{(t)}(\mu_N)$, $\mathscr{R}_{\tilde{U}^{(t)}(\mu_N)}(z)$, should satisfy the relation

$$\frac{P_N}{\left(\mathscr{R}_{\tilde{U}^{(t)}(\mu_N)}(z) + z^{-1}\right) - A_N} + \frac{Q_N}{\left(\mathscr{R}_{\tilde{U}^{(t)}(\mu_N)}(z) + z^{-1}\right) - B_N} = z,$$

which can be solved as

$$\mathcal{R}_{\tilde{U}^{(t)}(\mu_N)}(z) = \frac{1}{2Nz}\big((z-1)N + a\lambda z - \lambda z - \{(z-1)^2 N^2$$
$$+ 2(1+a-z-az+2bz)\lambda zN + (2a-4b+\lambda+a^2\lambda)\lambda z^2\}^{1/2}\big)$$
$$= \frac{\lambda(az-bz-a)}{(z-1)N} + O\left(\frac{1}{N^2}\right).$$

Since we know that

$$\mathcal{R}_{(\tilde{U}^{(t)}(\mu_N))^{\boxplus N}}(z) = N\mathcal{R}_{\tilde{U}^{(t)}(\mu_N)}(z),$$

taking the limit $N\to\infty$, we obtain

$$\mathcal{R}_{\tilde{U}^{(t)}(p_\lambda)}(z) = \frac{\lambda(az-bz-a)}{z-1} = (a-b)\lambda + \frac{b\lambda}{1-z},$$

which implies that the free cumulants of $t(=(a, b))$-transformed measure $\tilde{U}^{(t)}(p_\lambda)$ can be given by

$$r_1(\tilde{U}^{(t)}(p_\lambda)) = a\lambda,$$
$$r_n(\tilde{U}^{(t)}(p_\lambda)) = b\lambda \quad \text{for } n \geqslant 2.$$

Thus it follows that the t-deformed free cumulants series of the t-deformed free Poisson law p_λ becomes

$$\mathcal{R}^{(t)}_{p_\lambda}(z) = \frac{\lambda}{1-z},$$

because, by the relation (3.6), we have

$$R^{(t)}_n(p_\lambda) = \lambda \quad \text{for } n \geqslant 1,$$

which is still consistent with the characterization of the Poisson law, that is, the Poisson law of parameter λ should be characterized as the law all the cumulants of which are equal to λ.

Now let us determine the measure of the $t(=(a, b))$-deformed free Poisson law, p_λ, exactly. By the formula (3.4) for the conditionally free convolution, the Cauchy transformation of p_λ should satisfy the relation

$$\frac{1}{G_{p_\lambda}(z)} = z - \mathcal{R}^{(t)}_{p_\lambda}(G_{\tilde{U}^{(t)}(p_\lambda)}) = z - \frac{\lambda}{1-G_{\tilde{U}^{(t)}(p_\lambda)}},$$

that is,

$$\frac{1}{G_{p_\lambda}(z)} = z - \frac{\lambda}{1-\dfrac{1}{b/G_{p_\lambda}(z)+(1-b)z+(b-a)\lambda}},$$

which implies the quadratic equation

$$AG_{p_\lambda}(z)^2 + BG_{p_\lambda}(z) + C = 0,$$

where

$$(4.1)\qquad \begin{aligned} A &= (b-1)z^2 + (\lambda + a\lambda + 1 - 2b\lambda)z - (a-b)\lambda^2, \\ B &= (1-2b)z + (2b-a)\lambda - 1, \qquad C = b. \end{aligned}$$

Although we can determine the probability measure p_λ by solving the above quadratic equation and using the Stieltjes inversion formula, we shall here give the measure with the help of the orthogonal polynomials.

LEMMA 4.1. *The Cauchy transform of the t-deformed free Poisson measure p_λ can be expanded into the following continued fraction*:

$$G_{p_\lambda}(z) = \cfrac{1}{z-\lambda-\cfrac{\lambda}{z-(a\lambda+1)-\cfrac{b\lambda}{z-(a\lambda+1)-\cfrac{b\lambda}{z-(a\lambda+1)-\cfrac{b\lambda}{\ddots}}}}}.$$

Proof. First we give the function $H(z)$ by the relation

$$H(z) = z-(a\lambda+1) - \frac{b\lambda}{H(z)},$$

which yields the equation

$$(4.2)\qquad H(z)^2 - \big(z-(a\lambda+1)\big)H(z) + b\lambda = 0.$$

If we put the function $G(z)$ as

$$G(z) = \frac{1}{z-\lambda-\lambda/H(z)},$$

then it can be easily checked by using the equation (4.2) that $G(z)$ satisfies the same quadratic equation as in (4.1). ∎

Applying the theory of Stieltjes expansion (see, for instance, [10]), it can be claimed that the t-deformed free Poisson measure p_λ has the following orthogonal polynomials:

PROPOSITION 4.2. *We define the sequence $\{Q_n^{(p_\lambda)}(X)\}_{n\geqslant 0}$ of polynomials by the following recurrence relations*:

$$\begin{aligned} &Q_0^{(p_\lambda)}(X) = 1, \qquad Q_1^{(p_\lambda)}(X) = X-\lambda, \\ &Q_2^{(p_\lambda)}(X) = \big(X-(\alpha\lambda+1)\big)Q_1^{(p_\lambda)}(X) - \lambda Q_0^{(p_\lambda)}(X), \\ &Q_{n+1}^{(p_\lambda)}(X) = \big(X-(\alpha\lambda+1)\big)Q_n^{(p_\lambda)}(X) - b\lambda Q_{n-1}^{(p_\lambda)}(X) \qquad \textit{for } n \geqslant 2. \end{aligned}$$

Then $\{Q_n^{(p_\lambda)}(X)\}_{n\geqslant 0}$ *makes an orthogonal system with respect to the* **t**-*deformed free Poisson measure* p_λ, *that is,*

$$\int_{t\in\boldsymbol{R}} Q_k^{(p_\lambda)}(t)\,Q_m^{(p_\lambda)}(t)\,dp_\lambda(t) = 0 \qquad \textit{if } k \neq m.$$

We can reformulate our orthogonal polynomials in a constant recurrence type of Cohen–Trenholme (see [6]) in the following way:

$$Q_0(X) = b^{-1}, \qquad Q_1(X) = X-\lambda,$$
$$Q_{n+1}(X) = \big(X-(a\lambda+1)\big)Q_n(X)-b\lambda Q_{n-1}(X) \qquad \text{for } n\geqslant 1.$$

The unique probability measure orthogonalizing the above system of polynomials has been calculated in [8] (see also [6]), which is compactly supported and it has the absolutely continuous part and the discrete part in general. Using the result in [8], we can give the probability measure p_λ exactly as follows:

PROPOSITION 4.3. *Let*

$$f(x) = (b-1)x^2+(\lambda+a\lambda+1-2b\lambda)x-(a-b)\lambda^2.$$

Then the absolutely continuous part p_λ^C *of the* **t**-*deformed free Poisson measure* p_λ *is given by*

$$dp_\lambda^C(x) = \frac{\sqrt{4b\lambda-(x-a\lambda-1)^2}}{2\pi f(x)}\,\chi_{[1+a\lambda-2\sqrt{b\lambda},1+a\lambda+2\sqrt{b\lambda}]}(x)\,dx,$$

and the discrete part p_λ^D *is* 0 *except possibly in the following cases*:

Case 1. $f(x)$ *has two real roots* y_1 *and* y_2. *Then*

$$dp_\lambda^D(x) = w_1\delta_{y_1}+w_2\delta_{y_2},$$

where

$$w_i = \frac{1}{\sqrt{(\lambda-a\lambda-1)^2-4\lambda(b-1)}} \times \max\left(0, \frac{\lambda}{|y_i-(\lambda-a\lambda-1)|}-b\,|y_i-(\lambda-a\lambda-1)|\right).$$

In this case, the parameters should satisfy the inequality

$$(\lambda+1)^2+a\lambda(a\lambda-2\lambda+2)-4\lambda b > 0,$$

and two real roots can be given by

$$y_i = \frac{2b\lambda-\lambda-a\lambda-1\pm\sqrt{(\lambda+1)^2+a\lambda(a\lambda-2\lambda+2)-4\lambda b}}{2(b-1)}.$$

Case 2. $b = 1$ *and* $\lambda \neq a\lambda + 1$ *so that* $f(x)$ *has one real root*

$$y = \lambda + \frac{\lambda}{\lambda - a\lambda - 1}.$$

Then

$$dp_\lambda^D(x) = \max\left(0,\, 1 - \frac{b\lambda}{(\lambda - a\lambda - 1)^2}\right)\delta_y.$$

5. DEFORMED FERMI CONVOLUTION

In this section, we consider another special case of the $\boldsymbol{t}$-transformation, which will give an interpolation between free and Fermi convolutions in the deformed free case.

DEFINITION 5.1. For $\tau \geqslant 0$, we shall call the $\boldsymbol{t}(= (1, \tau))$-deformed free convolution the *τ-free convolution.*

The τ-free convolution will give a deformation of Fermi convolution introduced in [7]. Actually, it follows from Theorem 3.2 that the moment-cumulant formula for the τ-free convolution can be given by

$$m_n(\mu) = \sum_{\pi \in \mathrm{NC}(n)} \tau^{\mathrm{nsi}(\pi)} \prod_{B \in \pi} R_{|B|}^{(\tau)}(\mu),$$

where $\mathrm{nsi}(\pi)$ is the number of non-singletonic inner blocks of non-crossing partition π. In the case of $\tau = 0$, this formula will be reduced to the following one for the Fermi convolution:

$$m_n(\mu) = \sum_{\pi \in \mathrm{AIP}(n)} \prod_{B \in \pi} R_{|B|}^{(\mathrm{Fermi})}(\mu),$$

where $\mathrm{AIP}(n)$ denotes the set of *almost interval partitions* of n elements, which are non-crossing partitions that do not contain inner blocks other than singletons (see [7]). Thus the τ-free convolution interpolates between free and Fermi convolutions at $\tau = 1$ and at $\tau = 0$, respectively.

As we have mentioned before, the τ-free Gaussian law is the same as for the t-free case in [4] and the model of the τ-free Gaussian random variables would be also realized as in [5] on the $\tau(=t)$-free Fock space. Here, we shall give the model of the τ-free Poisson processes on the τ-free Fock space using the knowledge of the τ-free Poisson law, which can be easily obtained from the previous section.

For our convenience, we shall begin with recalling the definition of the τ-free Fock space that is a deformed full Fock space introduced in [5].

DEFINITION 5.2. For $\tau \geqslant 0$ and a given Hilbert space $\mathscr{H}$ with the scalar product $\langle | \rangle$ (later, we will specialize to $\mathscr{H} = L^2(\boldsymbol{R}_+)$), the *$\tau$-free Fock space* is defined as the full Fock space

$$\mathscr{F}^{(\tau)}(\mathscr{H}) = \boldsymbol{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathscr{H}^{\otimes n}$$

completed with respect to the following scalar product:

$$(\xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n | \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_m)_\tau = \delta_{n,m} \tau^{n-1} \prod_{j=1}^{n} \langle \xi_j | \eta_j \rangle,$$

$$(\Omega | \Omega)_\tau = 1,$$

where Ω is the distinguished unit vector called *vacuum.*

For a vector $\xi \in \mathscr{H}$, we define the *τ-creation operator* $a^{(\tau)*}(\xi)$ on $\mathscr{F}^{(\tau)}(\mathscr{H})$ by

$$a^{(\tau)*}(\xi)\, \xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n = \xi \otimes \xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n \quad (n \geqslant 1),$$

$$a^{(\tau)*}(\xi)\, \Omega = \xi,$$

and the *τ-annihilation operator* $a^{(\tau)}(\xi)$ on $\mathscr{F}^{(\tau)}(\mathscr{H})$ by

$$a^{(\tau)}(\xi)\, \xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n = \tau \langle \xi_1 | \xi \rangle\, \xi_2 \otimes \ldots \otimes \xi_n \quad (n \geqslant 2),$$

$$a^{(\tau)}(\xi)\, \xi_1 = \langle \xi_1 | \xi \rangle\, \Omega, \quad a^{(\tau)}(\xi)\, \Omega = 0,$$

where $\xi_1, \xi_2, \ldots, \xi_n$ are arbitrary vectors in $\mathscr{H}$.

Remark 5.3. The operators $a^{(\tau)}(\xi)$ and $a^{(\tau)*}(\xi)$ are adjoint of each other with respect to the scalar product $(|)_\tau$, that is, $a^{(\tau)*}(\xi) = a^{(\tau)}(\xi)^*$.

The *vacuum state* φ on all bounded operators b on the Fock space $\mathscr{F}^{(\tau)}(\mathscr{H})$ can be defined as

$$\varphi(b) = (b\Omega | \Omega)_\tau.$$

Remark 5.4. The position operators

$$g^{(\tau)}(\xi) = a^{(\tau)}(\xi) + a^{(\tau)}(\xi)^* \quad (\xi \in \mathscr{H})$$

are the model of the τ-free Gaussian random variables, that is, the distribution of the operator $g^{(\tau)}(\xi)$ with respect to the vacuum expectation can be given by the τ-free Gaussian law (see [5]).

Furthermore, in order to give the τ-free Poisson processes, we shall adopt the analogue $p(T)$ of the *gauge operator* for $T \in B(\mathscr{H})$ introduced in [9], which is defined as

$$p(T)\, \Omega = 0,$$

$$p(T)(\xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n) = (T\xi_1) \otimes \xi_2 \otimes \ldots \otimes \xi_n.$$

The operator $p(T)$ can be also regarded as the differential second quantization operator for the free case.

Here we take $\mathscr{H} = L^2(\boldsymbol{R}_+)$ with the canonical inner product, and consider the τ-free Fock space $\mathscr{F}^{(\tau)}(L^2(\boldsymbol{R}_+))$. For $h \in L^\infty(\boldsymbol{R}_+)$, we define the multiplication operator T_h by $T_h(f) = hf$, where $f \in L^2(\boldsymbol{R}_+)$, and write $p(h) = p(T_h)$.

Let us consider the sum of the basic processes $a_x = a^{(\tau)}(\chi_{[0,x)})$, $a_x^* = a^{(\tau)}(\chi_{[0,x)})^*$, $p_x = p(\chi_{[0,x)})$, and the scalar $x\mathbb{1}$,

$$c_x = p_x + a_x + a_x^* + x\mathbb{1} \qquad (x \in \boldsymbol{R}_+)$$

on the τ-free Fock space, $\mathscr{F}^{(\tau)}(L^2(\boldsymbol{R}_+))$, where $\chi_{[0,x)}$ is the characteristic function on the interval $[0, x)$.

We shall see that the process c_x is our desired τ-free Poisson process.

LEMMA 5.5. *For $x \geqslant 0$, let $\{Q_n^{(\tau,x)}(X)\}_{n=0}^\infty$ be the orthogonal polynomials with respect to the τ-free Poisson measure of parameter x* (cf. Proposition 4.2). *Then we have*

$$Q_n^{(\tau,x)}(c_x)\Omega = \chi_{[0,x)}^{\otimes n} \qquad (n \geqslant 0),$$

where $\chi_{[0,x)}^{\otimes 0}$ means Ω.

Proof. We shall show the lemma by induction on n. It is clear that

$$Q_0^{(\tau,x)}(c_x)\Omega = \mathbb{1}\Omega = \Omega, \qquad Q_1^{(\tau,x)}(c_x)\Omega = a_x^*\Omega - x\mathbb{1}\Omega = \chi_{[0,x)},$$

$$\begin{aligned} Q_2^{(\tau,x)}(c_x)\Omega &= (c_x - (x+1)\mathbb{1})Q_1^{(\tau,x)}(c_x)\Omega - xQ_0^{(\tau,x)}(c_x)\Omega \\ &= (p_x + a_x + a_x^* - \mathbb{1})\chi_{[0,x)} - x\Omega \\ &= \chi_{[0,x)} + x\Omega + \chi_{[0,x)}^{\otimes 2} - \chi_{[0,x)} - x\Omega = \chi_{[0,x)}^{\otimes 2}. \end{aligned}$$

Assume $Q_k^{(\tau,x)}(c_x)\Omega = \chi_{[0,x)}^{\otimes k}$ for $k \leqslant n$. Then we obtain, for $n > 2$,

$$\begin{aligned} Q_{n+1}^{(\tau,x)}(c_x)\Omega &= (c_x - (x+1)\mathbb{1})Q_n^{(\tau,x)}(c_x)\Omega - \tau x Q_{n-1}^{(\tau,x)}(c_x)\Omega \\ &= (p_x + a_x + a_x^* - \mathbb{1})\chi_{[0,x)}^{\otimes n} - \tau x \chi_{[0,x)}^{\otimes(n-1)} \\ &= \chi_{[0,x)}^{\otimes n} + \tau x \chi_{[0,x)}^{\otimes(n-1)} + \chi_{[0,x)}^{\otimes(n+1)} - \chi_{[0,x)}^{\otimes n} - \tau x \chi_{[0,x)}^{\otimes(n-1)} = \chi_{[0,x)}^{\otimes(n+1)}. \quad \blacksquare \end{aligned}$$

It follows from Lemma 5.5 that if $k \neq m$, then we have

$$(Q_m^{(\tau,x)}(c_x)Q_k^{(\tau,x)}(c_x)\Omega \mid \Omega)_\tau = (Q_k^{(\tau,x)}(c_x)\Omega \mid Q_m^{(\tau,x)}(c_x)\Omega)_\tau = (\chi_{[0,x)}^{\otimes k} \mid \chi_{[0,x)}^{\otimes m})_\tau = 0$$

because the operator c_x is self-adjoint with respect to the inner product $(\,|\,)_\tau$. This means that, for any polynomial f, we have

$$(f(c_x)\Omega \mid \Omega)_\tau = \int_{y \in \boldsymbol{R}} f(y)\, dp_x^{(\tau)}(y),$$

where $p_x^{(\tau)}$ is the τ-free Poisson measure of parameter x. Here we have obtained the following theorem:

THEOREM 5.6. *The moments of the process c_x $(x \geqslant 0)$ with respect to the vacuum expectation can be given by the τ-free Poisson law of parameter x, namely,*

$$\varphi(c_x^n) = \sum_{k=1}^{n} \Big(\sum_{\pi \in \mathrm{NC}(n,k)} \tau^{\mathrm{nsi}(\pi)} \Big) x^k,$$

where $\mathrm{NC}(n, k)$ *is the set of non-crossing partitions of n elements with precisely k blocks.*

Remark 5.7. In [9], it can be found that the free Poisson process on the full Fock space is of the same form as c_x, which corresponds to the case of $\tau = 1$. Therefore the $(\tau = 0)$-free Poisson process can be regarded as the model of the fermionic Poisson process.

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Received on 12.3.2004