PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 24, Fasc. 1 (2004), pp. 165–172

ON ARMA(1, q) MODELS WITH BOUNDED AND PERIODICALLY CORRELATED SOLUTIONS

BY

A. WERON AND A. WYŁOMAŃSKA (WROCŁAW)

Abstract. In this paper, motivated by [2], we derive necessary and sufficient conditions for bounded and periodically correlated solutions to the system of equations described by ARMA(1, q) model.

2000 Mathematics Subject Classification: 62M10.

Key words and phrases: Periodically correlated, ARMA model, periodic coefficients.

1. INTRODUCTION

We consider a system ARMA(1, q) given by the formula

$$X_n - b_n X_{n-1} = a_n \xi_n + a_{n-1} \xi_{n-1} + \ldots + a_{n-(q-1)} \xi_{n-(q-1)}, \tag{1}$$

where

- (X_n) is a sequence of complex random variables with mean 0 and finite variance in the space with the inner product $(\cdot, \cdot)$ and $M_X = \overline{\text{sp}}\{X_k\colon k \in Z\}$,
- (b_n) and (a_n) are sequences of non-zero complex numbers,
- (ξ_n) is a sequence of uncorrelated complex random variables with mean 0 and variance 1 and $M_\xi = \overline{\text{sp}}\{\xi_k\colon k \in Z\}$.

In a recent paper Hurd et al. [2] gave necessary and sufficient conditions for boundedness in the general case of AR(1) model, and then specifically for periodic and almost periodic coefficients (a_n). The present effort is an attempt to understand the situation in case of ARMA(1, q) models which is important for applications, see [1] and [3]. Such systems arise in climatology, economics, hydrology, electrical engineering and other disciplines. In Section 2 we discuss the relationship between existence of bounded solutions to the system equations described by ARMA(1, q) model and conditions on their coefficients (Theorem 1). Next, periodically correlated solutions are examined (Theorem 2). In Section 3 we simplify the consideration for $q = 2$.

Let us put

$$B_r^s = \prod_{j=r}^{s} b_j$$

with the convention that $B_r^s = 1$ if $r > s$. It is easy to show that iterating k times equation (1) we obtain:

$$X_{n+k} = B_{n+1}^{n+k} X_n + \sum_{j=1}^{k} B_{n+j+1}^{n+k} \sum_{s=0}^{q-1} \xi_{n+j-s} a_{n+j-s}, \tag{2}$$

$$X_{n-k} = \frac{X_n}{B_{n-k+1}^{n}} - \sum_{j=1}^{k} \frac{1}{B_{n-k+1}^{n-k+j}} \sum_{s=0}^{q-1} \xi_{n-k+j-s} a_{n-k+j-s}. \tag{3}$$

2. THE ARMA(1, q) MODEL

DEFINITION 1. A stochastic sequence is called *bounded* if

$$\sup_n \|X_n\| = \infty.$$

LEMMA 1. *If* $\sup_r |B_1^r| = \infty$ *and system* (1) *has a bounded solution in* $M\xi$, *then*:

$$\sup_{n \in Z} \left[\sum_{s=2-q}^{0} \left| \sum_{j=1}^{q-1+s} \frac{1}{B_{n+1}^{n+j}} \right|^2 |a_{n+s}|^2 + \sum_{s=1}^{\infty} \left| \sum_{j=s}^{q-1+s} \frac{1}{B_{n+1}^{n+j}} \right|^2 |a_{n+s}|^2 \right] < \infty. \tag{4}$$

Proof. If $\sup_r |B_1^r| = \infty$, then there exists a subsequence k_r of positive integers such that

$$\lim_r |B_1^{k_r}| = \infty.$$

So we have for all $n \in Z$:

$$\lim_r |B_{n+1}^{n+k_r}| = \infty.$$

If system (1) has a bounded solution, then from (2) we obtain

$$X_n + \sum_{j=1}^{k_r} \frac{B_{n+j+1}^{n+k_r}}{B_{n+1}^{n+k_r}} \sum_{s=0}^{q-1} \xi_{n+j-s} a_{n+j-s} = \frac{X_{n+k_r}}{B_{n+1}^{n+k_r}} \xrightarrow{r} 0.$$

Hence

$$X_n = -\lim_r \left[\sum_{j=1}^{k_r} \frac{1}{B_{n+1}^{n+j}} \sum_{s=0}^{q-1} \xi_{n+j-s} a_{n+j-s} \right].$$

We obtain

$$\sup_{n\in Z}\left[\lim_r\left\|-\sum_{j=1}^{k_r}\frac{1}{B_{n+1}^{n+j}}\sum_{s=0}^{q-1}\xi_{n+j-s}a_{n+j-s}\right\|^2\right]$$

$$=\sup_{n\in Z}\left[\sum_{s=2-q}^{0}\left|\sum_{j=1}^{q-1+s}\frac{1}{B_{n+1}^{n+j}}\right|^2|a_{n+s}|^2+\sum_{s=1}^{\infty}\left|\sum_{j=s}^{q-1+s}\frac{1}{B_{n+1}^{n+j}}\right|^2|a_{n+s}|^2\right].$$

Since X_n is a bounded solution of system (1), we obtain

$$\sup_{n\in Z}\left[\lim_r\left\|-\sum_{j=1}^{k_r}\frac{1}{B_{n+1}^{n+j}}\sum_{s=0}^{q-1}\xi_{n+j-s}a_{n+j-s}\right\|^2\right]=\sup_{n\in Z}\|X_n\|^2<\infty;$$

hence

$$\sup_{n\in Z}\left[\sum_{s=2-q}^{0}\left|\sum_{j=1}^{q-1+s}\frac{1}{B_{n+1}^{n+j}}\right|^2|a_{n+s}|^2+\sum_{s=1}^{\infty}\left|\sum_{j=s}^{q-1+s}\frac{1}{B_{n+1}^{n+j}}\right|^2|a_{n+s}|^2\right]<\infty. \blacksquare$$

LEMMA 2. *If* $\sup_r|B_r^0|^{-1}=\infty$ *and system* (1) *has a bounded solution in* $M\xi$, *then*:

$$\sup_{n\in Z}\Big[\sum_{s=0}^{q-2}\Big|\sum_{j=-s}^{0}B_{n+j+1}^{n}\Big|^2|a_{n-s}|^2+\sum_{s=q-1}^{\infty}\Big|\sum_{j=-s}^{q-1-s}B_{n+j+1}^{n}\Big|^2|a_{n-s}|^2\Big]<\infty. \tag{5}$$

Proof. If $\sup_r|B_r^0|^{-1}=\infty$, then there is a subsequence (k_r) of positive integers such that

$$\lim_r|B_{k_r}^0|^{-1}=\infty.$$

For all $n\in Z$ we have

$$\lim_r|B_{n+k_r}^n|^{-1}=\infty.$$

Since system (1) has the bounded solution, we get from (3):

$$X_n-\sum_{j=1}^{k_r}\sum_{s=0}^{q-1}B_{n-k_r+1+j}^{n}\xi_{n-k_r+j-s}a_{n-k_r+j-s}=X_{n-k_r}B_{n-k_r+1}^{n}\xrightarrow{r}0.$$

So we obtain

$$X_n=\lim_r\Big[\sum_{j=1}^{k_r}\sum_{s=0}^{q-1}B_{n-k_r+1+j}^{n}\xi_{n-k_r+j-s}a_{n-k_r+j-s}\Big]$$

$$=\lim_r\Big[\sum_{j=-k_r+1}^{0}\sum_{s=0}^{q-1}B_{n+1-j}^{n}\xi_{n-j-s}a_{n-j-s}\Big].$$

Since X_n is the bounded solution of system (1) and ξ_n is the orthonormal basis in $M\xi$, we have

$$\sup_{n\in Z}\|X_n\|^2 = \sup_{n\in Z}\Big[\lim_r \Big\|\sum_{j=-k_r+1}^{0}\sum_{s=0}^{q-1} B^n_{n+1-j}\xi_{n-j-s}a_{n-j-s}\Big\|^2\Big]$$
$$= \sup_{n\in Z}\Big[\sum_{s=0}^{q-2}\Big|\sum_{j=-s}^{0} B^n_{n+j+1}\Big|^2|a_{n-s}|^2+\sum_{s=q-1}^{\infty}\Big|\sum_{j=-s}^{q-1-s} B^n_{n+j+1}\Big|^2|a_{n-s}|^2\Big].$$

We obtain then

$$\sup_{n\in Z}\Big[\sum_{s=0}^{q-2}\Big|\sum_{j=-s}^{0} B^n_{n+j+1}\Big|^2|a_{n-s}|^2+\sum_{s=q-1}^{\infty}\Big|\sum_{j=-s}^{q-1-s} B^n_{n+j+1}\Big|^2|a_{n-s}|^2\Big] < \infty. \blacksquare$$

If $\sup_r|B^r_1| = \infty$ and $\sup_r|B^0_r|^{-1} = \infty$, then system (1) has a bounded solution. But there is a third possible condition, which gives a bounded solution of equation (1):

$$\sup_r|B^r_n| < \infty \quad \text{and} \quad \sup_r|B^0_r|^{-1} < \infty. \tag{6}$$

LEMMA 3. *If condition* (6) *holds and system* (1) *has a bounded solution, then*:

$$\sup_{n\in Z}\sum_{s=0}^{n+q-2}\Big|\sum_{j=\max(1-n,-s)}^{\min(0,q-1-s)} B^n_{n+1+j}\Big|^2|a_{n-s}|^2 < \infty \tag{7}$$

and

$$\sup_{n\in Z}\sum_{s=2-q}^{-n}\Big|\sum_{j=\max(1,s)}^{\min(-n,q-1+s)}\frac{1}{B^{n+j}_{n+1}}\Big|^2|a_{n+s}|^2 < \infty. \tag{8}$$

Proof. We use (2) and (3) (and provide $n = 0$). For all $k\in Z$ and some C we assume that $|B^k_1| < C$ and $|B^0_{-k}|^{-1} < C$. For all $k > 0$ we then have

$$\sup_{n\in Z}\sum_{s=0}^{n+q-2}\Big|\sum_{j=\max(1-n,-s)}^{\min(0,q-1-s)} B^n_{n+1+j}\Big|^2|a_{n-s}|^2$$
$$= \sup_{n\in Z}\|X_n-B^n_1X_0\|^2 \leqslant \sup_{n\in Z}\|X_n\|^2(1+C)^2 < \infty,$$

$$\sup_{n\in Z}\sum_{s=2-q}^{-n}\Big|\sum_{j=\max(1,s)}^{\min(-n,q-1+s)}\frac{1}{B^{n+j}_{n+1}}\Big|^2|a_{n+s}|^2$$
$$= \sup_{n\in Z}\Big\|X_{-n}-\frac{X_0}{B^0_{-n+1}}\Big\|^2 \leqslant \sup_{n\in Z}\|X_{-n}\|^2(1+C)^2 < \infty.$$

The solution of system (1) is given by

$$X_k = \begin{cases} B^k_1X+\sum_{j=1}^{\infty}B^k_{j+1}\sum_{s=0}^{q-1}\xi_{j-s}a_{j-s} & \text{if } k>0,\\ X & \text{if } k=0,\\ X/B^0_{k+1}-\sum_{j=k+1}^{0}B^{-j}_{k+1}\sum_{s=0}^{q-1}\xi_{j-s}a_{j-s} & \text{if } k<0,\end{cases} \tag{9}$$

where X is a random variable in $M\xi$. $\blacksquare$

THEOREM 1. *System* (1) *has a bounded solution if and only if one of the following conditions holds*:

(I) $\sup_r |B_1^r| = \infty$ *and*

$$\sup_{n\in Z}\left[\sum_{s=2-q}^{0}\left|\sum_{j=1}^{q-1+s}\frac{1}{B_{n+1}^{n+j}}\right|^2 |a_{n+s}|^2+\sum_{s=1}^{\infty}\left|\sum_{j=s}^{q-1+s}\frac{1}{B_{n+1}^{n+j}}\right|^2 |a_{n+s}|^2\right]<\infty.$$

(II) $\sup_r |B_r^0|^{-1} = \infty$ *and*

$$\sup_{n\in Z}\Big[\sum_{s=0}^{q-2}\Big|\sum_{j=-s}^{0}B_{n+j+1}^{n}\Big|^2 |a_{n-s}|^2+\sum_{s=q-1}^{\infty}\Big|\sum_{j=-s}^{q-1-s}B_{n+j+1}^{n}\Big|^2 |a_{n-s}|^2\Big]<\infty.$$

(III) $\sup_r |B_1^r| < \infty$, $\sup_r |B_r^0|^{-1} < \infty$, *and*

$$\sup_{n\in Z}\sum_{s=0}^{n+q-2}\Big|\sum_{j=\max(1-n,-s)}^{\min(0,q-1-s)}B_{n+1+j}^{n}\Big|^2 |a_{n-s}|^2<\infty,$$

$$\sup_{n\in Z}\sum_{s=2-q}^{-n}\left|\sum_{j=\max(1,s)}^{\min(-n,q-1+s)}\frac{1}{B_{n+1}^{n+j}}\right|^2 |a_{n+s}|^2<\infty.$$

Proof. If condition (I) holds, then the solution of system (1), given by the formula

$$X_n=-\sum_{j=1}^{\infty}\frac{1}{B_{n+1}^{n+j}}\sum_{s=0}^{q-1}\xi_{n+j-s}a_{n+j-s},\tag{10}$$

is bounded.

If condition (II) holds, then X_n defined by

$$X_n=\sum_{j=1}^{\infty}B_{n+1-j}^{n}\sum_{s=0}^{q-1}\xi_{n-j-s}a_{n-j-s}\tag{11}$$

is the bounded solution of system (1).

If condition (III) holds, then X_n given by formula (9) is bounded and is a solution of system (1). In Lemmas 1, 2 and 3 it is shown that if X_n is a bounded solution of system (1), then one of the conditions (I), (II) or (III) holds. ∎

DEFINITION 2. A stochastic sequence (X_n) is called *periodically correlated* with period T if for all k a sequence (X_{n+k}, X_n) is periodic in n with period T, i.e. $(X_{n+k}, X_n) = (X_{n+k+T}, X_{n+T})$.

THEOREM 2. *If* (b_n) *and* (a_n) *are periodic with the same period* T *and* $P = b_1 b_2 \ldots b_T$, *then system* (1) *has a bounded solution if and only if* $|P| \neq 1$. *Moreover, the solution is periodically correlated with the same period* T *and*:

(i) *if* $|P| > 1$, *then the solution is given by* (10);

(ii) *if* $|P| < 1$, *then the solution is given by* (11).

Proof. (i) If $|P| > 1$, then for all $n \in Z$, we have:

$$\sum_{s=2-q}^{0} \left| \sum_{j=1}^{q-1+s} \frac{1}{B_{n+1}^{n+j}} \right|^2 |a_{n+s}|^2 + \sum_{s=1}^{\infty} \left| \sum_{j=s}^{q-1+s} \frac{1}{B_{n+1}^{n+j}} \right|^2 |a_{n+s}|^2$$

$$= \sum_{s=2-q}^{0} \left| \sum_{j=1}^{q-1+s} \frac{1}{B_{n+1}^{n+j}} \right|^2 |a_{n+s}|^2 + \sum_{N=0}^{\infty} \sum_{w=1}^{T} \left| \sum_{j=w}^{q-1+w} \frac{1}{B_{n+1}^{n+j+NT}} \right|^2 |a_{n+NT+w}|^2$$

$$= \sum_{s=2-q}^{0} \left| \sum_{j=1}^{q-1+s} \frac{1}{B_{n+1}^{n+j}} \right|^2 |a_{n+s}|^2 + \sum_{N=0}^{\infty} |P|^{-2N} \sum_{w=1}^{T} \left| \sum_{j=w}^{q-1+w} \frac{1}{B_{n+1}^{n+j}} \right|^2 |a_{n+w}|^2$$

$$= \sum_{s=2-q}^{0} \left| \sum_{j=1}^{q-1+s} \frac{1}{B_{n+1}^{n+j}} \right|^2 |a_{n+s}|^2 + \frac{1}{1-|P|^{-2}} \sum_{w=1}^{T} \left| \sum_{j=w}^{q-1+w} \frac{1}{B_{n+1}^{n+j}} \right|^2 |a_{n+w}|^2 < \infty.$$

Therefore (4) holds and X_n defined by (10) is the bounded solution of system (1).

(ii) If $|P| < 1$, then for all $n \in Z$ we obtain:

$$\sum_{s=0}^{q-2} \Big| \sum_{j=-s}^{0} B_{n+j+1}^{n} \Big|^2 |a_{n-s}|^2 + \sum_{s=q-1}^{\infty} \Big| \sum_{j=-s}^{q-1-s} B_{n+j+1}^{n} \Big|^2 |a_{n-s}|^2$$

$$= \sum_{s=0}^{q-2} \Big| \sum_{j=-s}^{0} B_{n+j+1}^{n} \Big|^2 |a_{n-s}|^2 + \sum_{N=0}^{\infty} \sum_{w=1}^{T} \Big| \sum_{j=-w}^{q-1-w} B_{n+NT+w+1}^{n} \Big|^2 |a_{n-NT-w}|^2$$

$$= \sum_{s=0}^{q-2} \Big| \sum_{j=-s}^{0} B_{n+j+1}^{n} \Big|^2 |a_{n-s}|^2 + \sum_{N=0}^{\infty} |P|^{2N} \sum_{w=1}^{T} \Big| \sum_{j=-w}^{q-1-w} B_{n+w+1}^{n} \Big|^2 |a_{n-w}|^2$$

$$= \sum_{s=0}^{q-2} \Big| \sum_{j=-s}^{0} B_{n+j+1}^{n} \Big|^2 |a_{n-s}|^2 + \frac{1}{1-|P|^2} \sum_{w=1}^{T} \Big| \sum_{j=-w}^{q-1-w} B_{n+w+1}^{n} \Big|^2 |a_{n-w}|^2 < \infty.$$

Therefore (5) holds. X_n defined by formula (11) is bounded and satisfies formula (1).

In the next section it is shown for $q = 2$ that (X_n) defined by formulas (10) or (11) is periodically correlated and the condition $|P| = 1$ violates the conditions (I), (II) and (III) of Theorem 1. Therefore, system (1) has no bounded solution if $|P| = 1$. ∎

3. THE ARMA(1, 2) MODEL

For simplicity of the notation we consider here only the ARMA(1, 2) case:

(12) $$X_n - b_n X_{n-1} = a_n \xi_n + a_{n-1} \xi_{n-1}.$$

THEOREM 3. *If (b_n) and (a_n) are periodic with the same period T and $P = b_1 b_2 \ldots b_T$, then system* (12) *has a bounded solution if and only if $|P| \neq 1$. Moreover, the solution is periodically correlated with the same period T and is given by* (10) *if $|P| > 1$ and is given by* (11) *if $|P| < 1$.*

Proof. We will split the proof into 3 cases.

(i) In view of Theorem 2 we infer that if $|P| > 1$, then for all $n \in Z$ condition (4) holds. Hence there is a bounded solution of (12). The solution is given by formula (10) for $q = 2$.

Now we want to show that the stochastic sequences (X_n) in formula (10) are periodically correlated with period T. For any $k > 0$ and n we have

$$(13)\qquad (X_{n+k}, X_n) = \frac{1}{B_{n+1}^{n+k}} \sum_{j=1}^{\infty} \left|\frac{a_{n+k+j}}{B_{n+k+1}^{n+k+j}}\right|^2 \left(1 + \left|\frac{1}{b_{n+k+j+1}}\right|^2 + \frac{1}{b_{n+k+j+1}} + \frac{1}{\overline{b_{n+k+j+1}}}\right) + \frac{|a_{n+k}|^2}{B_{n+1}^{n+k}} \left(\frac{1}{b_{n+k+1}} + \left|\frac{1}{b_{n+k+1}}\right|^2\right).$$

The correlation function is bounded, and since coefficients (b_n) and (a_n) are periodic with period T, from (13) we obtain

$$(X_{n+k}, X_n) = (X_{n+T+k}, X_{n+T}).$$

(ii) Similarly, from Theorem 2 we infer that if $|P| < 1$, then for all $n \in Z$ condition (5) holds. Hence there is a bounded solution of (12). The solution is given by formula (11) for $q = 2$. The correlation function for $k > 0$ and n is given by

$$(14)\qquad (X_{n+k}, X_n) = \frac{1}{B_{n+1}^{n+k}} \sum_{j=2}^{\infty} |a_{n+k-j} B_{n+k-j+2}^{n+k}|^2 (1 + |b_{n+k-j+1}|^2 + b_{n+k-j+1} + \overline{b_{n+k-j+1}}) + \frac{|a_{n+k-1}|^2}{B_{n+1}^{n+k}} (|b_{n+k}|^2 + b_{n+k}).$$

Since the correlation function is bounded and (b_n) and (a_n) are periodic with period T, from (14) we obtain

$$(X_{n+k}, X_n) = (X_{n+T+k}, X_{n+T}).$$

Thus, by the above conditions, (X_n) is periodically correlated with period T.

(iii) If $|P| = 1$, then

$$\left|\frac{1}{b_{n+1}}\right|^2 |a_n|^2 + \sum_{s=1}^{\infty} \left|\frac{1}{B_{n+1}^{n+s}} + \frac{1}{B_{n+1}^{n+s+1}}\right|^2 |a_{n+s}|^2$$
$$= \left|\frac{1}{b_{n+1}}\right|^2 |a_n|^2 + \sum_{N=0}^{\infty} \sum_{w=1}^{T} \left|\frac{1}{B_{n+1}^{n+w+NT}} + \frac{1}{B_{n+1}^{n+w+1+NT}}\right|^2 |a_{n+NT+w}|^2$$
$$= \left|\frac{1}{b_{n+1}}\right|^2 |a_n|^2 + \sum_{N=0}^{\infty} |P|^{-2N} \sum_{w=1}^{T} \left|\frac{1}{B_{n+1}^{n+w}} + \frac{1}{B_{n+1}^{n+w+1}}\right|^2 |a_{n+w}|^2 \to \infty$$

and

$$|a_n|^2+\sum_{s=1}^{\infty}|B^n_{n-s+1}+B^n_{n-s+2}|^2\,|a_{n-s}|^2$$
$$=|a_n|^2+\sum_{N=0}^{\infty}|P|^{2N}\sum_{w=1}^{T}|B^n_{n+w+1}+B^n_{n+w+2}|^2\,|a_{n-NT-w}|^2\to\infty,$$

which violates the conditions (I) and (II) of Theorem 1. Since

$$\sum_{s=0}^{NT}\Big|\sum_{j=\max(1-NT,-s)}^{\min(0,1-s)}B^{NT}_{NT+1+j}\Big|^2\,|a_{NT-s}|^2$$
$$=\sum_{s=1}^{NT-1}|B^{NT}_{NT+1-s}+B^{NT}_{NT+2-s}|^2\,|a_{NT-s}|^2+|a_{NT}|^2+|B^{NT}_2|^2\,|a_0|^2$$
$$\geqslant\sum_{k=1}^{N-1}|B^{NT}_{(N-k)T+1}+B^{NT}_{(N-k)T+2}|^2\,|a_{(N-k)T}|^2=\sum_{k=1}^{N-1}|P|^k\,|1+1/b_1|^2\,|a_0|^2\to\infty,$$

the condition (III) of Theorem 1 is not satisfied. Therefore, in view of Theorem 1, system (12) has no bounded solution if $|P|=1$. ∎

Acknowledgments. The work was supported in part by the KBN grant PBZ–KBN 016/P03/99. The authors would like to thank Professor Andrzej Makagon for his critical comments leading to essential improvements of an initial version of the paper.

REFERENCES

[1] J. D. Hamilton, *Time Series Analysis*, Princeton University Press, Princeton 1994.

[2] H. Hurd, A. Makagon and A. G. Miamee, *On AR(1) models with periodic and almost periodic coefficients*, Stochastic Process. Appl. 100 (2002), pp. 167–185.

[3] M. Pourahmadi, *Foundations of Time Series Analysis and Prediction Theory*, Wiley, New York 2001.

Hugo Steinhaus Center
Institute of Mathematics
Wrocław University of Technology
50-370 Wrocław, Poland
E-mail: weron@im.pwr.wroc.pl, wyloman@im.pwr.wroc.pl

Received on 19.9.2003;
revised version on 22.4.2004