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LAW OF THE ITERATED LOGARITHM FOR SUBSEQUENCES OF PARTIAL SUMS WHICH ARE IN THE DOMAIN OF PARTIAL ATTRACTION OF A SEMISTABLE LAW

BY

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Abstract. Let $(X_n, n \geqslant 1)$ be a sequence of independent identically distributed random variables with a common distribution function F and let $S_n = \sum_{j=1}^{n} X_j$, $n \geqslant 1$. When F belongs to the domain of partial attraction of a semistable law with index α, $0 < \alpha < 2$, Chover's form of the law of the iterated logarithm has been obtained for subsequences of (S_n), along with some boundary crossing problems.

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1. INTRODUCTION

Let $\{X_n, n \geqslant 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables (r.v.'s) with common distribution function (d.f.) F. Set $S_n = \sum_{j=1}^{n} X_j$, $n \geqslant 1$. Let $(n_k, k \geqslant 1)$ be a strictly increasing subsequence of positive integers such that $n_{k+1}/n_k \to r$ $(r \geqslant 1)$ as $k \to \infty$. Kruglov (1972) has established that if there exist sequences (a_k) and (b_k) of real constants, $b_k \to \infty$ as $k \to \infty$, such that

$$(1) \qquad \operatorname*{Lim}_{k \to \infty} P\left(\frac{S_{n_k}}{b_k} - a_k \leqslant x\right) = G_\alpha(x)$$

at all continuity points x of G_α, then G_α is necessarily a semistable d.f. with characteristic exponent α, $0 < \alpha \leqslant 2$. Here F is said to belong to the *domain of partial attraction of a semistable distribution* G_α and the same is written as $F \in DP(\alpha)$, $0 < \alpha \leqslant 2$.

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We assume that $a_k = 0$ in (1). When $\alpha < 1$, a_k can always be chosen to be zero. When $\alpha > 1$, a_k becomes $n_k EX_1$. Here one can make $a_k = 0$ by shifting EX_1 to zero. Consequently, the condition $a_k = 0$ is no condition at all when $\alpha \neq 1$, $0 < \alpha < 2$. However, when $\alpha = 1$, this assumption restricts only to symmetric d.f.'s $F \in DP(1)$.

When $EX_n^2 < \infty$, Gut (1986) established the classical law of iterated logarithm (LIL) for geometrically fast increasing subsequences of (S_n). In fact, he showed that

$$\operatorname*{Lim\,sup}_{k \to \infty} \frac{S_{n_k}}{\sqrt{2n_k \log\log n_k}} = \begin{cases} 1 \text{ a.s.} & \text{if } \operatorname*{Lim\,sup}_{k \to \infty} (n_{k+1}/n_k) < \infty, \\ \varepsilon^* \text{ a.s.} & \text{if } \operatorname*{Lim\,inf}_{k \to \infty} (n_{k+1}/n_k) > 1, \end{cases}$$

where $\varepsilon^* = \inf\{\varepsilon > 0: \sum_{k=1}^{\infty} (\log n_k)^{-\varepsilon^2/2} < \infty\}$. Torrång (1987) extended the same to random subsequences. Observe that, when $n_k = 2^{2^{\cdot^{\cdot^{\cdot^{k}}}}}$, then $\varepsilon^* = 0$, and we have

$$\operatorname*{Lim\,sup}_{k \to \infty} \frac{S_{n_k}}{\sqrt{2n_k \log\log n_k}} = 0 \text{ a.s.}$$

That is, for such cases the norming sequence $\sqrt{2n_k \log\log n_k}$ will not be precise enough to give an almost sure bound for (S_{n_k}). In general, whenever $n_{k+1}/n_k \to \infty$ as $k \to \infty$, Schwabe and Gut (1996) have pointed out that $\sqrt{2n_k \log\log n_k}$ is no longer the proper normalizing sequence and it has to be replaced by $\sqrt{2n_k \log k}$.

When $n_k = n$, Chover (1966) observed that in the case of stable r.v.'s LIL involving Lim sup cannot be obtained under linear normalization and that it is possible under power normalization only. In fact, when X_n's are i.i.d. symmetric stable r.v.'s, Chover (1966) established the LIL for (S_n) by normalizing in the power. This means that

$$\operatorname*{Lim\,sup}_{n \to \infty} |S_n/n^{1/\alpha}|^{1/(\log\log n)} = e^{1/\alpha} \text{ a.s.}$$

Later Vasudeva (1984) proved the same for $F \in DA(\alpha)$, $0 < \alpha < 2$, and Divanji and Vasudeva (1989) extended the same to the case of $F \in DP(\alpha)$, $0 < \alpha < 2$.

Observations made by Gut (1986) and Schwabe and Gut (1996) motivated us to examine whether Chover's form of LIL for (S_{n_k}), when $F \in DP(\alpha)$, $0 < \alpha < 2$, can be obtained. We answer in the affirmative.

In the sequel, we use the following known facts. This can be referred to Divanji and Vasudeva (1989).

LEMMA 1. *Let $F \in DP(\alpha)$, $0 < \alpha < 2$. Then there exists a slowly varying function L and a function θ bounded in between two constants b_1, b_2, $0 < b_1 \leqslant b_2 < \infty$, such that*

$$\operatorname*{Lim}_{x \to \infty} \frac{x^{\alpha}\big(1 - F(x) + F(-x)\big)}{L(x)\,\theta(x)} = 1.$$

LEMMA 2. *Let $F \in DP(\alpha)$, $0 < \alpha < 2$, and let B_n be the smallest root of the equation $n(1-F(x)+F(-x)) = 1$. Then $B_n = n^{1/\alpha} l(n)\, \eta(n)$, where l is a function slowly varying at ∞ and η is bounded in between two positive constants.*

LEMMA 3. *Let L be any slowly varying function and let (x_n) and (y_n) be sequences of real constants tending to ∞ as $n \to \infty$. Then, for any $\delta > 0$,*

$$\operatorname*{Lim}_{n\to\infty} y_n^{\delta} \frac{L(x_n y_n)}{L(x_n)} = \infty \quad \text{and} \quad \operatorname*{Lim}_{n\to\infty} y_n^{-\delta} \frac{L(x_n y_n)}{L(x_n)} = 0.$$

The lemma follows from Karamata's representation of a slowly varying function (see Seneta (1976)).

In the next section we present our main results, and in the last section we discuss some boundary crossing problems. In the sequel, i.o., a.s. and s.v. mean "infinitely often", "almost surely" and "slowly varying", respectively. C, ε, k and n, with or without a superscript or subscript, denote positive constants with k and n confined to be integers.

2. MAIN RESULTS

THEOREM 1. *Let $F \in DP(\alpha)$, $0 < \alpha < 2$. Let (n_k) be an integer subsequence such that*

$$\operatorname*{Lim\,inf}_{k\to\infty} (n_{k+1}/n_k) > 1. \tag{2}$$

Then

$$\operatorname*{Lim\,sup}_{k\to\infty} (S_{n_k}/B_{n_k})^{1/\log\log n_k} = e^{\varepsilon^*/\alpha} \quad a.s., \tag{3}$$

where $\varepsilon^* = \inf\{\varepsilon_1 > 0\colon \sum_{k=1}^{\infty} (\log n_k)^{-\varepsilon_1} < \infty\}$.

Proof. To prove the assertion, it suffices to show that, for any $\varepsilon_1 \in (0, \varepsilon^*)$,

$$P\big(S_{n_k} \geqslant B_{n_k} (\log n_k)^{(\varepsilon^*+\varepsilon_1)/\alpha} \text{ i.o.}\big) = 0 \tag{4}$$

and

$$P\big(S_{n_k} \geqslant B_{n_k} (\log n_k)^{(\varepsilon^*-\varepsilon_1)/\alpha} \text{ i.o.}\big) = 1. \tag{5}$$

To prove (4), let

$$A_k = \{S_{n_k} \geqslant B_{n_k} (\log n_k)^{(\varepsilon^*+\varepsilon_1)/\alpha}\} \quad \text{and} \quad x_{n_k} = B_{n_k} (\log n_k)^{(\varepsilon^*-\varepsilon_1)/\alpha}.$$

By the theorem in Heyde (1967), one can find a C_2 and a k_1 such that, for all $k \geqslant k_1$,

$$P(A_k) \leqslant C_2\, n_k\, P(X \geqslant x_{n_k}).$$

Using Lemma 1, one can find a k_2 ($\geqslant k_1$) such that for all $k \geqslant k_2$

$$P(A_k) \leqslant C_2 n_k x_{n_k}^{-\alpha} L(x_{n_k}) \theta(x_{n_k}) = C_2 n_k \frac{L(B_{n_k}) \theta(B_{n_k})}{B_{n_k}^{\alpha} (\log n_k)^{\varepsilon^* + \varepsilon}} \frac{L(x_{n_k})}{L(B_{n_k})} \frac{\theta(x_{n_k})}{\theta(B_{n_k})}.$$

Applying Lemma 3 with $\delta = \varepsilon/2$ and using the boundedness of θ, one can find a k_3 ($\geqslant k_2$) such that, for all k ($\geqslant k_3$), $P(A_k) \leqslant C_3 (\log n_k)^{-(\varepsilon^* + \varepsilon/2)}$ for some $C_3 > 0$. Consequently, $\sum_{k=k_3}^{\infty} P(A_k) < \infty$ and (4) follows from the Borel–Cantelli lemma.

To establish (5) we first assume that $\varepsilon^* > 0$. The case of $\varepsilon^* = 0$ will be considered later. Use the relation $S_{n_k} = S_{n_k} - S_{n_{k-1}} + S_{n_{k-1}}$, $k \geqslant 1$, and define, for large k,

$$m_k = \min\{j\colon\ n_j \geqslant \beta^{(k-1)^{\delta}}\}, \tag{6}$$

where $\beta > 1$ and $\delta > 0$. In order to establish (5) it is enough to show that, for $\varepsilon \in (0, \varepsilon^*)$,

$$P\big(S_{n_{m_k}} - S_{n_{m_{k-1}}} \geqslant 2B_{n_{m_k}} (\log n_{m_k})^{(\varepsilon^* - \varepsilon)/\alpha} \text{ i.o.}\big) = 1 \tag{7}$$

and

$$P\big(S_{n_{m_{k-1}}} \geqslant B_{n_{m_k}} (\log n_{m_k})^{(\varepsilon^* - \varepsilon)/\alpha} \text{ i.o.}\big) = 0. \tag{8}$$

Define

$$z_n = B_n (\log n)^{(\varepsilon^* - \varepsilon)/\alpha} \quad \text{and} \quad D_k = S_{n_{m_k}} - S_{n_{m_{k-1}}} \geqslant z_{n_{m_k}},\ k \geqslant 1.$$

Note that $S_{n_{m_k}} - S_{n_{m_{k-1}}} \overset{d}{=} S_{n_{m_k} - n_{m_{k-1}}}$, $k \geqslant 1$. Hence, by the theorem in Heyde (1967), one can find a k_4 such that, for all k ($\geqslant k_4$),

$$P(D_k) \geqslant C_5 (n_{m_k} - n_{m_{k-1}}) P(X \geqslant 2z_{n_{m_k}}) = C_5 n_{m_k} (1 - n_{m_{k-1}}/n_{m_k}) P(X \geqslant 2z_{n_{m_k}}).$$

Since $\operatorname{Lim\,inf}_{k\to\infty} (n_{k+1}/n_k) > 1$ implies that there exists $\lambda < 1$ such that $n_{m_{k-1}}/n_{m_k} < \lambda < 1$ for all $k \geqslant k_4$,

$$P(D_k) \geqslant C_5 n_{m_k} P(X \geqslant 2z_{n_{m_k}}) \quad \text{for some } C_5 > 0.$$

Now, following the steps similar to those used to get an upper bound of $P(A_k)$, one can find a k_5 such that, for all k ($\geqslant k_5$),

$$P(D_k) \geqslant C_6 (\log n_k)^{-(\varepsilon^* - \varepsilon/2)} \quad \text{for some } C_6 > 0.$$

Hence $\sum_{k=k_5}^{\infty} P(D_k) = \infty$. In view of the fact that D_k's are mutually independent, applying the Borel–Cantelli lemma we establish (7). Observe that

$$P\big(S_{n_{m_{k-1}}} \geqslant B_{n_{m_k}} (\log n_{m_k})^{(\varepsilon^* - \varepsilon)/\alpha}\big) = \left(S_{n_{m_{k-1}}} \geqslant B_{n_{m_{k-1}}} \frac{B_{n_{m_k}}}{B_{n_{m_{k-1}}}} (\log n_{m_k})^{(\varepsilon^* - \varepsilon)/\alpha}\right).$$

Again, by Heyde (1967), one can find a k_6 such that, for all $k \geqslant k_6$,

$$P\left(S_{n_{m_{k-1}}} \geqslant B_{n_{m_k}} (\log n_{m_k})^{(\varepsilon^* - \varepsilon)/\alpha}\right) \leqslant C_2 n_{m_{k-1}} P\left(X_1 \geqslant B_{n_{m_k}} (\log n_{m_k})^{(\varepsilon^* - \varepsilon)/\alpha}\right).$$

Again following the steps similar to those used to get an upper bound of $P(A_k)$, one can find a k_7 such that, for all k ($\geqslant k_7$),

$$P\left(S_{n_{m_{k-1}}} \geqslant B_{n_{m_k}} (\log n_{m_k})^{(\varepsilon^* - \varepsilon)/\alpha}\right) \leqslant C_7 \frac{n_{m_{k-1}}}{n_{m_k}} \frac{1}{(\log n_{m_k})^{\varepsilon^* - 3\varepsilon/2}}.$$

By (6) we infer that $n_{m_k} \geqslant \beta^{(k-1)^\delta}$ implies $n_{n_{k+1}} \geqslant \beta^{k^\delta} \geqslant n_{m_k}$, and since $\operatorname{Lim\,inf}_{k\to\infty} (n_{k+1}/n_k) > 1$, there exists $\lambda > 1$ such that $n_{k+1} \geqslant \lambda n_k$. Therefore,

$$n_{m_{k+1}} \geqslant \beta^{k^\delta} \geqslant n_{m_k} \geqslant \lambda n_{m_{k-1}} \Rightarrow \lambda n_{m_{k-1}} \leqslant \beta^{k^\delta} \Rightarrow n_{m_{k-1}} \leqslant \lambda^{-1} \beta^{k^\delta} = \lambda_1 \beta^{k^\delta},$$

where $\lambda_1 = \lambda^{-1}$. Hence

$$\frac{n_{m_{k-1}}}{n_{m_k}} \leqslant \frac{\lambda_1 \beta^{k^\delta}}{\beta^{(k-1)^\delta}} \cong \frac{\lambda_1}{\beta^{k^{\delta_1}}}$$

and

$$\sum_{k=k_5}^{\infty} \frac{n_{m_{k-1}}}{n_{m_k}} \frac{1}{(\log n_{m_k})^{\varepsilon^* - 3\varepsilon/2}} \leqslant \lambda_1 \sum_{k=k_5}^{\infty} \frac{1}{\beta^{k^{\delta_1}} (\log n_{m_k})^{\varepsilon^* - 3\varepsilon/2}} < \infty.$$

Therefore $P\left(S_{n_{m_{k-1}}} \geqslant B_{n_{m_k}} (\log n_{m_k})^{(\varepsilon^* - \varepsilon)/\alpha} \text{ i.o.}\right) = 0$, which implies (5), follows from (7) and (8). Thus the proof of the theorem is completed.

THEOREM 2. *Let $F \in DP(\alpha)$, $0 < \alpha < 2$. Let (n_k) be an integer subsequence such that*

$$\operatorname*{Lim\,sup}_{k\to\infty} (n_{k+1}/n_k) < \infty. \tag{9}$$

Then

$$\operatorname*{Lim\,sup}_{k\to\infty} (S_{n_k}/B_{n_k})^{1/\log\log n_k} = e^{1/\alpha} \quad a.s.$$

Proof. Proceeding as in Theorem 1, it is enough to show that, for any $\varepsilon_1 \in (0, 1)$,

$$P\left(S_{n_k} \geqslant B_{n_k} (\log n_k)^{(1+\varepsilon_1)/\alpha} \text{ i.o.}\right) = 0 \tag{10}$$

and

$$P\left(S_{n_k} \geqslant B_{n_k} (\log n_k)^{(1-\varepsilon_1)/\alpha} \text{ i.o.}\right) = 1. \tag{11}$$

One can notice that (10) is a consequence of the theorem of Divanji and Vasudeva (1989), i.e.,

$$\operatorname*{Lim\,sup}_{k\to\infty} (S_{n_k}/B_{n_k})^{1/\log\log n_k} \leqslant \operatorname*{Lim\,sup}_{n\to\infty} (S_n/B_n)^{1/\log\log n} = e^{1/\alpha} \text{ a.s.}$$

From (9) we see that the sequences are at most geometrically increasing, which implies that there exists $\theta > 1$ such that

(12) $$n_{k+1} \leqslant \theta n_k.$$

Now define

(13) $$v_j = \min\{k\colon n_k > M^j\}, \quad j = 1, 2, \ldots,$$

where M is chosen such that $\theta/M < 1$. Proceeding as in Gut (1986) one can show that $M^j < n_{v_j} < \theta M^j$ and $1/\theta M \leqslant n_{v_{j-1}}/n_{v_j} \leqslant \theta/M < 1$. Consequently, (n_{v_j}) satisfies the condition $\operatorname{Lim\,sup}_{j\to\infty}(n_{v_{j-1}}/n_{v_j}) < 1$ of Theorem 1 and also the relation $\sum_{j=1}^{\infty}(\log n_{v_j})^{-\varepsilon_1} < \infty$ holds for all $\varepsilon_1 > 1$ (i.e. $\varepsilon^* = 1$). Now (11) follows from Theorem 1. Hence the proof of the theorem is completed.

Remark. The results by Schwabe and Gut (1996) for $\varepsilon^* = 0$ motivated us to examine whether Chover's form of LIL for (S_{n_k}) can be obtained for these rapidly increasing subsequences. Interestingly, we answer the question in the following theorem. Note that $n_{k+1}/n_k \to \infty$ as $k \to \infty$ comes under the class of at least geometrically increasing subsequences.

THEOREM 3. *Let* $F \in DP(\alpha)$, $0 < \alpha < 2$. *Let* (n_k) *be an integer subsequence such that*

(14) $$\operatorname*{Lim}_{k\to\infty}(n_{k+1}/n_k) = \infty.$$

Then

$$\operatorname*{Lim\,sup}_{k\to\infty}(S_{n_k}/B_{n_k})^{1/\log k} = e^{1/\alpha} \ \ a.s.$$

Proof. To prove the assertion it suffices to show that there exists ε, $0 < \varepsilon < 1$, such that

(15) $$P(S_{n_k} \geqslant B_{n_k} k^{(1+\varepsilon)/\alpha} \text{ i.o.}) = 0$$

and

(16) $$P(S_{n_k} \geqslant B_{n_k} k^{(1-\varepsilon)/\alpha} \text{ i.o.}) = 1.$$

To prove (15), let $E_k = \{S_{n_k} \geqslant B_{n_k} k^{(1+\varepsilon)/\alpha}\}$ and $y_k = B_{n_k} k^{(1+\varepsilon)/\alpha}$. By the theorem in Heyde (1967), one can find a C_8 and a k_8 such that, for all $k \geqslant k_8$,

$$P(E_k) \leqslant C_8 n_k P(X \geqslant y_k).$$

Using Lemma 1, one can find a k_9 $(\geqslant k_8)$ such that, for all $k \geqslant k_9$,

$$P(E_k) \leqslant C_8 n_k y_k^{-\alpha} L(y_k)\theta(y_k) = C_8 n_k \frac{L(B_{n_k})\theta(B_{n_k})}{B_{n_k}^{\alpha} k^{1+\varepsilon}} \frac{L(y_k)}{L(B_{n_k})} \frac{\theta(y_k)}{\theta(B_{n_k})}.$$

Applying Lemma 3 with $\delta = \varepsilon/2$ and using the boundedness of θ, one can find a k_{10} ($\geqslant k_9$) such that, for all k ($\geqslant k_{10}$), $P(E_k) \leqslant C_9 k^{-(1+\varepsilon/2)}$ for some $C_9 > 0$. Consequently, $\sum_{k=k_{10}}^{\infty} P(E_k) < \infty$ and (15) follows from the Borel–Cantelli lemma.

To prove (16) define for large k

$$m_k = \min\{j\colon\ n_j \geqslant \beta^{(k-1)^\delta}\}, \tag{17}$$

where $\beta > 1$ and $\delta > 0$, and use the relation $S_{n_k} = S_{n_k} - S_{n_{k-1}} + S_{n_{k-1}}$, $k \geqslant 1$. We are going to show that, for any $\varepsilon \in (0, 1)$,

$$P(S_{n_{m_k}} - S_{n_{m_{k-1}}} \geqslant 2B_{n_{m_k}} k^{(1-\varepsilon)/\alpha} \text{ i.o.}) = 1 \tag{18}$$

and

$$P(S_{n_{m_{k-1}}} \geqslant B_{n_{m_k}} k^{(1-\varepsilon)/\alpha} \text{ i.o.}) = 0. \tag{19}$$

Note that $S_{n_{m_k}} - S_{n_{m_{k-1}}} \overset{d}{=} S_{n_{m_k} - n_{m_{k-1}}}$, $k \geqslant 1$. Define $z_k = B_{n_{m_k}} k^{(1-\varepsilon)/\alpha}$ and $T_k = (S_{n_{m_k} - n_{m_{k-1}}}) \geqslant 2z_k$, $k \geqslant 1$. Hence, by the theorem in Heyde (1967), one can find a k_{11} such that, for all k ($\geqslant k_{11}$),

$$\begin{aligned} P(S_{n_{m_k}} - S_{n_{m_{k-1}}} \geqslant 2B_{n_{m_k}} k^{(1-\varepsilon)/\alpha}) &= P(T_k) \geqslant C_{10}(n_{m_k} - n_{m_{k-1}}) P(X \geqslant 2z_{n_{m_k}}) \\ &= C_{10}\, n_{m_k}(1 - n_{m_{k-1}}/n_{m_k}) P(X \geqslant 2z_{n_{m_k}}) \quad \text{for some } C_{10} > 0. \end{aligned}$$

Since (14) is at least geometrically fast, there exists $\lambda < 1$ such that

$$n_{m_{k-1}}/n_{m_k} < \lambda < 1 \quad \text{for all } k \geqslant k_{11}, \tag{20}$$

and, consequently, $P(T_k) \geqslant C_{10}\, n_{m_k} P(X \geqslant 2z_{n_{m_k}})$.

Now, following the steps similar to those used to get an upper bound of $P(A_k)$, one can find a k_{12} such that, for all k ($\geqslant k_{12}$), $P(T_k) \geqslant C_{11} k^{-(1-\varepsilon/2)}$ for some $C_{11} > 0$. Hence $\sum_{k=k_{12}}^{\infty} P(T_k) = \infty$. In view of the fact that T_k's are mutually independent, applying the Borel–Cantelli lemma we establish (18). Observe that

$$P(S_{n_{m_{k-1}}} \geqslant B_{n_{m_k}} k^{(1-\varepsilon)/\alpha}) = P\left(S_{n_{m_{k-1}}} \geqslant B_{n_{m_{k-1}}} \frac{B_{n_{m_k}}}{B_{n_{m_{k-1}}}} k^{(1-\varepsilon)/\alpha}\right).$$

Using Lemma 2 and (20), we get $B_{n_{m_k}}/B_{n_{m_{k-1}}} \approx C_{13}$ for some $C_{13} > 0$. Again by Heyde (1967), one can find some $C_{14} > 0$ and k_{13} such that, for all $k \geqslant k_{13}$,

$$P(S_{n_{m_{k-1}}} \geqslant B_{n_{m_k}} k^{(1-\varepsilon)/\alpha}) \leqslant C_{14}\, n_{m_{k-1}} P(X_1 \geqslant B_{n_{m_k}} k^{(1-\varepsilon)/\alpha}).$$

Again following the steps similar to those used to get (8), we can find a k_{14} such that, for all k ($\geqslant k_{14}$),

$$P(S_{n_{m_{k-1}}} \geqslant B_{n_{m_k}} k^{(1-\varepsilon)/\alpha}) \leqslant C_{14} \sum_{k=k_{14}}^{\infty} \frac{1}{\beta^{k^{\delta_1}} k^{1-3\varepsilon/2}} < \infty.$$

By (6) we infer that $n_{m_k} \geqslant \beta^{(k-1)^\delta}$ implies $n_{m_{k+1}} \geqslant \beta^{k^\delta} \geqslant n_{m_k}$, and since $\operatorname{Lim}_{k\to\infty}(n_{k+1}/n_k) < 1$, there exists $\lambda > 1$ such that $n_{k+1} \geqslant \lambda n_k$. Therefore,

$$n_{m_{k+1}} \geqslant \beta^{k^\delta} \geqslant n_{m_k} \geqslant \lambda n_{m_{k-1}} \Rightarrow \lambda n_{m_{k-1}} \leqslant \beta^{k^\delta} \Rightarrow n_{m_{k-1}} \leqslant \lambda^{-1}\beta^{k^\delta} = \lambda_1 \beta^{k^\delta},$$

where $\lambda_1 = \lambda^{-1}$. Hence

$$\frac{n_{m_{k-1}}}{n_{m_k}} \leqslant \frac{\lambda_1 \beta^{k^\delta}}{\beta^{(k-1)^\delta}} \cong \frac{\lambda_1}{\beta^{k^{\delta_1}}} \quad \text{and} \quad \sum_{k=k_5}^{\infty} \frac{n_{m_{k-1}}}{n_{m_k}} \frac{1}{k^{\varepsilon^*-3\varepsilon/2}} \leqslant \lambda_1 \sum_{k=k_5}^{\infty} \frac{1}{\beta^{k^{\delta_1}} k^{\varepsilon^*-3\varepsilon/2}} < \infty.$$

Hence

$$P(S_{n_{m_{k-1}}} \geqslant B_{n_{m_k}} k^{(\varepsilon^*-\varepsilon)/\alpha} \text{ i.o.}) = 0,$$

and consequently (16) follows from (18) and (19). The proof of the theorem is completed.

3. BOUNDARY CROSSING PROBLEMS

Here we study some boundary crossing random variables related to Theorems 1 and 2. Define, for any $\varepsilon > 0$,

$$Y_{n_k}(\varepsilon) = \begin{cases} 1 & \text{if } S_{n_k} \geqslant B_{n_k}(\log n_k)^{(\theta-\varepsilon)/\alpha}, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\theta = \begin{cases} \varepsilon^* & \text{if } (n_k) \text{ is at least geometrically fast,} \\ 1 & \text{if } (n_k) \text{ is at most geometrically fast,} \end{cases}$$

and

$$\varepsilon^* = \inf\{\varepsilon_1 > 0\colon \sum_{k=1}^{\infty} (\log n_k)^{-\varepsilon_1} < \infty\}.$$

Let, for any $\varepsilon > 0$, $N_{m_k}(\varepsilon)$ be a partial sum sequence of $Y_{n_k}(\varepsilon)$, i.e.,

$$N_{m_k}(\varepsilon) = \sum_{k=1}^{m_k} Y_{n_k}(\varepsilon).$$

Observe that, by (10), $N_\infty(\varepsilon)$ is a proper random variable. We study this problem as corollaries to Theorems 1, 2 and 3. Here we show that all the moments in $0 < \lambda \leqslant 1$ are finite for $N_\infty(\varepsilon)$. This proper random variable $N_\infty(\varepsilon)$ was studied by various authors; see e.g. Slivka (1969) and Slivka and Savero (1970).

COROLLARY 1. *Let* $F \in DP(\alpha)$, $0 < \alpha < 2$. *Let* $\{n_k, k \geqslant 1\}$ *be an increasing* *subsequence of positive integers. Then for* $\varepsilon > 0$ *and for any* λ, $0 < \lambda \leqslant 1$,

$$EN_\infty^\lambda < \infty \quad \textit{if} \ \sum_{k=1}^{\infty} n_k^{\lambda-1} P\left(S_{n_k} > B_{n_k} (\log n_k)^{(\theta+\varepsilon)/\alpha}\right) < \infty.$$

Proof. First we show that, for $\lambda = 1$, $EN_\infty(\varepsilon) < \infty$, and then claim that the existence of lower moments follows from that of the higher moments. Observe that

$$EN_\infty(\varepsilon) = \sum_{k=1}^{\infty} P\left(S_{n_k} > B_{n_k} (\log n_k)^{(\varepsilon^*+\varepsilon)/\alpha}\right).$$

Following similar steps of the proof of (3), we can find some constant $C_1 > 0$ and some $k_1 > 0$ such that, for all $k \geqslant k_1$,

$$EN_\infty(\varepsilon) \leqslant C_1 \sum_{k=k_1}^{\infty} \frac{1}{(\log n_k)^{-(\theta+\varepsilon/2)}} < \infty.$$

To show this we use the definition

$$\theta = \begin{cases} \varepsilon^* & \text{if } (n_k) \text{ is at least geometrically fast,} \\ 1 & \text{if } (n_k) \text{ is at most geometrically fast,} \end{cases}$$

and then the proof of (4) and (10). Consequently, $EN_\infty(\varepsilon) < \infty$ for $\lambda = 1$, and therefore $EN_\infty^\lambda < \infty$ for $\lambda < 1$. Thus the proof of the corollary is completed.

COROLLARY 2. *Let* $F \in DP(\alpha)$, $0 < \alpha < 2$. *Let* $\{n_k, k \geqslant 1\}$ *be an increasing subsequence of positive integers. Then for* $\varepsilon > 0$ *and for any* λ, $0 < \lambda \leqslant 1$,

$$EN_\infty^\lambda < \infty \quad \textit{if} \ \sum_{k=1}^{\infty} n_k^{\lambda-1} P\left(S_{n_k} > B_{n_k} k^{(1+\varepsilon)/\alpha}\right) < \infty.$$

Proof. First we show that, for $\lambda = 1$, $EN_\infty(\varepsilon) < \infty$, and then claim that the existence of lower moments follows from that of the higher moments. Observe that

$$EN_\infty(\varepsilon) = \sum_{k=1}^{\infty} P\left(S_{n_k} > B_{n_k} k^{(1+\varepsilon)/\alpha}\right).$$

Following similar steps of the proof of (15), we can find some constant $C_1 > 0$ and some $k_1 > 0$ such that, for all $k \geqslant k_1$,

$$EN_\infty(\varepsilon) \leqslant C_1 \sum_{k=k_1}^{\infty} \frac{1}{k^{1+\varepsilon/2}} < \infty.$$

Consequently, $EN_\infty(\varepsilon) < \infty$ for $\lambda = 1$, and therefore $EN_\infty^\lambda < \infty$ for $\lambda < 1$. Thus the proof of the corollary is completed.

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