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LIMITING BEHAVIOR OF WEIGHTED SUMS OF HEAVY-TAILED RANDOM VECTORS AND APPLICATIONS

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Abstract. We present an integral test to determine the limiting behavior of weighted sums of i.i.d. $\boldsymbol{R}^d$-valued random vectors belonging to the (generalized) domain of operator semistable attraction of some nonnormal law, and deduce a version of Chover's law of the iterated logarithm for them. As applications, the corresponding limit results for some classical summability methods are also established.

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1. INTRODUCTION AND MAIN RESULTS

Let $X, X_1, X_2, \ldots$ be i.i.d. $\boldsymbol{R}^d$-valued random vectors. We assume that X belongs to the strict generalized domain of semistable attraction of a full operator semistable Y having nonnormal component (see [11] for details). Then, by definition, there exists a constant $c > 1$ and a sequence (k_n) of natural numbers tending to infinity with $k_{n+1}/k_n \to c$ as $n \to \infty$ and linear operators $A_n \in GL(\boldsymbol{R}^d)$ such that for $S_n = \sum_{i=1}^n X_i$ we have

$$A_n S_{k_n} \Rightarrow Y \quad \text{as } n \to \infty. \tag{1.1}$$

Here $\Rightarrow$ denotes convergence in distribution. The distribution ν of the limit Y is then strictly (c^E, c)-operator semistable (E an invertible $d \times d$ matrix), that is

$$\nu^c = (c^E \nu), \tag{1.2}$$

where ν^c denotes the c-fold convolution power and $(c^E \nu)(A) = \nu(c^{-E} A)$ is the image measure. Note that if ν is strictly operator stable with exponent E, then (1.2) holds for any $c > 1$, but the class of operator semistable laws is much larger than that of operator stable laws.

Then it is shown in [15] that there exists a sequence $(B_n) \subset GL(\mathbf{R}^d)$ regularly varying with exponent $-E$, that is, $B_{[\lambda n]} B_n^{-1} \to \lambda^{-E}$ as $n \to \infty$, such that

$$B_{k_n} S_{k_n} \Rightarrow Y \quad \text{as } n \to \infty. \tag{1.3}$$

Moreover, the whole sequence $(B_n S_n)_n$ is stochastically compact with limit distributions in $\{\lambda^{-E} \nu^{\lambda}\colon \lambda \in [1, c]\}$. Given any unit vector $\theta \in \mathbf{R}^d$, we can project the random walk (S_n) onto the direction θ, that is we consider the one-dimensional random walk

$$\langle S_n, \theta\rangle = \sum_{i=1}^{n} \langle X_i, \theta\rangle.$$

Then it is shown in [15] that for any $\|\theta\| = 1$ there exists a sequence $r_n = r_n(\theta) > 0$ such that $(r_n \langle S_n, \theta\rangle)_n$ is stochastically compact. The norming sequence (r_n) behaves roughly like $n^{-1/\alpha(\theta)}$, where the tail index $0 < \alpha(\theta) < 2$ depends on the exponent E in (1.2). More precisely, for every $\delta > 0$ there exists an $n_0 \geqslant 1$ such that

$$n^{-1/\alpha(\theta) - \delta} \leqslant r_n \leqslant n^{-1/\alpha(\theta) + \delta} \tag{1.4}$$

whenever $n \geqslant n_0$. See [11], Remark 8.3.21, for details.

The tail behavior and the asymptotic behavior of truncated moments of $\langle X, \theta\rangle$ are well understood. In fact, if we let $V_0(t, \theta) = P\{|\langle X, \theta\rangle| > t\}$, it follows from Theorem 6.4.15 of [11] that for any $\delta > 0$ there exist constants $C_1, C_2 > 0$ and a $t_0 > 0$ such that

$$C_1 \lambda^{-\alpha(\theta) - \delta} \leqslant \frac{V_0(\lambda t, \theta)}{V_0(t, \theta)} \leqslant C_2 \lambda^{-\alpha(\theta) + \delta} \tag{1.5}$$

for any $t \geqslant t_0$ and any $\lambda \geqslant 1$. If we let $U_b(t, \theta) = E\big(|\langle X, \theta\rangle|^b I(|\langle X, \theta\rangle| \leqslant t)\big)$, where $b > \alpha(\theta)$, it is shown in Corollary 6.4.16 of [11] that there exists a $t_0 > 0$ and constants $C_3, C_4 > 0$ such that

$$C_3 \leqslant \frac{t^b V_0(t, \theta)}{U_b(t, \theta)} \leqslant C_4 \quad \text{for all } t \geqslant t_0. \tag{1.6}$$

Some technical estimates on $nP(|\langle X, \theta\rangle| > r_n^{-1})$ as in (9.21) and (9.24) of [11] together with some asymptotic results on r_n as in Lemma 4.1 of [13] are also needed. In fact,

$$0 < \inf_{n \geqslant 1} nP(|\langle X, \theta\rangle| > r_n^{-1}) \leqslant \sup_{n \geqslant 1} nP(|\langle X, \theta\rangle| > r_n^{-1}) < \infty. \tag{1.7}$$

The law of the iterated logarithm for sums of α-stable random variables was first discovered in [8] and then generalized in various ways. See e.g. [1]–[6] and [13]. In particular, [5] established some result on the limiting behavior of weighted sums of heavy-tailed random vectors when the weights are of uniform bounded variation. In this paper we generalize the results in [4] partly in the following way, extending the results in [5]:

Let X belong to the strict generalized domain of semistable attraction of some full (c^E, c) operator semistable Y having no normal component. Then we have

THEOREM 1.1. *Let* f: $[1, \infty) \to (0, \infty)$ *be nondecreasing with* $\lim_{x\to\infty} f(x) = \infty$. *Then*:

(a) *If there exists an* $\varepsilon_0 > 0$ *such that*

$$\int_2^\infty \frac{dx}{xf(x)^{1-\varepsilon_0}} < \infty,$$

then for any array of real numbers $\{a_{nk},\ 1 \leqslant k \leqslant k_n,\ n \geqslant 1\}$ *with* $k_n \leqslant Mn$, *for all* $n \geqslant 1$, $\sup_{n,k} |a_{nk}| \leqslant M$ *and* $\sum_{k=1}^{k_n} a_{nk}^2 = O(n^{\delta_0})$ *for some* $\delta_0 < 1$, *where* M *is a positive constant not depending on* n, *for any* $\|\theta\| = 1$ *we have, for* $r_n = r_n(\theta)$ *and* $\alpha(\theta)$ *as above,*

$$\limsup_{n\to\infty} \frac{\left|r_n \sum_{k=1}^{k_n} a_{nk} \langle X_k, \theta\rangle\right|}{f(n)^{1/\alpha(\theta)}} = 0 \ \text{a.s.} \tag{1.8}$$

and especially for any $\delta > 0$

$$\limsup_{n\to\infty} \frac{\left|r_n \sum_{k=1}^{k_n} a_{nk} \langle X_k, \theta\rangle\right|}{(\log n)^{(1+\delta)/\alpha(\theta)}} = 0 \ \text{a.s.} \tag{1.9}$$

(b) *If there exists an* $\varepsilon_0 > 0$ *such that*

$$\int_2^\infty \frac{dx}{xf(x)^{1+\varepsilon_0}} = \infty,$$

then for any array of real numbers $\{a_{nk},\ 1 \leqslant k \leqslant k_n,\ n \geqslant 1\}$ *such that there exist two strictly increasing sequences of positive integers* $l(n)$, $m(n)$, $n \geqslant 1$, *with*

$$\sup_{n\geqslant 1} \big(l(n+1)-l(n)\big) < \infty \quad \text{and} \quad \liminf_{n\to\infty} |a_{l(n),m(n)}| > 0$$

and any $\|\theta\| = 1$ *we have*

$$\limsup_{n\to\infty} \frac{\left|r_n \sum_{k=1}^{k_n} a_{nk} \langle X_k, \theta\rangle\right|}{f(n)^{1/\alpha(\theta)}} = \infty \ \text{a.s.} \tag{1.10}$$

and especially for any $0 < \delta < 1$

$$\limsup_{n\to\infty} \frac{\left|r_n \sum_{k=1}^{k_n} a_{nk} \langle X_k, \theta\rangle\right|}{(\log n)^{(1-\delta)/\alpha(\theta)}} = \infty \ \text{a.s.} \tag{1.11}$$

As a corollary the following law of the iterated logarithm (LIL) holds true:

COROLLARY 1.2. *Let* $\{a_{nk}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ *fulfill conditions* (a) *and* (b) *of Theorem* 1.1. *Then for any* $\|\theta\| = 1$ *we have*

$$\limsup_{n\to\infty} \Big|r_n \sum_{k=1}^{k_n} a_{nk} \langle X_k, \theta\rangle\Big|^{1/\log\log n} = e^{1/\alpha(\theta)} \ a.s., \tag{1.12}$$

where r_n *is as above.*

Complementarily to our results on the limiting behavior of weighted sums of $\langle X_i, \theta\rangle$ given above, we also consider the behavior of the norm of the weighted sum of the X_i's. Recall that the distribution ν of Y is a full (c^E, c) operator semistable law without normal component and let $\boldsymbol{R}^d = V_1 \oplus \ldots \oplus V_p$ denote the spectral decomposition of $\boldsymbol{R}^d$ with respect to E. Recall that $E = E^{(1)} \oplus \ldots \oplus E^{(p)}$ and that every eigenvalue of $E^{(i)}$ has real part $1/\alpha_i$ for $1 \leqslant i \leqslant p$. Then Theorem 1 in [7] implies that $0 < \alpha_p < \ldots < \alpha_1 < 2$.

In the following let X belong to the strict generalized domain of semistable attraction of a (c^E, c) semistable law ν such that (1.3) holds. In view of Theorem 8.3.7 of [11] we can assume without loss of generality that the distribution of X is spectrally compatible with ν. Then the spaces V_i are B_n-invariant for all n and all $1 \leqslant i \leqslant p$, so that $B_n = B_n^{(1)} \oplus \ldots \oplus B_n^{(p)}$. We write $X = X^{(1)} + \ldots + X^{(p)}$ with respect to the spectral decomposition of $\boldsymbol{R}^d$ obtained above and for $1 \leqslant i \leqslant p$ set $X^{(1,\ldots,i)} = X^{(1)} + \ldots + X^{(i)}$ and $B_n^{(1,\ldots,i)} = B_n^{(1)} \oplus \ldots \oplus B_n^{(i)}$.

THEOREM 1.3. *Suppose that* X *is in the strict generalized domain of semistable attraction of some full* (c^E, c) *operator semistable law without normal component, where* $c > 1$. *Moreover, let* $f\colon [1, \infty) \to (0, \infty)$ *be nondecreasing with* $\lim_{x\to\infty} f(x) = \infty$. *Then:*

(a) *If there exists an* $\varepsilon_0 > 0$ *such that*

$$\int_2^\infty \frac{dx}{x f(x)^{1-\varepsilon_0}} < \infty,$$

then for any $1 \leqslant i \leqslant p$, *for any array of real numbers* $\{a_{nk}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ *with* $k_n \leqslant Mn$, *for all* $n \geqslant 1$, $\sup_{n,k} |a_{nk}| \leqslant M$ *and* $\sum_{k=1}^{k_n} a_{nk}^2 = O(n^{\delta_0})$ *for some* $\delta_0 < 1$, *where* M *is a positive constant not depending on* n, *we have*

$$\limsup_{n\to\infty} \frac{\|B_n^{(1,\ldots,i)} \sum_{k=1}^{k_n} a_{nk} X_k^{(1,\ldots,i)}\|}{f(n)^{1/\alpha_i}} = 0 \ a.s. \tag{1.13}$$

and especially for any $\delta > 0$

$$\limsup_{n\to\infty} \frac{\|B_n^{(1,\ldots,i)} \sum_{k=1}^{k_n} a_{nk} X_k^{(1,\ldots,i)}\|}{(\log n)^{(1+\delta)/\alpha_i}} = 0 \ a.s. \tag{1.14}$$

(b) *If there exists an* $\varepsilon_0 > 0$ *such that*

$$\int_2^\infty \frac{dx}{x f(x)^{1+\varepsilon_0}} = \infty,$$

then for any $1 \leqslant i \leqslant p$, *for any array of real numbers* $\{a_{nk}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ *such that there exist two strictly increasing sequences of positive integers* $l(n)$, $m(n)$, $n \geqslant 1$, *with*

$$\sup_{n \geqslant 1} \big(l(n+1) - l(n)\big) < \infty \quad \text{and} \quad \liminf_{n \to \infty} |a_{l(n),m(n)}| > 0$$

we have

$$\limsup_{n \to \infty} \frac{\big\|B_n^{(1,\ldots,i)} \sum_{k=1}^{k_n} a_{nk} X_k^{(1,\ldots,i)}\big\|}{f(n)^{1/\alpha(\theta)}} = \infty \ \text{a.s.} \tag{1.15}$$

and especially for any $0 < \delta < 1$

$$\limsup_{n \to \infty} \frac{\big\|B_n^{(1,\ldots,i)} \sum_{k=1}^{k_n} a_{nk} X_k^{(1,\ldots,i)}\big\|}{(\log n)^{(1-\delta)/\alpha_i}} = \infty \ \text{a.s.} \tag{1.16}$$

COROLLARY 1.4. *Under the assumptions of Theorem* 1.3 *we have*

$$\limsup_{n \to \infty} \Big\|B_n^{(1,\ldots,i)} \sum_{k=1}^{k_n} a_{nk} X_n^{(1,\ldots,i)}\Big\|^{1/\log\log n} = e^{1/\alpha_i} \ \text{a.s.} \tag{1.17}$$

2. PROOFS

To prove the convergent parts of Theorems 1.1 and 1.3 we need the following preliminary results.

LEMMA 2.1 (see [2]). *Let* $f > 0$ *be a nondecreasing function with*

$$\int_2^\infty \frac{dx}{x f(x)} < \infty.$$

Then there exists a nondecreasing function $g > 0$ *such that*

$$g(x) \leqslant f(x), \quad \limsup_{x \to \infty} \frac{g(2x)}{g(x)} < \infty \quad \text{and} \quad \int_2^\infty \frac{dx}{x g(x)} < +\infty.$$

LEMMA 2.2. *Let* $(Z_i)_{i \leqslant N}$ *be independent random variables. Set* $S_k = \sum_{i=1}^k Z_i$, $k \leqslant N$. *Then for any integer* $j \geqslant 2$ *there exist positive numbers* C_j *and* D_j *depending only on* j *such that for all* $t > 0$

$$P\{\max_{k \leqslant N} |S_k| > 4^{j-1} t\} \leqslant C_j P\{\max_{i \leqslant N} |X_i| > t\} + D_j (P\{\max_{k \leqslant N} |S_k| > t\})^j.$$

Proof. The assertion follows from Proposition 6.7 of [9] by induction. ∎

We also need the next lemma to prove the divergent parts of our main theorems.

LEMMA 2.3. *Let f be as in* (b) *of Theorem* 1.1. *Then there exists a non-decreasing function g: $[1, \infty) \to (0, \infty)$ such that*

$$\lim_{n\to\infty} g(x) = \infty \quad \text{and} \quad \int_2^\infty \frac{dx}{x\big(f(x)g(x)\big)^{1+\varepsilon_0}} = \infty.$$

Proof. The assertion follows from Lemma 2.2 of [1]. ∎

Proof of Theorem 1.1. (a) Without loss of generality we can assume that $k_n \geqslant n$, and by Lemma 2.1 we also can assume that $\limsup_{x\to\infty} f(2x)/f(x) < \infty$. Furthermore, we can also assume that M is an even integer. Directly from (4.6) of [13] we have $\sup_{n\geqslant 1} r_{n-1} r_n^{-1} < \infty$, and hence

$$\sup_{n\geqslant 1} r_n r_{k_n}^{-1} \leqslant \max\{1, (\sup_{n\geqslant 1} r_{n-1} r_n^{-1})^M\} < \infty.$$

Consequently, we have

$$\limsup_{n\to\infty} r_n f(n)^{-1/\alpha(\theta)} \big(r_{k_n} f(k_n)^{-1/\alpha(\theta)}\big)^{-1} < \infty.$$

Then to prove (1.8), it is enough to show that

$$\limsup_{n\to\infty} \frac{\left|r_{k_n} \sum_{k=1}^{k_n} a_{nk} \langle X_k, \theta\rangle\right|}{f(k_n)^{1/\alpha(\theta)}} = 0 \text{ a.s.} \tag{2.1}$$

By the same argument as in [10], one can assume, without loss of generality, that $k_n = n$ for every $n \geqslant 1$. Hence (2.1) follows from

$$\limsup_{n\to\infty} \frac{\left|r_n \sum_{k=1}^{n} a_{nk} \langle X_k, \theta\rangle\right|}{f(n)^{1/\alpha(\theta)}} = 0 \text{ a.s.} \tag{2.2}$$

Choose an integer $j \geqslant 2$ with $j(1-\delta_0) > 1$. Then (2.2) holds if we can show that for any $\varepsilon > 0$ we have

$$\limsup_{n\to\infty} \frac{\left|r_n \sum_{k=1}^{n} a_{nk} \langle X_k, \theta\rangle\right|}{f(n)^{1/\alpha(\theta)}} \leqslant 4^{j-1}\cdot 3M\varepsilon \text{ a.s.} \tag{2.3}$$

Note that, by (1.5) and (1.7) and our assumptions of f, for any $b > 0$ and some constant $C > 0$ we have

$$\sum_{n=1}^\infty P\{r_n|\langle X_n, \theta\rangle| > b^{-1}\varepsilon f(n)^{1/\alpha(\theta)}\} \leqslant C\sum_{n=1}^\infty f(n)^{-1+\varepsilon_0} P\{r_n|\langle X_n, \theta\rangle| > 1\}$$

$$\leqslant C\sup_{n\geqslant 1} nP\{r_n|\langle X_n, \theta\rangle| > 1\} \sum_{n=1}^\infty \big(nf(n)^{1-\varepsilon_0}\big)^{-1} < \infty.$$

Note that by (4.7) of [13] we have $b = \sup_{n\geq 1}\sup_{1\leq k\leq n} r_n r_k^{-1} < \infty$. Then the monotonicity of f together with the Borel–Cantelli lemma implies that

$$\sum_{k=1}^{n} r_n |\langle X_k, \theta\rangle| I\big(r_n |\langle X_k, \theta\rangle| > \varepsilon f(n)^{1/\alpha(\theta)}\big)$$

is bounded almost surely. Therefore

$$\frac{r_n \sum_{k=1}^{n} a_{nk} |\langle X_k, \theta\rangle| I\big(r_n |\langle X_k, \theta\rangle| > \varepsilon f(n)^{1/\alpha(\theta)}\big)}{f(n)^{1/\alpha(\theta)}} \to 0 \text{ a.s.}$$

Hence to prove (2.3) it enough to show that

$$\text{(2.4)} \quad \limsup_{n\to\infty} \frac{r_n \big|\sum_{k=1}^{n} a_{nk} \langle X_k, \theta\rangle I\big(r_n |\langle X_k, \theta\rangle| \leq \varepsilon f(n)^{1/\alpha(\theta)}\big)\big|}{f(n)^{1/\alpha(\theta)}} \leq 4^{j-1}\cdot 3M\varepsilon \text{ a.s.}$$

Before we prove (2.4) we first show that

(2.5)

$$\frac{r_n \sum_{k=1}^{n} \langle X_k, \theta\rangle - n r_n E\langle X, \theta\rangle I\big(r_n |\langle X, \theta\rangle| \leq \varepsilon f(n)^{1/\alpha(\theta)}\big)}{f(n)^{1/\alpha(\theta)}} \to 0 \text{ in probability.}$$

In fact, for $\tilde{\varepsilon} > 0$ decompose

$$\begin{aligned}
&P\Big\{\Big|\sum_{k=1}^{n} r_n \langle X_k, \theta\rangle - n r_n E\langle X, \theta\rangle I\big(r_n |\langle X, \theta\rangle| \leq \varepsilon f(n)^{1/\alpha(\theta)}\big)\Big| \geq \tilde{\varepsilon} f(n)^{1/\alpha(\theta)}\Big\}\\
&\leq P\Big(\bigcup_{k=1}^{n} \{r_n |\langle X_k, \theta\rangle| > \varepsilon f(n)^{1/\alpha(\theta)}\}\Big)\\
&\quad + P\Big\{\Big|\sum_{k=1}^{n} r_n \langle X_k, \theta\rangle I\big(r_n |\langle X_k, \theta\rangle| \leq \varepsilon f(n)^{1/\alpha(\theta)}\big)\\
&\qquad - n r_n E\langle X, \theta\rangle I\big(r_n |\langle X, \theta\rangle| \leq \varepsilon f(n)^{1/\alpha(\theta)}\big)\Big| \geq \tilde{\varepsilon} f(n)^{1/\alpha(\theta)}\Big\}\\
&= I_1 + I_2.
\end{aligned}$$

Now, by (1.5) and (1.7),

$$\begin{aligned}
I_1 &\leq nP\{r_n |\langle X, \theta\rangle| > \varepsilon f(n)^{1/\alpha(\theta)}\}\\
&\leq C_2 \big(\sup_{n\geq 1} nP\{r_n |\langle X, \theta\rangle| > 1\}\big) f(n)^{-(1-\varepsilon_0)} \to 0.
\end{aligned}$$

Moreover, by Chebyshev's inequality together with (1.6) we conclude that for some constants $C_1, C_2 > 0$ we have

$$\begin{aligned}
I_2 &\leq C_1 n r_n^2 f(n)^{-2/\alpha(\theta)} U_2\big(r_n^{-1} f(n)^{1/\alpha(\theta)} \varepsilon, \theta\big)\\
&\leq C_2 nP\{r_n |\langle X, \theta\rangle| > \varepsilon f(n)^{1/\alpha(\theta)}\} \to 0,
\end{aligned}$$

proving (2.5). Since $\left(r_n \sum_{k=1}^{n} \langle X_k, \theta \rangle\right)$ is stochastically compact, (2.5) implies

$$\frac{nr_n E\langle X, \theta\rangle I\left(r_n |\langle X, \theta\rangle| \leqslant \varepsilon f(n)^{1/\alpha(\theta)}\right)}{f(n)^{1/\alpha(\theta)}} \to 0 \quad \text{as } n \to \infty.$$

It follows that

$$(2.6) \quad \frac{\left|r_n \sum_{k=1}^{n} a_{nk} E\langle X_k, \theta\rangle I\left(r_n |\langle X, \theta\rangle| \leqslant \varepsilon f(n)^{1/\alpha(\theta)}\right)\right|}{f(n)^{1/\alpha(\theta)}}$$

$$\leqslant \frac{Mnr_n \left|E\langle X, \theta\rangle I\left(r_n |\langle X, \theta\rangle| \leqslant \varepsilon f(n)^{1/\alpha(\theta)}\right)\right|}{f(n)^{1/\alpha(\theta)}} \to 0 \quad \text{as } n \to \infty.$$

Let $U_{nk} = r_n \langle X_k, \theta\rangle I\left(r_n |\langle X_k, \theta\rangle| \leqslant \varepsilon f(n)^{1/\alpha(\theta)}\right)$. In view of (2.6), to prove (2.4), it is enough to show that

$$(2.7) \quad \limsup_{n\to\infty} \frac{\left|\sum_{k=1}^{n} a_{nk}(U_{nk} - EU_{nk})\right|}{f(n)^{1/\alpha(\theta)}} \leqslant 4^{j-1} \cdot 3M\varepsilon \text{ a.s.}$$

By the Borel–Cantelli lemma, it is enough to prove that

$$(2.8) \quad \sum_{n=1}^{\infty} P\left\{\left|\sum_{k=1}^{n} a_{nk}(U_{nk} - EU_{nk})\right| > 4^{j-1} \cdot 3M\varepsilon f(n)^{1/\alpha(\theta)}\right\} < \infty.$$

In view of Lemma 2.2, (2.8) follows from

$$(2.9) \quad \sum_{n=1}^{\infty} P\left\{\max_{1 \leqslant k \leqslant n} |a_{nk}(U_{nk} - EU_{nk})| > 3M\varepsilon f(n)^{1/\alpha(\theta)}\right\} < \infty$$

and

$$(2.10) \quad \sum_{n=1}^{\infty} \left(P\left\{\sum_{k=1}^{n} |a_{nk}(U_{nk} - EU_{nk})| > 3M\varepsilon f(n)^{1/\alpha(\theta)}\right\}\right)^j < \infty.$$

Since $\max_{1 \leqslant k \leqslant n} |a_{nk}(U_{nk} - EU_{nk})| \leqslant 2M\varepsilon f(n)^{1/\alpha(\theta)}$, for every $n \geqslant 1$, we know that $P\{\max_{1 \leqslant k \leqslant n} |a_{nk}(U_{nk} - EU_{nk})| > 3M\varepsilon f(n)^{1/\alpha(\theta)}\} = 0$, so (2.9) holds true.

By Chebyshev's inequality together with (1.5) and (1.6), we have for some constant $C > 0$

$$P\left\{\sum_{k=1}^{n} |a_{nk}(U_{nk} - EU_{nk})| > 3M\varepsilon f(n)^{1/\alpha(\theta)}\right\}$$

$$\leqslant C\left(\sum_{k=1}^{n} a_{nk}^2\right) r_n^2 f(n)^{-2/\alpha(\theta)} U_2\left(r_n^{-1} f(n)^{1/\alpha(\theta)} \varepsilon, \theta\right)$$

$$\leqslant n^{\delta_0} P\{r_n |\langle X, \theta\rangle| > \varepsilon f(n)^{1/\alpha(\theta)}\} \leqslant n^{\delta_0 - 1}.$$

Since $j(1-\delta_0) > 1$, (2.10) follows at once. Hence (2.7) holds true.

(b) The proof is similar to the proof of Theorem 1.1 (b) in [5], so we omit it. See also the proof of Theorem 1.5 in [5]. ∎

Before we give a proof of Theorem 1.3 and its corollary, similar to that in [13], we first prove a special case sufficient for our purpose. Recall from [11] that a (c^E, c) operator semistable law is called *spectrally simple* if every eigenvalue of E has the same real part.

PROPOSITION 2.4. *Let the distribution of Y be a full (c^E, c) operator semistable, spectrally simple, nonnormal law on a finite-dimensional vector space V and let X belong to the strict generalized domain of semistable attraction of Y, i.e.* (1.3) *holds. Let $f\colon [1, \infty] \to (0, \infty)$ be nondecreasing with $\lim_{x\to\infty} f(x) = \infty$. Then:*

(a) *If there exists an $\varepsilon_0 > 0$ such that*

$$\int_2^\infty \frac{dx}{xf(x)^{1-\varepsilon_0}} < \infty,$$

then for any array of real numbers $\{a_{nk},\ 1 \leqslant k \leqslant k_n,\ n \geqslant 1\}$ with $k_n \leqslant Mn$, for all $n \geqslant 1$, $\sup_{n,k} |a_{nk}| \leqslant M$ and $\sum_{k=1}^{k_n} a_{nk}^2 = O(n^{\delta_0})$ for some $\delta_0 < 1$, where M is a positive constant not depending on n, we have

$$\limsup_{n\to\infty} \frac{\|B_n \sum_{k=1}^{k_n} a_{nk} X_k\|}{f(n)^{1/\alpha}} = 0 \ \text{a.s.}$$

(b) *If there exists an $\varepsilon_0 > 0$ such that*

$$\int_2^\infty \frac{dx}{xf(x)^{1+\varepsilon_0}} = \infty,$$

then for any array of real numbers $\{a_{nk},\ 1 \leqslant k \leqslant k_n,\ n \geqslant 1\}$ such that there exist two strictly increasing sequences of positive integers $l(n)$, $m(n)$, $n \geqslant 1$, with

$$\sup_{n\geqslant 1} \big(l(n+1) - l(n)\big) < \infty \quad \text{and} \quad \liminf_{n\to\infty} |a_{l(n),m(n)}| > 0$$

we have

$$\limsup_{n\to\infty} \frac{\|B_n \sum_{k=1}^{k_n} a_{nk} X_k\|}{f(n)^{1/\alpha}} = \infty \ \text{a.s.},$$

where $B_n \in RV(-E)$ is the embedding sequence and $1/\alpha$ is the real part of the eigenvalues of E.

Proof. (a) By the same argument as in the proof of Theorem 1.1, we can assume that $k_n = n$ for every $n \geqslant 1$. Hence it is enough to prove that

$$\limsup_{n\to\infty} \frac{\|B_n \sum_{k=1}^{n} a_{nk} X_k\|}{f(n)^{1/\alpha}} = 0 \ \text{a.s.} \tag{2.11}$$

Let $\{\theta^{(1)}, \ldots, \theta^{(m)}\}$ be an orthonormal basis of V. Since

$$\|B_n \sum_{k=1}^{n} a_{nk} X_k\|^2 = |\langle B_n \sum_{k=1}^{n} a_{nk} X_k,\ \theta^{(1)}\rangle|^2 + \ldots + |\langle B_n \sum_{k=1}^{n} a_{nk} X_k,\ \theta^{(m)}\rangle|^2,$$

to prove (2.11) it suffices to show that for any $1 \leqslant j \leqslant m$ we have

$$\frac{|\langle B_n \sum_{k=1}^{k_n} a_{nk} X_k, \theta^{(j)} \rangle|}{f(n)^{1/\alpha}} \to 0 \text{ a.s.} \quad \text{as } n \to \infty. \tag{2.12}$$

Fix any $1 \leqslant j \leqslant m$, write $B_n^* \theta^{(j)} = r_n \theta_n$ for some $r_n > 0$ and $\|\theta_k\| = 1$. Hence to prove (2.12) it is enough to show that

$$\frac{r_n |\langle \sum_{k=1}^{n} a_{nk} X_k, \theta_n \rangle|}{f(n)^{1/\alpha}} \to 0 \text{ a.s.} \quad \text{as } n \to \infty. \tag{2.13}$$

Following the proof of Theorem 1.1, using the uniform R-O variation results obtained in [14] instead of (1.5), (1.6) and (1.7), we get (2.13). We leave the details to the reader.

(b) The proof is similar to the proof of Theorem 1.1. See also [5]. ■

Proof of Theorem 1.3. Using Proposition 2.4, we obtain the result of Theorem 1.3 along the lines of the proof of Theorem 2.6 in [13]. ■

3. APPLICATIONS

In this section, as applications of Theorem 1.1, we will discuss the corresponding results for some classical summability methods. For the Cesàro method, Riesz method, by the same argument as in [10] we have:

THEOREM 3.1 (Cesàro method). *Let $0 < \alpha < 1$ and $f: [1, \infty) \to (0, \infty)$ be nondecreasing with $\lim_{x \to \infty} f(x) = \infty$. Then:*

(a) *If there exists an $\varepsilon_0 > 0$ such that*

$$\int_2^\infty \frac{dx}{x f(x)^{1-\varepsilon_0}} < \infty,$$

then for any $\|\theta\| = 1$ we have, for $r_n = r_n(\theta)$ and $\alpha(\theta)$ as above,

$$\limsup_{n \to \infty} \frac{|r_n \sum_{k=0}^{n} A_{n-k}^{\alpha-1} \langle X_k, \theta \rangle|}{f(n)^{1/\alpha(\theta)}} = 0 \text{ a.s.} \tag{3.1}$$

and especially for any $\delta > 0$

$$\limsup_{n \to \infty} \frac{|r_n \sum_{k=0}^{n} A_{n-k}^{\alpha-1} \langle X_k, \theta \rangle|}{(\log n)^{(1+\delta)/\alpha(\theta)}} = 0 \text{ a.s.} \tag{3.2}$$

(b) *If there exists an $\varepsilon_0 > 0$ such that*

$$\int_2^\infty \frac{dx}{x f(x)^{1+\varepsilon_0}} = \infty,$$

then for any $\|\theta\| = 1$ *we have*

(3.3) $$\limsup_{n\to\infty} \frac{|r_n \sum_{k=0}^{n} A_{n-k}^{\alpha-1} \langle X_k, \theta\rangle|}{f(n)^{1/\alpha(\theta)}} = \infty \ \text{a.s.}$$

and especially for any $0 < \delta < 1$

(3.4) $$\limsup_{n\to\infty} \frac{|r_n \sum_{k=0}^{n} A_{n-k}^{\alpha-1} \langle X_k, \theta\rangle|}{(\log n)^{(1-\delta)/\alpha(\theta)}} = \infty \ \text{a.s.},$$

where for any $\beta > -1$, $A_0^\beta = 1$ *and* $A_j^\beta = (\beta+1)\ldots(\beta+j)/j!$ *for every* $j \geq 1$.

COROLLARY 3.2. *For any* $\|\theta\| = 1$ *we have*

(3.5) $$\limsup_{n\to\infty} \left|r_n \sum_{k=0}^{n} A_{n-k}^{\alpha-1} \langle X_k, \theta\rangle\right|^{1/\log\log n} = e^{1/\alpha(\theta)} \ \text{a.s.},$$

where r_n *is as above.*

THEOREM 3.3 (Riesz method or delayed method). *Let* $p > 1$ *and* $f\colon [1, \infty) \to (0, \infty)$ *be nondecreasing with* $\lim_{x\to\infty} f(x) = \infty$. *Then*

(a) *If there exists an* $\varepsilon_0 > 0$ *such that*

$$\int_2^\infty \frac{dx}{x f(x)^{1-\varepsilon_0}} < \infty,$$

then for any $\varepsilon > 0$ *and for any* $\|\theta\| = 1$ *we have, for* $r_n = r_n(\theta)$ *and* $\alpha(\theta)$ *as above,*

(3.6) $$\limsup_{n\to\infty} \frac{|r_n \sum_{k=n}^{n+\varepsilon n^{1/p}} \langle X_k, \theta\rangle|}{f(n)^{1/\alpha(\theta)}} = 0 \ \text{a.s.}$$

and especially for any $\delta > 0$

(3.7) $$\limsup_{n\to\infty} \frac{|r_n \sum_{k=n}^{n+\varepsilon n^{1/p}} \langle X_k, \theta\rangle|}{(\log n)^{(1+\delta)/\alpha(\theta)}} = 0 \ \text{a.s.}$$

(b) *If there exists an* $\varepsilon_0 > 0$ *such that*

$$\int_2^\infty \frac{dx}{x f(x)^{1+\varepsilon_0}} = \infty,$$

then for any $\varepsilon > 0$ *and for any* $\|\theta\| = 1$ *we have*

(3.8) $$\limsup_{n\to\infty} \frac{|r_n \sum_{k=n}^{n+\varepsilon n^{1/p}} \langle X_k, \theta\rangle|}{f(n)^{1/\alpha(\theta)}} = \infty \ \text{a.s.}$$

and especially for any $0 < \delta < 1$

(3.9) $$\limsup_{n\to\infty} \frac{|r_n \sum_{k=n}^{n+\varepsilon n^{1/p}} \langle X_k, \theta\rangle|}{(\log n)^{(1-\delta)/\alpha(\theta)}} = \infty \ \text{a.s.}$$

COROLLARY 3.4. *For any $p > 1$, any $\varepsilon > 0$ and for any $\|\theta\| = 1$ we have*

$$\limsup_{n\to\infty} \left| r_n \sum_{k=n}^{n+\varepsilon n^{1/p}} \langle X_k, \theta \rangle \right|^{1/\log\log n} = e^{1/\alpha(\theta)} \quad a.s., \tag{3.10}$$

where r_n is as above.

For the Euler method, we have

THEOREM 3.5 (Euler method). *Let $0 < q < 1$ and $f\colon [1, \infty) \to (0, \infty)$ be nondecreasing with $\lim_{x\to\infty} f(x) = \infty$. Then:*

(a) *If there exists an $\varepsilon_0 > 0$ such that*

$$\int_2^\infty \frac{dx}{x f(x)^{1-\varepsilon_0}} < \infty,$$

then for any $\|\theta\| = 1$ we have, for $r_n = r_n(\theta)$ and $\alpha(\theta)$ as above,

$$\limsup_{n\to\infty} \frac{\left| r_n \sqrt{n} \sum_{k=0}^{n} C_n^k q^k (1-q)^{n-k} \langle X_k, \theta \rangle \right|}{f(n)^{1/\alpha(\theta)}} = 0 \quad a.s. \tag{3.11}$$

and especially for any $\delta > 0$

$$\limsup_{n\to\infty} \frac{\left| r_n \sqrt{n} \sum_{k=0}^{n} C_n^k q^k (1-q)^{n-k} \langle X_k, \theta \rangle \right|}{(\log n)^{(1+\delta)/\alpha(\theta)}} = 0 \quad a.s. \tag{3.12}$$

(b) *If there exists an $\varepsilon_0 > 0$ such that*

$$\int_2^\infty \frac{dx}{x f(x)^{1+\varepsilon_0}} = \infty,$$

then for any $\|\theta\| = 1$ we have

$$\limsup_{n\to\infty} \frac{\left| r_n \sqrt{n} \sum_{k=0}^{n} C_n^k q^k (1-q)^{n-k} \langle X_k, \theta \rangle \right|}{f(n)^{1/\alpha(\theta)}} = \infty \quad a.s. \tag{3.13}$$

and especially for any $0 < \delta < 1$

$$\limsup_{n\to\infty} \frac{\left| r_n \sqrt{n} \sum_{k=0}^{n} C_n^k q^k (1-q)^{n-k} \langle X_k, \theta \rangle \right|}{(\log n)^{(1-\delta)/\alpha(\theta)}} = \infty \quad a.s., \tag{3.14}$$

where $C_n^k = n!/(k!(n-k)!)$ for any $n \geqslant 1$ and $0 \leqslant k \leqslant n$.

COROLLARY 3.6. *For any $0 < q < 1$ and any $\|\theta\| = 1$ we have*

$$\limsup_{n\to\infty} \left| r_n \sqrt{n} \sum_{k=0}^{n} C_n^k q^k (1-q)^{n-k} \langle X_k, \theta \rangle \right|^{1/\log\log n} = e^{1/\alpha(\theta)} \quad a.s., \tag{3.15}$$

where r_n is as above.

For Borel's method we have

THEOREM 3.7 (Borel method). *Let $f\colon [1, \infty) \to (0, \infty)$ be nondecreasing with $\lim_{x\to\infty} f(x) = \infty$. Then:*

(a) *If there exists an $\varepsilon_0 > 0$ such that*

$$\int_2^\infty \frac{dx}{x f(x)^{1-\varepsilon_0}} < \infty,$$

then for any $\|\theta\| = 1$ we have, for $r_n = r_n(\theta)$ and $\alpha(\theta)$ as above,

$$\limsup_{\lambda\to\infty} \frac{\left|r_{[\lambda]}\sqrt{\lambda} e^{-\lambda} \sum_{k=0}^\infty (\lambda^k/k!) \langle X_k, \theta\rangle\right|}{f(\lambda)^{1/\alpha(\theta)}} = 0 \ a.s. \tag{3.16}$$

and especially for any $\delta > 0$

$$\limsup_{\lambda\to\infty} \frac{\left|r_{[\lambda]}\sqrt{\lambda} e^{-\lambda} \sum_{k=0}^\infty (\lambda^k/k!) \langle X_k, \theta\rangle\right|}{(\log \lambda)^{(1+\delta)/\alpha(\theta)}} = 0 \ a.s. \tag{3.17}$$

(b) *If there exists an $\varepsilon_0 > 0$ such that*

$$\int_2^\infty \frac{dx}{x f(x)^{1+\varepsilon_0}} = \infty,$$

then for any $\|\theta\| = 1$ we have

$$\limsup_{\lambda\to\infty} \frac{\left|r_{[\lambda]}\sqrt{\lambda} e^{-\lambda} \sum_{k=0}^\infty (\lambda^k/k!) \langle X_k, \theta\rangle\right|}{f(\lambda)^{1/\alpha(\theta)}} = \infty \ a.s. \tag{3.18}$$

and especially for any $0 < \delta < 1$

$$\limsup_{\lambda\to\infty} \frac{\left|r_{[\lambda]}\sqrt{\lambda} e^{-\lambda} \sum_{k=0}^\infty (\lambda^k/k!) \langle X_k, \theta\rangle\right|}{(\log \lambda)^{(1-\delta)/\alpha(\theta)}} = \infty \ a.s., \tag{3.19}$$

where $[x]$ denotes the largest integer less than or equal to x.

Proof. It is enough to prove part (a). By Lemma 2.1, we can assume that $\limsup_{x\to\infty} f(2x)/f(x) < \infty$. Since

$$\sup_{n\leqslant\lambda<n+1} \left|\sum_{k=0}^\infty \frac{\lambda^k}{k!} \langle X_k, \theta\rangle\right| \leqslant \sum_{k=0}^\infty \frac{(n+1)^k}{k!} |\langle X_k, \theta\rangle|$$

and $\sup_{n\geqslant 1} r_n r_{n+1}^{-1} < \infty$, it is enough to prove that

$$\limsup_{n\to\infty} \frac{r_n \sqrt{n} e^{-n} \sum_{k=0}^\infty \left((n+1)^k/k!\right) |\langle X_k, \theta\rangle|}{f(n)^{1/\alpha(\theta)}} = 0 \text{ a.s.} \tag{3.20}$$

Take any $0 < t < \min\{1, \alpha(\theta)\}$. Then, by Theorem 16 of [12], there exists an integer $M > 1$ such that

$$\sum_{k\geqslant Mn+1} \left(e^{-n}\frac{n^k}{k!}\right)^t \leqslant C n^{-(1+t/2)}.$$

Hence, by Markov's inequality, for any $\varepsilon > 0$ and some constant $C > 0$

$$P\left\{r_n\sqrt{n}e^{-n}\sum_{k\geqslant Mn+1}\frac{n^k}{k!}|\langle X_k, \theta\rangle| > \varepsilon f(n)^{1/\alpha(\theta)}\right\} \leqslant Cn^{-1}r_n^t E|\langle X, \theta\rangle|^t.$$

Since $E|\langle X, \theta\rangle|^t < \infty$ and (1.4) holds true, by the Borel–Cantelli lemma, we get

$$\limsup_{n\to\infty}\frac{r_n\sqrt{n}e^{-n}\sum_{k\geqslant Mn+1}|\langle X_k, \theta\rangle|}{f(n)^{1/\alpha(\theta)}} = 0 \text{ a.s.}$$

Then (3.20) follows from

$$\limsup_{n\to\infty}\frac{r_n\sqrt{n}e^{-n}\sum_{k=0}^{Mn}(n^k/k!)|\langle X_k, \theta\rangle|}{f(n)^{1/\alpha(\theta)}} = 0 \text{ a.s.} \tag{3.21}$$

Let $a_{nk} = \sqrt{n}e^{-n}(n^k/k!)$, $n \geqslant 1$, $0 \leqslant k \leqslant Mn$. A slight modification of the proof of Theorem 1.1 yields (3.21). This completes the proof of Theorem 3.7. ∎

As a corollary the following law of the iterated logarithm (LIL) holds true:

COROLLARY 3.8. *For any* $\|\theta\| = 1$ *we have*

$$\limsup_{\lambda\to\infty}\left|r_{[\lambda]}\sqrt{\lambda}e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\langle X_k, \theta\rangle\right|^{1/\log\log\lambda} = e^{1/\alpha(\theta)} \text{ a.s.,} \tag{3.22}$$

where r_n *is as above.*

Results similar to Theorems 3.1, 3.3, 3.5, and 3.7 and respective corollaries also hold true for $X^{(1,\ldots,i)}$. We leave the formulation and the proofs to the interested reader.

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