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LOCAL LARGE DEVIATION THEOREM FOR SUMS OF I.I.D. RANDOM VECTORS WHEN THE CRAMÉR CONDITION HOLDS IN THE WHOLE SPACE

BY

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Abstract. A class of multidimensional distributions is considered. This class contains all the elliptically contoured distributions having sup-exponential weight function. Each representative of the class determines a family of the so-called exponential or conjugate distributions. It is established that the conjugate distribution is asymptotically normal. On the basis of this normality a large deviation local limit theorem is proved. The theorem assumes no restrictions on the order of deviations.

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1. INTRODUCTION

Let $\xi, \xi^{(1)}, \xi^{(2)}, \ldots$ be a sequence of i.i.d. random vectors taking values in $\mathbf{R}^d$, $d \geqslant 1$. Denote by B the covariance matrix of ξ. We assume that this matrix exists and, furthermore, is positive definite, i.e. the underlying distribution is strictly d-dimensional. Moreover, we assume that $E\xi_k = 0$, $k = 1, \ldots, d$, where $\xi = (\xi_1, \ldots, \xi_d)$. Consider the sums $\zeta^{(n)} = \xi^{(1)} + \ldots + \xi^{(n)}$, $n = 1, 2, \ldots$ By the CLT, as $n \to \infty$

$$(1.1)\qquad P(n^{-1/2}\zeta^{(n)} \in A) \to \int_A \varphi_{(0,B)}(x)\,\lambda_d(dx)$$

for any set A being the set of continuity of the Lebesgue measure λ_d. Here $\varphi_{(0,B)}(x)$ denotes the density of the normal distribution in $\mathbf{R}^d$ having the zero vector of expectation and the covariance matrix B.

If in (1.1) the set $A = A_n$, i.e. A varies as $n \to \infty$, and

$$n^{-1/2}\inf(|x|:\ x \in A_n) \to \infty,$$

then from (1.1) it follows that $P(n^{-1/2}\zeta^{(n)}\in A_n)\to 0$. The problem arises: how to establish an asymptotic formula for $P(n^{-1/2}\zeta^{(n)}\in A_n)$? Such a problem is called the *large deviation problem.*

The large deviation theory for the sums of i.i.d. scalar random variables is well developed. The overview of [14] gives an impression about the state of the theory to the end of the seventies. Among the works which appeared later on, the series of papers written by Rozovskii (see e.g. [16]) are worth mentioning.

As to the sums of i.i.d. random vectors such works as [17], [6], [2], [15] should be mentioned in the first turn. The volume of this paper does not allow us to analyze the present state of that part of the large deviation theory which deals with i.i.d. random vectors. We emphasize only that our attention is focused on an extension of the local theorem proved in [6].

Thus, we stay within the frame of the classical large deviation theory that has not very much in common with the modern theory presented in Dembo and Zeitouni [7], Deuschel and Stroock [8], Ellis [9], Bahadur and Zabell [1], etc.

As in [6] we assume that the underlying distribution is absolutely continuous and the so-called *Cramér condition* holds, i.e.

$$f(s) = Ee^{\langle s,\xi\rangle} = \int_{\boldsymbol{R}^d} e^{\langle s,x\rangle} p(x)\lambda_d(dx) < \infty, \qquad s\in S, \tag{1.2}$$

where S is an open subset of $\boldsymbol{R}^d$ having $\boldsymbol{0}=(0,\ldots,0)$ as an interior point.

Denote by $M(s)$ and $\bar{M}(s)$, respectively, the gradient and the matrix of the second partial derivatives (hessian) of $\ln f(s)$. Let X be the image of S under the mapping $M(s)$.

The following form of the local limit theorem was established in [6].

THEOREM 1.1. *Let the underlying density $p(x)$ be uniformly bounded. Let, further, $p_n(x)$ be the n-th convolution of $p(x)$. If (1.2) is fulfilled, then as $n\to\infty$*

$$\sup_{x\in X_1}\left|\frac{p_n(nx)}{\varrho^n(x)\psi_n(x)}-1\right| = o(1). \tag{1.3}$$

Here X_1 is a compact subset of X while

$$\varrho(x) = \inf_{s\in S} e^{-\langle s,x\rangle} f(s)$$

and

$$\psi_n(x) = (2\pi n)^{-d/2}\left(\det\bar{M}(s(x))\right)^{-1/2}, \qquad s(x) = M^{-1}(x).$$

Such a fact allows asymptotic analysis of the large deviation probability $P(n^{-1/2}\zeta^{(n)}\in A_n)$ for various sequences $\{A_n\}$.

The question is: how to extend the relation (1.3) to the whole X? It should be emphasized that the situation admits huge variety of possible configurations of S and X. So one should try to classify them somehow. We set off the following special cases:

(1) both S and X coincide with $\boldsymbol{R}^d$;
(2) S is bounded while $X = \boldsymbol{R}^d$;
(3) X is bounded while $S = \boldsymbol{R}^d$.

The present paper concerns the first of these cases. As to the second one it is analyzed in [13] (see also [12]). The third case was studied in [18].

It is evident that the desired extension requires additional restrictions on the underlying density $p(x)$. They concern certain regularity of the asymptotic behavior of $p(x)$ as $|x| \to \infty$. Such a regularity aims at proving a version of the Abel type theorem which establishes the asymptotic of $f(s)$ as $s \to \partial S$. In the process of proving the Abel theorem we simultaneously predict asymptotic properties of the so-called conjugate density. So, the Abel theorem plays a key role within the framework of the considered problem.

We assume that as $|x| \to \infty$

$$p(x) = e^{-r(|x|)}\big(1+\tau(x)\big), \tag{1.4}$$

where

$$\lim_{t\to\infty} \frac{r(t)}{t} = \infty, \qquad |\tau(x)| \leqslant \omega(|x|), \qquad \lim_{t\to\infty} \omega(t) = 0,$$

and $r(t)$ is sufficiently smooth (see Section 2).

Obviously, (1.4) implies $S = \boldsymbol{R}^d$. It is easy to show that $X = \boldsymbol{R}^d$ as well. Of course, the condition does not exhaust the whole case $S = X = \boldsymbol{R}^d$. Nevertheless, it determines a very rich class of distributions which contains, in particular, the spherically invariant densities $p(x) = q(|x|)$, $q(t) = e^{-r(t)}$.

Consider the elliptically contoured density

$$p^{(A)}(x) = (\det A)^{1/2}\, q\big((x^T Ax)^{1/2}\big),$$

where A is a positive definite matrix. Denote by $p_n^{(A)}(x)$ the n-th convolution of $p^{(A)}(x)$. For $x = A^{-1/2}u$ we have

$$p_n^{(A)}(x) = (\det A)^{-1/2}\, p_n^{(A)}(A^{-1/2}u) = p_n(u),$$

where $p_n(u)$ is the n-th convolution of $p(u) = q(|u|)$. It means that the relation

$$\sup_{x\in\boldsymbol{R}^d} \left| \frac{p_n(nx)}{\varrho^n(x)\,\psi_n(x)} - 1 \right| = o(1), \tag{1.5}$$

being proved for a spherically invariant underlying density $p(x) = q(|x|)$, is immediately extended to all elliptically contoured densities which are determined by the weight function $q(t)$.

Obviously, it remains true for densities which are spherically invariant only asymptotically (see (1.4)). So, the condition (1.4) is well motivated.

The paper is organized as follows. In Section 2 we state our basic result. Section 3 contains auxiliary facts concerning the class $\mathscr{H}$. The proof of the Abel theorem is given in Section 4. In Section 5 we prove the local limit theorem for the conjugate distributions. The basic result is derived in Section 6 as an immediate corollary to that theorem. Some statistical applications and other concluding remarks are given in Section 7.

2. REGULARITY CONDITIONS AND THE BASIC RESULT

It is convenient to state the regularity conditions imposed on $r(t)$ in terms of its derivative. Let us put $h(t) = r'(t)$. We have to distinguish between the following three classes:

(1) $h(t)$ is of slow variation as $t \to \infty$;

(2) $h(t)$ is of regular variation as $t \to \infty$;

(3) $h(t)$ tends to infinity faster than any power of t.

These three cases cover all the possible rates of growth of $r(t)$ which are compatible with the assumption that $S = \boldsymbol{R}^d$.

Recall that a positive measurable function $l(t)$ defined on $(0, \infty)$ is of slow variation as $t \to \infty$ if for any fixed $c > 0$

$$\lim_{t\to\infty} \frac{l(ct)}{l(t)} = 1.$$

It is well known that a slow varying function admits the Karamata representation

$$l(t) = a(t)\exp\left(\int_1^t \frac{\varepsilon(u)}{u}\,du\right),$$

where $\lim_{t\to\infty} a(t) = a_0 > 0$, $\lim_{t\to\infty} \varepsilon(t) = 0$ (see e.g. [5], Theorem 1.3.1).

Denote by $\mathscr{L}$ the class of slowly varying functions as $t \to \infty$ admitting the following representation:

$$l(t) = \exp\left(\int_1^t \frac{\varepsilon(u)}{u}\,du\right), \quad t \geqslant 1, \tag{2.1}$$

where $\varepsilon(t)$ is differentiable.

The following definition relates to the first of the above cases.

Definition 2.1. We say that $h(t) \in \mathscr{S} \subset \mathscr{L}$ if the function $\varepsilon(t)$ in (2.1) satisfies the following conditions:

$$\lim_{t\to\infty} \frac{t\varepsilon'(t)}{\varepsilon(t)} = 0, \quad \int_1^\infty \frac{\varepsilon(t)}{t}\,dt = \infty \tag{2.2}$$

and for some $\eta \in (0, 1/4)$

$$\varliminf_{t\to\infty} t^\eta \varepsilon(t) > 0. \tag{2.3}$$

Obviously, if $h(t) \in \mathscr{S}$, then it monotonically increases for all sufficiently large t and $h(\infty) = \infty$.

DEFINITION 2.2. We say that $h(t) \in \mathscr{R}$ if it is of regular variation with exponent $\alpha > 0$ and admits the representation

$$h(t) = t^{\alpha} l(t), \tag{2.4}$$

where $l(t) \in \mathscr{L}$ and additionally in (2.1)

$$\overline{\lim_{t\to\infty}}\, t\,|\varepsilon'(t)| < \infty. \tag{2.5}$$

The following definition specifies the third case.

DEFINITION 2.3. We say that a continuous function $h(t) \in \mathscr{F}$ if it strictly monotonically increases and its inverse function $m(t)$ belongs to $\mathscr{S}$.

Set $\mathscr{H} = \mathscr{S} \cup \mathscr{R} \cup \mathscr{F}$. It is easily seen that the class $\mathscr{H}$ restricts the regularity but not the rate of growth of the functions which belong to it. The class contains, for example, such different functions as, say, the k-th iterations of $\ln x$ and e^x. Practically, the whole spectrum of possible growth rates is covered by $\mathscr{H}$.

The following theorem contains our basic result.

THEOREM 2.4. *If the underlying density is uniformly bounded and satisfies* (1.4) *where* $r'(t) \in \mathscr{H}$, *then the relation* (1.5) *holds.*

3. ASYMPTOTIC PROPERTIES OF THE FUNCTIONS BELONGING TO $\mathscr{H}$

It is convenient to adopt the following notation. We write $\alpha(t) \asymp \beta(t)$ if

$$0 < \underline{\lim_{t\to\infty}} \frac{\alpha(t)}{\beta(t)} \leqslant \overline{\lim_{t\to\infty}} \frac{\alpha(t)}{\beta(t)} < \infty.$$

By e_x we denote the unit vector which is collinear to $x \in \boldsymbol{R}^d$, i.e. $e_x = |x|^{-1} x$. Let c be any positive constant whose concrete value is of no importance, i.e. c is, strictly speaking, not the same in different places. By $\omega(t)$ we denote any nonnegative function such that $\lim_{t\to\infty} \omega(t) = 0$. Finally, θ is any variable taking values in $[-1, 1]$.

In order to study the asymptotic properties of the moment generating function we have to make use of the so-called Laplace method. This method is based on the approximation of the integrand in a certain neighborhood of its point of maximum. The asymptotic itself depends both on the maximum of the integrand and on how acute is that maximum. The latter, in view of (1.4), requires knowing the analytic properties of the function $r(|x|)$. Straightforward calculation shows that

$$\operatorname{grad} r(|x|) = h(|x|)\, e_x \tag{3.1}$$

and

$$\text{hess}\, r(|x|) = \frac{h(|x|)}{|x|}(I - e_x e_x^T) + h'(|x|)\, e_x e_x^T, \tag{3.2}$$

where I is the unit matrix.

The derivatives of the third order are

$$r_{iii}(|x|) = \frac{\partial^3}{\partial x_i^3} r(|x|) = \frac{h(|x|)}{|x|^2}(3e_{x_i}^3 - 3e_{x_i}) + \frac{h'(|x|)}{|x|}(3e_{x_i} - 3e_{x_i}^3)$$
$$+ h''(|x|)\, e_{x_i}^3, \quad i = 1, \ldots, d,$$

$$r_{iij}(|x|) = \frac{\partial^3}{\partial x_i^2 \partial x_j} r(|x|) = \frac{h(|x|)}{|x|^2}(3e_{x_i}^2 e_{x_j} - e_{x_j}) + \frac{h'(|x|)}{|x|}(e_{x_j} - 3e_{x_i}^2 e_{x_j}) \tag{3.3}$$
$$+ h''(|x|)\, e_{x_i}^2 e_{x_j}, \quad i \neq j,\ i, j = 1, \ldots, d,$$

$$r_{ijk}(|x|) = \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} r(|x|) = \frac{h(|x|)}{|x|^2}(3e_{x_i} e_{x_j} e_{x_k}) + \frac{h'(|x|)}{|x|}(-3e_{x_i} e_{x_j} e_{x_k})$$
$$+ h''(|x|)\, e_{x_i} e_{x_j} e_{x_k}, \quad i \neq j,\ j \neq k,\ i \neq k,\ i, j, k = 1, \ldots, d.$$

It is worth noting that for all $h(t) \in \mathscr{H}$ the second derivative $h''(t)$ exists, and if $h(t) \in \mathscr{F}$, then $h''(t) \to \infty$, $t \to \infty$. Let

$$V = V(x) = \text{hess}\, r(|x|).$$

It is easily seen that for any $x \in \boldsymbol{R}^d$ the matrix V is positive definite with eigenvalues

$$\lambda_1(|x|) = \frac{h(|x|)}{|x|}, \qquad \lambda_2(|x|) = h'(|x|). \tag{3.4}$$

Furthermore, $V(x)e = \lambda_1(|x|)e$ for all $e \perp e_x$ while $V(x)e_x = \lambda_2(|x|)e_x$. It means that the multiplicity of $\lambda_1(|x|)$ equals $d-1$ and the eigenvector e_x corresponds to $\lambda_2(|x|)$.

LEMMA 3.1. *Let* $t \to \infty$. *If* $h(t) \in \mathscr{S}$, *then*

$$\lambda_2(t) = o(\lambda_1(t)); \tag{3.5}$$

if $h(t) \in \mathscr{R}$, *then*

$$\lambda_1(t) \asymp \lambda_2(t); \tag{3.6}$$

but if $h(t) \in \mathscr{F}$, *then*

$$\lambda_1(t) = o(\lambda_2(t)). \tag{3.7}$$

Proof. Let $h(t) \in \mathscr{S}$. By (2.1) we have

$$h'(t) = \frac{h(t)}{t}\varepsilon(t),$$

and therefore (3.5) follows. Let $h(t) \in \mathscr{R}$. In view of (2.1) and (2.4) we obtain

$$h'(t) = \frac{h(t)}{t}\big(\varepsilon(t)+\alpha\big),$$

whence (3.6) follows. If $h(t) \in \mathscr{F}$, then (3.7) follows from the evident relation

$$\frac{h(t)}{th'(t)} = \frac{um'(u)}{m(u)},$$

where $u = h(t)$. The lemma is proved.

The statements (3.5) and (3.7) mean that in the case where either $h(t)$ or $m(t)$ is of slow variation the eigenvalues of $V(x)$ behave in a quite different way as $|x| \to \infty$. This seriously complicates the asymptotic analysis of $f(s)$.

LEMMA 3.2. *Let $\lambda_+ = \max\big(\lambda_1(t), \lambda_2(t)\big)$. If $h(t) \in \mathscr{H}$, then as $t \to \infty$*

$$t\lambda_+^{1/2} \to \infty, \qquad t\lambda_+^{1/2} = o\Big(\int_1^{h(t)} m(u)\,du\Big).$$

Proof. Let $h(t) \in \mathscr{R}$. From (3.6) it follows that

$$\lambda_+ \asymp \frac{h(t)}{t} \asymp h'(t).$$

By virtue of (2.4) we have $m(t) \sim t^{1/\alpha} l_1(t)$, where $l_1(t)$ slowly varies as $t \to \infty$. It implies that (see e.g. [10], Chapter VIII, Theorem 1)

$$\int_1^u m(v)\,dv \sim u^{1+1/\alpha} l_1(u), \qquad u \to \infty.$$

Then

$$t\lambda_+^{1/2} = \big(th(t)\big)^{1/2} = \big(um(u)\big)^{1/2} \sim u^{1/2+1/(2\alpha)} l_2(u) = o\Big(\int_1^u m(v)\,dv\Big),$$

where $t = m(u)$ and $l_2(u)$ is of slow variation. Thus the lemma follows.

Now let $h(t) \in \mathscr{F}$. By (3.7) we have $\lambda_+ \sim h'(t) \sim \big(m'(u)\big)^{-1}$, $t = m(u)$, where, we remind, $m(u)$ slowly varies as $u \to \infty$. Since (see e.g. [10], Chapter VIII, Theorem 1)

$$\int_1^u m(v)\,dv \sim um(u),$$

we obtain, taking into account (2.3),

$$t\lambda_+^{1/2} \sim \left(\frac{um(u)}{\varepsilon(u)}\right)^{1/2} \leqslant c\big(u^{1+\eta} m(u)\big)^{1/2} = o\Big(\int_1^u m(v)\,dv\Big).$$

It remains to consider the case $h(t) \in \mathscr{S}$. Here in view of (2.2) we have $\lambda_+ \sim h(t)/t$, and therefore $t\lambda_+^{1/2} \sim (th(t))^{1/2}$. Utilizing the l'Hospital rule and (2.2) we get

$$\lim_{t\to\infty} \frac{(th(t))^{1/2}}{\int_1^{h(t)} m(u)\,du} = \frac{1}{2}\lim_{t\to\infty} \frac{t^{-1/2}h(t)^{1/2}}{th'(t)} = \frac{1}{2}\lim_{t\to\infty} \frac{(th(t))^{-1/2}}{\varepsilon(t)} = 0.$$

In view of (2.3) the lemma follows.

In particular, from the lemma we obtain

$$m(|s|)\big(\max(\lambda_1(m(|s|)), \lambda_2(m(|s|)))\big)^{1/2} = o\Big(\int_1^{|s|} m(u)\,du\Big). \tag{3.8}$$

Notice that in (3.1)–(3.3) the functions $h(|x|)$, $h'(|x|)$, $h''(|x|)$ play a dominant role. The following lemma establishes asymptotic relations between these functions.

Lemma 3.3. *Let $t \to \infty$. If $h(t) \in \mathscr{S}$, then*

$$h'(t) = o\left(\frac{h(t)}{t}\right), \quad h''(t) = o\left(\frac{h(t)}{t^2}\right); \tag{3.9}$$

if $h(t) \in \mathscr{R}$, then

$$\frac{h'(t)}{t} \asymp \frac{h(t)}{t^2}, \quad |h''(t)| \leqslant c\frac{h(t)}{t^2}; \tag{3.10}$$

but if $h(t) \in \mathscr{F}$, then

$$\frac{h(t)}{t^2} = o(h''(t)), \quad \frac{h'(t)}{t} = o(h''(t)). \tag{3.11}$$

Proof. Let $h(t) \in \mathscr{S}$. The first of the relations (3.9) follows immediately from (3.5). In order to prove the second one notice that

$$h''(t) = \frac{h(t)}{t^2}\varepsilon(t)\left(\varepsilon(t) + \frac{t\varepsilon'(t)}{\varepsilon(t)} - 1\right)$$

and use (2.2).

Now let $h(t) \in \mathscr{R}$. In view of (2.1) and (2.4) we have

$$h''(t) = \frac{h(t)}{t^2}(\varepsilon^2(t) + (2\alpha - 1)\varepsilon(t) + t\varepsilon'(t) + \alpha^2 - \alpha).$$

Therefore, from (2.5) and (3.6) we obtain (3.10).

Finally, let $h(t) \in \mathscr{F}$. Putting $t = m(u)$ we obtain

$$\frac{h(t)}{t^2 h''(t)} = -\frac{u(m'(u))^3}{m^2(u)m''(u)} \quad \text{and} \quad \frac{h'(t)}{th''(t)} = -\frac{(m'(u))^2}{m(u)m''(u)}.$$

Since for $m(u) \in \mathcal{S}$

$$\frac{um'(u)}{m(u)} = o(1),$$

we have

$$\frac{h(t)}{t^2 h''(t)} = o\left(\frac{(m'(u))^2}{m(u)\,|m''(u)|}\right).$$

From (2.1) and (2.2) it follows that

$$\frac{(m'(u))^2}{m(u)\,|m''(u)|} = \frac{\varepsilon(u)}{|1-\varepsilon(u)-u\varepsilon'(u)/\varepsilon(u)|} = o(1).$$

Thus (3.11) follows. The proof is complete.

Let us put

$$\tilde{r}(|x|) = \max_{i,j,k=1,\dots,d} |r_{ijk}(|x|)|.$$

From Lemma 3.3 and the evident relation

$$\tilde{r}(|x|) \leqslant c \max\left(\frac{h(|x|)}{|x|^2}, \frac{h'(|x|)}{|x|}, |h''(|x|)|\right)$$

we conclude that

$$\tilde{r}(|x|) \leqslant c\frac{h(|x|)}{|x|^2}, \quad h(t) \in \mathcal{S} \cup \mathcal{R}, \tag{3.12}$$

and

$$\tilde{r}(|x|) \leqslant ch''(|x|), \quad h(t) \in \mathcal{F}. \tag{3.13}$$

Set

$$\sigma_1^2(t) = (\lambda_1(t))^{-1}, \quad \sigma_2^2(t) = (\lambda_2(t))^{-1}.$$

LEMMA 3.4. *If $h(t) \in \mathcal{S} \cup \mathcal{R}$, then $t^{-2}h(t)\sigma_2^3(t) = o(1)$.*

Proof. Let $h(t) \in \mathcal{S}$. By (3.4) and (2.3) for all sufficiently large t we have

$$t^{-2}h(t)\sigma_2^3(t) = (th(t))^{-1/2}(\varepsilon(t))^{-3/2} \leqslant ct^{-1/2+(3/2)\eta}(h(t))^{-1/2} = o(1).$$

If $h(t) \in \mathcal{R}$, then the required statement follows immediately from the obvious relation

$$t^{-2}h(t)\sigma_2^3(t) = (th(t))^{-1/2}(\varepsilon(t)+\alpha)^{-3/2}.$$

The lemma is proved.

LEMMA 3.5. *If $h(t) \in \mathcal{F}$, then there exists $l(t)$ such that $l(t) \to \infty$ as $t \to \infty$ and*

$$\sup_{|\tau| \leqslant \sigma_1(t) l(t)} h''(t+\tau) \leqslant ct^{-2}(h(t))^{1+2\eta}, \quad \eta \in (0, 1/4).$$

Proof. Since $h(t)\in\mathscr{F}$, we have

$$\sup_{|v|\leqslant u/2} h''\big(m(u+v)\big) = \sup_{|v|\leqslant u/2} \frac{|m''(u+v)|}{\big(m'(u+v)\big)^3} \leqslant \sup_{|v|\leqslant u/2} \frac{(u+v)^{1+2\eta}}{m^2(u+v)} \leqslant cu^{1+2\eta}\big(m(u)\big)^{-2}.$$

Define

$$\tau_+ = m(3u/2)-m(u), \qquad \tau_- = m(u)-m(u/2).$$

It is easily seen that

$$\tau_+ = m'(u+\tfrac{1}{2}\theta u)(u/2) = \frac{m(u+\frac{1}{2}\theta u)}{2+\theta}\varepsilon(u+\tfrac{1}{2}\theta u), \qquad 0\leqslant\theta\leqslant 1.$$

Recall that $m(u)$ is of slow variation. In view of (2.3) we obtain $\tau_+ \geqslant cu^{-\eta}m(u)$. On the other hand, $\sigma_1\big(m(u)\big) = u^{-1/2}\big(m(u)\big)^{1/2} = o(\tau_+)$. Similarly, $\sigma_1\big(m(u)\big) = o(\tau_-)$. Let $l(t)$, $t = m(u)$, be such that

$$\lim_{t\to\infty} l(t) = \infty, \qquad l(t)\sigma_1(t) = o\big(\min(\tau_+, \tau_-)\big).$$

Then

$$\sup_{|\tau|\leqslant\sigma_1(t)l(t)} h''(t+\tau) \leqslant \sup_{\tau\in[-\tau_-,\tau_+]} h''(t+\tau) = \sup_{|v|\leqslant u/2} h''\big(m(u+v)\big).$$

The lemma is proved.

Let

$$\sigma_+(t) = \max\big(\sigma_1(t), \sigma_2(t)\big).$$

From Lemmas 3.4 and 3.5 we arrive at the following corollary:

COROLLARY 3.6. *If $h(t)\in\mathscr{H}$, then there exists $l(|s|)$ such that $\lim_{|s|\to\infty} l(|s|) = \infty$ and*

$$\sup_{|u|\leqslant\sigma_+(m(|s|))l(|s|)} \tilde{r}\big(m(|s|)+u\big)\sigma_+^3\big(m(|s|)\big)l^3(|s|) = o(1). \tag{3.14}$$

Proof. Let us write for brevity $\sigma_1, \sigma_2, \sigma_+$ instead of $\sigma_1\big(m(|s|)\big)$, $\sigma_2\big(m(|s|)\big)$, $\sigma_+\big(m(|s|)\big)$, respectively. Let $h(t)\in\mathscr{S}\cup\mathscr{R}$. Note that in this case $\sigma_+\asymp\sigma_2$. It is easily seen that $\sigma_2 = \big(m'(|s|)\big)^{1/2} = o\big(m(|s|)\big)$. Thus, there exists $\mu(|s|)$ such that $\mu(|s|)\to\infty$ as $|s|\to\infty$ and

$$\sigma_2\,\mu(|s|) = o\big(m(|s|)\big). \tag{3.15}$$

From (3.12) we have

$$\sup_{|u|\leqslant\sigma_2\mu(|s|)} \tilde{r}\big(m(|s|)+u\big) \leqslant \sup_{|u|\leqslant\sigma_2\mu(|s|)} c\frac{h\big(m(|s|)+u\big)}{\big(m(|s|)+u\big)^2}.$$

Thus, by (3.15)

$$\sup_{|u|\leqslant \sigma_2\mu(|s|)} \tilde{r}(m(|s|)+u) \leqslant c\frac{|s|}{m^2(|s|)},$$

whence

$$\sup_{|u|\leqslant \sigma_2\mu(|s|)} \tilde{r}(m(|s|)+u)\sigma_2^3 \leqslant c\frac{|s|}{m^2(|s|)}\sigma_2^3.$$

In view of Lemma 3.4 there exists $v(|s|)\to\infty$ as $|s|\to\infty$ such that

$$\sup_{|u|\leqslant \sigma_2\mu(|s|)} \tilde{r}(m(|s|)+u)\sigma_2^3 v^3(|s|) = o(1).$$

Taking $l(|s|) = \min(\mu(|s|), v(|s|))$ we arrive at the statement for $h(t)\in\mathscr{S}\cup\mathscr{R}$.

Now let $h(t)\in\mathscr{F}$. Here $\sigma_+\asymp\sigma_1$. From Lemma 3.5 it follows that there exists $\mu(|s|)\to\infty$ as $|s|\to\infty$ and

$$\sup_{|u|\leqslant \sigma_1\mu(|s|)} h''(m(|s|)+u) \leqslant c|s|^{1+2\eta}(m(|s|))^{-2},$$

whence

$$\sup_{|u|\leqslant \sigma_1\mu(|s|)} h''(m(|s|)+u)\sigma_1^3 \leqslant c|s|^{-1/2+2\eta}(m(|s|))^{-1/2}.$$

Let $v(|s|)\to\infty$ as $|s|\to\infty$ and

$$\sup_{|u|\leqslant \sigma_1\mu(|s|)} h''(m(|s|)+u)\sigma_1^3 v^3(|s|) = o(1).$$

Thus, taking into account (3.13), we obtain

$$\sup_{|u|\leqslant \sigma_1\mu(|s|)} \tilde{r}(m(|s|)+u)\sigma_1^3 v^3(|s|) = o(1).$$

As before it remains to put $l(|s|) = \min(\mu(|s|), v(|s|))$. The corollary is proved.

COROLLARY 3.7. *Let us put $x(s) = m(|s|)e_s$. Then for any $h(t)\in\mathscr{H}$ there exists $l(|s|)\to\infty$ such that*

$$r(|x|) = r(m(|s|)) + \langle s, x-x(s)\rangle + \tfrac{1}{2}(x-x(s))^T V(x(s))(x-x(s)) + \delta(x, s), \tag{3.16}$$

where

$$\sup_{|x-x(s)|<\sigma_+ l(|s|)} |\delta(x, s)| \to 0 \quad \text{as } |s|\to\infty. \tag{3.17}$$

Proof. By Taylor's formula, $r(|x|)$ in a neighborhood of $x(s)$ admits the representation (3.16) where

$$\delta(x, s) = \tfrac{1}{6} \sum_{i,j,k=1}^{d} r_{ijk}\big(|x(s)+\Theta_{ijk}(x-x(s))|\big)\big(x_i-x_i(s)\big)\big(x_j-x_j(s)\big)\big(x_k-x_k(s)\big)$$

while Θ_{ijk} are diagonal matrices with non-zero entries lying in [0, 1]. Obviously,

$$\sup_{|x-x(s)|<\sigma_+ l(|s|)} |\delta(x, s)| \leqslant \sup_{|x-x(s)|<\sigma_+ l(|s|)} \tilde{r}\big(|x(s)+\Theta(x-x(s))|\big)|x-x(s)|^3$$

$$\leqslant \sup_{|x-x(s)|<\sigma_+ l(|s|)} \tilde{r}\big(m(|s|)+\theta|x-x(s)|\big)\sigma_+^3 l^3(|s|).$$

The statement follows immediately from (3.14). The corollary is proved.

4. A THEOREM OF THE ABEL TYPE

The following function plays a very important role:

$$H(x, s) = \langle s, x\rangle - r(|x|). \tag{4.1}$$

We regard $H(x, s)$ as a function of x taken s as a parameter. From (3.2) it follows that $H(x, s)$ as a function of x is concave for any $s \in \boldsymbol{R}^d$. Since in (3.1) the function $h(t)$ is strictly increasing, $H(x, s)$ attains its maximum at $x = x(s) = m(|s|)e_s$, where, we remind, $m(t)$ is inverse to $h(t)$. Corollary 3.7 implies that if $h(t) \in \mathscr{H}$, then there exists $l(|s|) \to \infty$ such that

$$H(x, s) = H\big(x(s), s\big) - \tfrac{1}{2}\big(x-x(s)\big)^T \bar{H}\big(x-x(s)\big) + \delta(x, s), \tag{4.2}$$

where $\bar{H} = V\big(x(s)\big)$ and $\delta(x, s)$ satisfies (3.17). The above formula plays a key role in the proof of the Abel theorem by means of the Laplace method. For the sake of brevity we will write l instead of $l(|s|)$.

Consider the ellipsoid region

$$A_s = \big\{x\colon\ H\big(x(s), s\big) - H\big(x(s)+x, s\big) \leqslant l^2/4\big\}. \tag{4.3}$$

The boundary of A_s is represented as

$$\partial A_s = \big\{x\colon\ x = r(e)e,\ e \in S^{d-1},\ H\big(x(s), s\big) - H\big(x(s)+x, s\big) = l^2/4\big\}.$$

Lemma 4.1. *Let σ_+ be as in Corollary 3.7. Then*

$$\sup_{e \in S^{d-1}} r(e) < l\sigma_+ \quad \textit{and} \quad r^2(e)e^T \bar{H} e \sim l^2/2.$$

Proof. For $r = l\sigma_+$ by (4.2) we have for all sufficiently large $|s|$

$$H\big(x(s), s\big) - H\big(x(s)+l\sigma_+ e, s\big) = \tfrac{1}{2} l^2 \sigma_+^2 e^T \bar{H} e + \theta\omega(|s|).$$

Obviously, $\lambda_- \leqslant e^T \bar{H} e \leqslant \lambda_+$, where $\lambda_- = \min(\lambda_1, \lambda_2)$, $\lambda_+ = \max(\lambda_1, \lambda_2)$. In particular, $\sigma_+^2 e^T \bar{H} e \geqslant \lambda_- \sigma_+^2 = 1$. Hence, for all sufficiently large $|s|$,

$$H(x(s), s) - H(x(s) + l\sigma_+ e, s) \geqslant \tfrac{1}{3} l^2.$$

It implies that $\sup_{e \in S^{d-1}} r(e) < l\sigma_+$. Further

$$H(x(s), s) - H(x(s) + r(e) e, s) = \tfrac{1}{2} r(e)^2 e^T \bar{H} e + \theta\omega(|s|),$$

and therefore $r^2(e) e^T \bar{H} e \sim l^2/2$. The lemma is proved.

Consider the ellipsoid

$$A_s' = \{x\colon\ x^T \bar{H} x \leqslant \tfrac{1}{3} l^2\}. \tag{4.4}$$

It is evident that for all sufficiently large $|s|$ we have $A_s' \subset A_s$. Let an orthogonal matrix C be such that

$$C^T \bar{H} C = D = \operatorname{diag}(\lambda_1, \ldots, \lambda_1, \lambda_2). \tag{4.5}$$

For any $x = (x_1, \ldots, x_{d-1}, x_d)$ we put $\bar{x} = (x_1, \ldots, x_{d-1})$.

The following theorem establishes the asymptotic of the moment generating function $f(s)$ as $|s| \to \infty$.

THEOREM 4.2. *Let the underlying density $p(x)$ be uniformly bounded and satisfy* (1.4) *where $h(t) = r'(t)$ belongs to $\mathscr{H}$. Then as $|s| \to \infty$*

$$f(s) = \int_{R^d} e^{\langle s,x\rangle} p(x)\,dx = (2\pi)^{d/2} \sigma_1^{d-1} \sigma_2 e^{H(x(s),s)} (1 + \theta\omega(|s|)),$$

where $H(x, s)$ is defined as in (4.1) *while $x(s)$ is the point of maximum of $H(x, s)$ as a function of x.*

Proof. Let Y be so large that for $|x| > Y$

$$p(x) < 2e^{-r(|x|)}. \tag{4.6}$$

Define $f(s)$ as follows:

$$f(s) = \int_{|x| \leqslant Y} + \int_{|x| > Y} = f_1(s) + f_2(s). \tag{4.7}$$

Obviously, as $|s| \to \infty$

$$f_1(s) = O(e^{Y|s|}). \tag{4.8}$$

Let A_s be determined by (4.3). Denote by $f_{21}(s)$ the corresponding part of $f_2(s)$, i.e.

$$f_{21}(s) = \int_{A_s + x(s)} e^{\langle s,x\rangle} p(x)\,dx. \tag{4.9}$$

It is evident that $\min(|x|: x \in A_s + x(s)) \to \infty$ as $|s| \to \infty$. By (1.4) and (4.1),

$$f_{21}(s) = \int_{A_s + x(s)} e^{H(x,s)}\, dx\,(1+\theta\omega(|s|)).$$

From (4.2) it follows that

$$\int_{A_s + x(s)} e^{H(x,s)}\, dx = e^{H(x(s),s)} \int_{A_s} \exp(-\tfrac{1}{2} x^T \bar{H} x)\, dx\,(1+\theta\omega(|s|)).$$

Further,

$$\int_{A_s} \exp(-\tfrac{1}{2} x^T \bar{H} x)\, dx = \int_{\mathbf{R}^d} \exp(-\tfrac{1}{2} x^T \bar{H} x)\, dx - \int_{x \notin A_s} \exp(-\tfrac{1}{2} x^T \bar{H} x)\, dx.$$

It is well known that

$$\int_{\mathbf{R}^d} \exp(-\tfrac{1}{2} x^T \bar{H} x)\, dx = (2\pi)^{d/2} \sigma_1^{d-1} \sigma_2.$$

In view of (4.4) we have

$$\int_{x \notin A_s} \exp(-\tfrac{1}{2} x^T \bar{H} x)\, dx \leqslant \int_{x \notin A_s'} \exp(-\tfrac{1}{2} x^T \bar{H} x)\, dx.$$

By (4.5) we obtain

$$\int_{x \notin A_s'} \exp(-\tfrac{1}{2} x^T \bar{H} x)\, dx \leqslant \int_{\lambda_1 |\bar{x}|^2 + \lambda_2 x_d^2 \geqslant (1/3) l^2} \exp\left(-\tfrac{1}{2}(\lambda_1 |\bar{x}|^2 + \lambda_2 x_d^2)\right) dx$$

$$= \sigma_1^{d-1} \sigma_2 \int_{|x| \geqslant cl} \exp(-\tfrac{1}{2}|x|^2)\, dx = o(\sigma_1^{d-1} \sigma_2).$$

Thus, as $|s| \to \infty$

$$f_{21}(s) = (2\pi)^{d/2} \sigma_1^{d-1} \sigma_2\, e^{H(x(s),s)} (1+\theta\omega(|s|)). \tag{4.10}$$

It remains to estimate

$$f_{22}(s) = \int_{|x| > Y,\, x - x(s) \notin A_s} e^{\langle s,x \rangle} p(x)\, dx.$$

For all sufficiently large $|s|$ we have

$$\tfrac{1}{4} r^2(e)\, e^T \bar{H} e \leqslant H(x(s), s) - H(x(s) + r(e) e, s).$$

Note that $H(x(s)+re, s)$, $r > 0$, as a function of r, is concave. Taking into account (4.2), for $r \geqslant r(e)$ and for all sufficiently large $|s|$ we obtain

$$H(x(s)+re, s) \leqslant H(x(s), s) - \frac{H(x(s), s) - H(x(s)+r(e)e, s)}{r(e)}\, r \tag{4.11}$$

$$\leqslant H(x(s), s) - \tfrac{1}{4} \cdot r \cdot r(e) \cdot e^T \bar{H} e \leqslant H(x(s), s) - \tfrac{1}{8} l \sqrt{x^T \bar{H} x}.$$

Therefore,

$$f_{22}(s) \leqslant 2 \int_{x \notin A_s} e^{H(x(s)+x,s)}\,dx \leqslant 2e^{H(x(s),s)} \int_{|x| \geqslant r(e_x)} \exp\left(-\tfrac{1}{8} l \sqrt{x^T \bar{H} x}\right) dx.$$

As before we get

$$\int_{|x| \geqslant r(e_x)} \exp\left(-\tfrac{1}{8} l \sqrt{x^T \bar{H} x}\right) dx \leqslant \int_{x \notin A_s'} \exp\left(-\tfrac{1}{8} l \sqrt{x^T \bar{H} x}\right) dx = o(\sigma_1^{d-1} \sigma_2).$$

Thus

$$f_{22}(s) = o(\sigma_1^{d-1} \sigma_2 e^{H(x(s),s)}). \tag{4.12}$$

It remains to combine (4.7)–(4.12). The theorem is proved.

Since

$$tm(t) - r\big(m(t)\big) = \int_1^t m(u)\,du + m(1) - r(1),$$

one can restate the Abel theorem as follows:

$$f(s) = (2\pi)^{d/2} \sigma_1^{d-1}\big(m(|s|)\big) \sigma_2\big(m(|s|)\big) \times \exp\Big(m(1) - r(1) + \int_1^{|s|} m(u)\,du\Big)\big(1 + \theta\omega(|s|)\big). \tag{4.13}$$

This formula is sometimes very convenient.

5. ASYMPTOTIC PROPERTIES OF THE CONJUGATE DISTRIBUTIONS

Consider the family of the so-called conjugate densities defined as

$$p_s(x) = \frac{e^{\langle s,x \rangle} p(x)}{f(s)}, \quad s \in S. \tag{5.1}$$

It turns out that the conjugate distribution is asymptotically normal as $|s| \to \infty$. Moreover, the convergence to the limiting normal distribution is very strong. Let us put

$$\hat{p}_s(x) = (\det \bar{H})^{-1/2} p_s\big(\bar{H}^{-1/2} x + x(s)\big). \tag{5.2}$$

The following statement is of independent interest.

LEMMA 5.1. *Under the conditions of Theorem* 4.2 *as* $|s| \to \infty$

$$\sup_{x \in \mathbf{R}^d} |\hat{p}_s(x) - \varphi_{(0,I)}(x)| = \omega(|s|),$$

where I *is the unit matrix.*

Proof. Let Y be chosen as in (4.6). In view of (1.4), (5.1) and (4.1) we have for $|\bar{H}^{-1/2}x+x(s)| > Y$

$$\hat{p}_s(x) = (\det \bar{H})^{-1/2}\big(f(s)\big)^{-1}\exp\big(H(\bar{H}^{-1/2}x+x(s),\, s)\big)\big(1+\theta\omega(|s|)\big). \tag{5.3}$$

Applying (4.2) and Theorem 4.2 yield

$$\hat{p}_s(x) = \varphi_{(0,I)}(x)\big(1+\theta\omega(|s|)\big) \tag{5.4}$$

at least for $|x| \leqslant 3^{-1/2}l$.

In particular, for any $Z > 0$ and all sufficiently large $|s|$ we have

$$\hat{p}_s(x) \leqslant c\exp(-\tfrac{1}{4}|x|^2) \tag{5.5}$$

provided $Z \leqslant |x| \leqslant 3^{-1/2}l$.

From (5.3), (4.11) and Theorem 4.2 we obtain

$$\hat{p}_s(x) \leqslant c\exp(-\tfrac{1}{8}l|x|) \tag{5.6}$$

provided $|x| \geqslant 3^{-1/2}l$, $|\bar{H}^{-1/2}x+x(s)| > Y$.

Now let $|\bar{H}^{-1/2}x+x(s)| \leqslant Y$. Since in (5.1) the underlying density is uniformly bounded, we have taking into account (5.2)

$$\hat{p}_s(x) \leqslant c(\det \bar{H})^{-1/2}e^{|s|Y}\big(f(s)\big)^{-1}. \tag{5.7}$$

By (5.4), for any fixed Z we have

$$\sup_{|x|\leqslant Z}|\hat{p}_s(x)-\varphi_{(0,I)}(x)| = \omega(|s|).$$

Utilizing (5.5)–(5.7), (4.13) and (3.8) yields

$$\sup_{|x|\geqslant Z}|\hat{p}_s(x)-\varphi_{(0,I)}(x)| \leqslant \sup_{|x|\geqslant Z}\varphi_{(0,I)}(x)+\sup_{\substack{|x|\geqslant Z,\\ |\bar{H}^{-1/2}x+x(s)|\geqslant Y}}\hat{p}_s(x)+\sup_{|\bar{H}^{-1/2}x+x(s)|\leqslant Y}\hat{p}_s(x)$$
$$= \omega(Z)+\theta\omega(Z)+\omega(|s|).$$

Since Z is arbitrary, the lemma follows.

The normalizing used in (5.2) does not quite fit our purpose. Actually, we should deal with

$$\bar{p}_s(x) = (\det \bar{M})^{1/2}p_s\big(\bar{M}^{1/2}x+M(s)\big), \tag{5.8}$$

where we write for brevity $\bar{M}$ instead of $\bar{M}(s)$. The density so normalized has the zero expectation and the unit covariance matrix. We expect that $\bar{p}_s(x)$ is also asymptotically normal. In order to justify this we should prove that $M(s)$ and $x(s)$ are asymptotically close as well as $\bar{M}$ and $\bar{H}^{-1}$. It is established in the following two statements.

LEMMA 5.2. *Let the unit vectors $e^{(0)}$ and e' be such that*

$$e^{(0)} = (0, \ldots, 0, 1), \quad e' = (e_1, \ldots, e_{d-1}, 0) = (\bar{e}, 0).$$

If an orthogonal matrix C is as in (4.5), *then*

$$|\langle e, C(M(s)-x(s))\rangle| = \theta\omega(|s|)(|\langle e, e'\rangle|\sigma_1+|\langle e, e^{(0)}\rangle|\sigma_2).$$

Proof. It is easily seen that

$$\langle e, C(M(s)-x(s))\rangle = \int_{R^d} \langle e, C(x-x(s))\rangle p_s(x)\,dx.$$

Split R^d as before, i.e.

(5.9) $R^d = (x\colon |x| \leqslant Y)\cup(x\colon x-x(s)\in A_s)\cup \sup(x\colon |x| > Y,\ x-x(s)\notin A_s)$,

where Y and A_s are defined as in (4.6) and (4.3), respectively. Denote by I_1, I_2 and I_3 the corresponding parts of the last integral, i.e.

$$\int_{R^d} \langle e, C(x-x(s))\rangle p_s(x)\,dx = I_1+I_2+I_3. \tag{5.10}$$

By (4.13) and (3.8) we have

$$I_1 \leqslant c\lambda_1^{(d-1)/2}\lambda_2^{1/2}m(|s|)\exp\Big(|s|Y-\int_1^{|s|} m(u)\,du\Big) = o(\lambda_+^{-1/2}). \tag{5.11}$$

By virtue of Theorem 4.2 and (4.2) we obtain

$$I_2 = (2\pi)^{-d/2}\lambda_1^{(d-1)/2}\lambda_2^{1/2}\int_{x\in A_s} \langle e, Cx\rangle \exp(-\tfrac{1}{2}x^T\bar{H}x)\,dx\,(1+o(1))$$

$$= -(2\pi)^{-d/2}\lambda_1^{(d-1)/2}\lambda_2^{1/2}\int_{x\notin A_s} \langle e, Cx\rangle \exp(-\tfrac{1}{2}x^T\bar{H}x)\,dx\,(1+o(1)).$$

In view of (4.5) and (4.4) we get

$$\int_{x\notin A_s} \langle e, Cx\rangle \exp(-\tfrac{1}{2}x^T\bar{H}x)\,dx$$

$$\leqslant \int_{\lambda_1|\bar{x}|^2+\lambda_2x_d^2\geqslant(1/3)l^2} |\langle e, x\rangle|\exp\big(-\tfrac{1}{2}(\lambda_1|\bar{x}|^2+\lambda_2x_d^2)\big)\,dx.$$

Put for the sake of brevity $\alpha = \langle e, e'\rangle$, $\beta = \langle e, e^{(0)}\rangle$. Then

$$\int_{\lambda_1|\bar{x}|^2+\lambda_2x_d^2\geqslant(1/3)l^2} |\langle e, x\rangle|\exp\big(-\tfrac{1}{2}(\lambda_1|\bar{x}|^2+\lambda_2x_d^2)\big)\,dx$$

$$= \lambda_1^{-(d-1)/2}\lambda_2^{-1/2}\int_{|x|\geqslant cl} |\alpha\sigma_1\langle \bar{e}, \bar{x}\rangle+\beta\sigma_2x_d|\exp(-\tfrac{1}{2}|x|^2)\,dx$$

$$\leqslant \lambda_1^{-(d-1)/2}\lambda_2^{-1/2}(|\alpha|\sigma_1+|\beta|\sigma_2)\int_{|x|\geqslant cl} |x|\exp(-\tfrac{1}{2}|x|^2)\,dx.$$

Thus,

(5.12) $$I_2 \leqslant \omega(|s|)(|\langle e, e'\rangle|\,\sigma_1 + |\langle e, e^{(0)}\rangle|\,\sigma_2).$$

As to I_3 in view of (4.11) we have

$$I_3 \leqslant c\lambda_1^{(d-1)/2}\lambda_2^{1/2} \int_{x\notin A_s'} |\langle e, x\rangle| \exp\big(-(l/8)\sqrt{x^T \bar{H} x}\big)\,dx.$$

Repeating the just utilized argument yields

$$I_3 \leqslant c(|\alpha|\,\sigma_1 + |\beta|\,\sigma_2) \int_{|x|\geqslant cl} |x|\, e^{-(l/8)|x|}\,dx.$$

Thus,

(5.13) $$I_3 \leqslant \omega(|s|)(|\langle e, e'\rangle|\,\sigma_1 + |\langle e, e^{(0)}\rangle|\,\sigma_2).$$

The desired statement follows from (5.10)–(5.13). The lemma is proved.

LEMMA 5.3. *Under the conditions of Theorem* 4.2 *as* $|s| \to \infty$

$$\sup_{e\in S^{d-1}} \left|\frac{e^T \bar{M} e}{e^T \bar{H}^{-1} e} - 1\right| = \omega(|s|).$$

Proof. It is easily seen that for any $e \in S^{d-1}$ we have

$$e^T \bar{M} e = Q - \langle e, M(s) - x(s)\rangle^2,$$

where Q is the quadratic form determined by the matrix $\|q_{ij}\|_{i,j=1,\dots,d}$ with

$$q_{ij} = \int_{\mathbf{R}^d} (x_i - x_i(s))(x_j - x_j(s))\, p_s(x)\,dx.$$

Split the right-hand side integral in accordance with (5.9):

(5.14) $$q_{ij} = q_{ij}' + q_{ij}'' + q_{ij}'''.$$

Then, by (4.2),

$$\begin{aligned} Q'' &= \sum_{i,j=1}^{d} q_{ij}'' e_i e_j \\ &= (2\pi)^{-d/2} \lambda_1^{(d-1)/2} \lambda_2^{1/2} \sum_{i,j=1}^{d} e_i e_j \int_{x\in A_s} x_i x_j \exp(-\tfrac{1}{2} x^T \bar{H} x)\,dx\,\big(1+\omega(|s|)\big) \\ &= Q^*\big(1+\omega(|s|)\big). \end{aligned}$$

Let C and D be as in (4.5). Let us put $Ce = \varepsilon = (\varepsilon_1, \dots, \varepsilon_d)$. Then

$$Q^* = \lambda_1^{(d-1)/2} \lambda_2^{1/2} \sum_{i,j=1}^{d} \varepsilon_i \varepsilon_j J_{ij},$$

where

$$J_{ij} = (2\pi)^{-d/2} \int_{Cx\in A_s} x_i x_j \exp\left(-\tfrac{1}{2}(\lambda_1 |\bar{x}|^2 + \lambda_2 x_d^2)\right) dx.$$

For $i \neq j$

$$J_{ij} \leqslant c \int_{\lambda_1|\bar{x}|^2 + \lambda_2 x_d^2 \geqslant cl^2} |x_i|\,|x_j| \exp\left(-\tfrac{1}{2}(\lambda_1 |\bar{x}|^2 + \lambda_2 x_d^2)\right) dx$$

or

$$J_{ij} \leqslant \begin{cases} c\lambda_1^{-(d+1)/2} \lambda_2^{-1/2}\, \omega(|s|) & \text{if } i \neq d,\ j \neq d, \\ c\lambda_1^{-d/2} \lambda_2^{-1}\, \omega(|s|) & \text{if } i \neq d,\ j = d. \end{cases}$$

For $i = j$

$$J_{ii} = (2\pi)^{-d/2} \int_{R^d} x_i^2 \exp\left(-\tfrac{1}{2}(\lambda_1 |\bar{x}|^2 + \lambda_2 x_d^2)\right) dx$$

$$+ c\theta \int_{\lambda_1|\bar{x}|^2 + \lambda_2 x_d^2 \geqslant cl^2} x_i^2 \exp\left(-\tfrac{1}{2}(\lambda_1 |\bar{x}|^2 + \lambda_2 x_d^2)\right) dx$$

or

$$J_{ii} = \begin{cases} \lambda_1^{-(d+1)/2} \lambda_2^{-1/2} \left(1 + \omega(|s|)\right) & \text{if } i \neq d, \\ \lambda_1^{-(d-1)/2} \lambda_2^{-3/2} \left(1 + \omega(|s|)\right) & \text{if } i = d. \end{cases}$$

So, we may write down

$$Q^* = |D^{-1/2}\varepsilon|^2 + \varepsilon^T D^{-1/2}\Theta D^{-1/2}\varepsilon,$$

where $\Theta = \|\theta_{ij}\|_{i,j=1,\ldots,d}$ with $\theta_{ij} = \theta\omega(|s|)$. We remind that, by our convention, the concrete form of θ and $\omega(t)$ plays no role. Thus,

(5.15) $$Q'' = |D^{-1/2}\varepsilon|^2 \left(1 + \omega(|s|)\right).$$

In view of (3.8) we have

$$q'_{ij} \leqslant c\lambda_1^{(d-1)/2} \lambda_2^{1/2} m^2(|s|) \exp\left(|s|\, Y - \int_1^{|s|} m(u)\, du\right) = o(\lambda_+^{-1}),$$

whence

(5.16) $$Q' = \sum_{i,j=1}^{d} q'_{ij} e_i e_j = o(\lambda_+^{-1}) = o(|D^{-1/2}\varepsilon|^2).$$

Further (cf. the estimate of I_3 in the proof of Lemma 5.2),

$$Q''' = \sum_{i,j=1}^{d} q'''_{ij} e_i e_j \leqslant c\lambda_1^{(d-1)/2} \lambda_2^{1/2} \sum_{i,j=1}^{d} \varepsilon_i \varepsilon_j J'_{ij},$$

where

$$J'_{ij} = \int_{\lambda_1|\bar{x}|^2+\lambda_2 x_d^2 \geqslant cl^2} |x_i|\,|x_j| \exp\left(-(l/8)\sqrt{\lambda_1|\bar{x}|^2+\lambda_2 x_d^2}\right) dx.$$

It is easily seen that

$$Q''' \leqslant \omega(|s|)\,|D^{-1/2}\varepsilon|^2. \tag{5.17}$$

From (5.14)–(5.17) it follows that $Q = |D^{-1/2}\varepsilon|^2(1+\omega(|s|))$. It remains to apply Lemma 5.2. The lemma is proved.

Now we can prove the basic statement of this section.

THEOREM 5.4. *Under the conditions of Theorem* 4.2 *as* $|s| \to \infty$

$$\sup_{x\in\boldsymbol{R}^d} |\bar{p}_s(x) - \varphi_{(0,I)}(x)| = \omega(|s|) \quad \textit{and} \quad \int_{\boldsymbol{R}^d} |\bar{p}_s(x) - \varphi_{(0,I)}(x)|\,dx = \omega(|s|).$$

Proof. We confine ourselves to the proof of the second statement. The first one is proved much easier. From Lemma 5.3 it follows that $\det\bar{M} = \det\bar{H}^{-1}(1+\omega(|s|))$. Determine y by the relation $\bar{M}^{1/2}x + M(s) = \bar{H}^{-1/2}y + x(s)$. By (5.4) we have

$$\bar{p}_s(x) = \varphi_{(0,I)}(y)\left(1+\theta\omega(|s|)\right) \tag{5.18}$$

provided $|y| \leqslant 3^{-1/2} l$. Since $y = \bar{H}^{1/2}\bar{M}^{1/2}x + \bar{H}^{1/2}(M(s)-x(s))$, we have

$$|y|^2 = x^T\bar{M}^{1/2}\bar{H}\bar{M}^{1/2}x + (M(s)-x(s))^T\bar{H}(M(s)-x(s)) + 2x^T\bar{M}^{1/2}\bar{H}(M(s)-x(s))$$
$$= x^T\bar{M}^{1/2}\bar{H}\bar{M}^{1/2}x + R^2 + 2\theta R(x^T\bar{M}^{1/2}\bar{H}\bar{M}^{1/2}x)^{1/2}.$$

From Lemma 5.2 and (4.5) we obtain

$$R^2 = (M(s)-x(s))^T\bar{H}(M(s)-x(s)) = \left(C(M(s)-x(s))\right)^T DC(M(s)-x(s))$$
$$= \lambda_1\langle e', C(M(s)-x(s))\rangle^2 + \lambda_2\langle e^{(0)}, C(M(s)-x(s))\rangle^2$$
$$= \omega(|s|)(\lambda_1\sigma_1^2 + \lambda_2\sigma_2^2) = \omega(|s|).$$

By Lemma 5.3 for $|x| \leqslant Z$ we have $x^T\bar{M}^{1/2}\bar{H}\bar{M}^{1/2}x = |x|^2 + \theta\omega(|s|)$. So, for $|x| \leqslant Z$ we have $|y|^2 = |x|^2 + \theta\omega(|s|)$, and therefore

$$\sup_{|x|\leqslant Z} |\bar{p}_s(x) - \varphi_{(0,I)}(x)| = \omega(|s|).$$

If Z is sufficiently large, then for $|x| \geqslant Z$ we have $|y|^2 \geqslant \frac{1}{4}|x|^2 - 2R|x| \geqslant \frac{1}{8}|x|^2$. Let Y be as in (4.6). Taking into account (5.5)–(5.7) we obtain

$$\bar{p}_s(x) \leqslant \begin{cases} c\exp(-c|x|^2) & \text{if } Z \leqslant |x| \leqslant cl, \\ c\exp(-cl|x|) & \text{if } |x| \geqslant cl,\ |\bar{M}^{1/2}x + M(s)| \geqslant Y, \\ c(\det\bar{M})^{1/2} e^{|s|Y}(f(s))^{-1} & \text{if } |\bar{M}^{1/2}x + M(s)| < Y. \end{cases} \tag{5.19}$$

It is easily seen that

(5.20) $$\int_{R^d} |\bar{p}_s(x)-\varphi_{(0,I)}(x)|\,dx$$

$$\leqslant \int_{|x|\leqslant Z} |\bar{p}_s(x)-\varphi_{(0,I)}(x)|\,dx+\int_{|x|\geqslant Z}\varphi_{(0,I)}(x)\,dx+\int_{|x|\geqslant Z}\bar{p}_s(x)\,dx = I_1+I_2+I_3.$$

We have already established that

(5.21) $$I_1=\omega(|s|).$$

Obviously, for any fixed Z

(5.22) $$I_2=\omega(Z).$$

It remains to estimate I_3. From (5.19) we easily obtain

$$I_{31}=\int_{Z\leqslant|x|<cl}\bar{p}_s(x)\,dx\leqslant c\int_{|x|\geqslant Z}\exp(-c|x|^2)\,dx.$$

So, we have

(5.23) $$I_{31}=\omega(Z).$$

Further,

$$I_{32}=\int_{|x|\geqslant cl,|\bar{M}^{1/2}x+M(s)|\geqslant Y}\leqslant c\int_{|x|\geqslant cl}\exp(-cl|x|)\,dx.$$

Thus,

(5.24) $$I_{32}=\omega(|s|).$$

Finally, by (5.1),

$$I_{33}=\int_{|\bar{M}^{1/2}x+M(s)|<Y}\bar{p}_s(x)\,dx\leqslant cX^d(\det\bar{M})^{1/2}\exp(|s|\,Y)\big(f(s)\big)^{-1}.$$

Taking into account (4.13) and Lemma 5.3 we obtain

$$I_{33}\leqslant cX^d\exp\Big(|s|\,Y-\int_1^{|s|}m(u)\,du\Big).$$

In view of (3.8) we get

(5.25) $$I_{33}=\omega(|s|).$$

It remains to combine (5.20)–(5.25). The theorem is proved.

6. PROOF OF THEOREM 2.4

By means of the Cramér transformation we obtain

$$p_n(nx) = f^n(s)\, e^{-n\langle s,x\rangle}\, p_{n,s}(nx),$$

where $p_{n,s}$ is the n-convolution of p_s defined in (5.1). Determine $s = s(x)$ by the equation $x = M(s)$. Then

$$p_n(nx) = n^{-d/2}\big(\det \bar{M}(s(x))\big)^{-1/2} \varrho^n(x)\, \bar{p}_{n,s(x)}(\mathbf{0}), \tag{6.1}$$

where $\bar{p}_{n,s}$ is the n-convolution of $\bar{p}_s$ defined in (5.8).

LEMMA 6.1. *Under the conditions of Theorem* 2.4 *as* $n \to \infty$

$$\sup_{s\in \mathbf{R}^d} \sup_{x\in \mathbf{R}^d} |\bar{p}_{n,s}(x) - \varphi_{(0,\mathbf{I})}(x)| = o(1).$$

Proof. Assume that the characteristic function ψ_s corresponds to $\bar{p}_s$, i.e.

$$\psi_s(t) = \int_{\mathbf{R}^d} e^{i\langle t,x\rangle}\, \bar{p}_s(x)\, dx.$$

By the Parseval identity we obtain

$$(2\pi)^{-d} \int_{\mathbf{R}^d} |\psi_s(t)|^2\, dt = \int_{\mathbf{R}^d} \bar{p}_s^2(x)\, dx \leqslant \sup_{x\in \mathbf{R}^d} \bar{p}_s(x).$$

Therefore, taking into account Theorem 5.4 we have

$$\sup_{s\in \mathbf{R}^d} \int_{\mathbf{R}^d} |\psi_s(t)|^2\, dt < \infty \tag{6.2}$$

and

$$\sup_{s\in \mathbf{R}^d} \sup_{|t|\geqslant \delta} |\psi_s(t)| < 1, \qquad \delta > 0. \tag{6.3}$$

The lemma follows immediately from (6.2), (6.3) and Theorem 5.4 (see e.g. [4], Section 4.19).

From Lemma 6.1 it follows that as $n \to \infty$

$$\bar{p}_{n,s}(\mathbf{0}) = (2\pi)^{-d/2}\big(1 + o(1)\big) \tag{6.4}$$

uniformly in $s \in \mathbf{R}^d$. In view of (6.1) and (6.4) the theorem follows.

7. CONCLUDING REMARKS

1. It is readily seen that the crucial role in the proof of Theorem 2.4 is played by the asymptotic normality of the conjugate densities, which is established in Theorem 5.4. The latter theorem may be regarded as a contribution to

the theory of the so-called *natural exponential families*, at least to that part of the theory which analyzes the limit distributions arising as $s \to \partial S$ (see e.g. [3]).

2. It should be emphasized that the case $h(t) \in \mathscr{R}$ is rather simple. Here the eigenvalues λ_1 and λ_2 of the matrix $\bar{H}$ are of the same order. This simplifies the asymptotic analysis of the conjugate densities. It is the case $h(t) \in \mathscr{S} \cup \mathscr{F}$ that requires much more efforts.

3. It is easily seen that the method that allowed us to prove Theorem 2.4 works efficiently in more general situations as well. For example, instead of the basic condition (1.4) we could assume that, say,

$$p(x) = \exp\left(-r(|x|)\, a(e_x)\right)\left(1 + \omega(|x|)\right),$$

where $a(e)$ is a continuous strictly positive function defined on the unit sphere S^{d-1}. This case requires no principal changes in the proof, though it leads to much more cumbersome formulae adding less to the point.

4. Within the theory probability context the established results are of use in the following cases:

- the integral large deviation theorems held uniformly in very rich families of sets;
- testing the quality of the simpler upper bounds for large deviation probabilities;
- testing the convergence of the integral functionals of $\zeta^{(n)}$;
- asymptotic analysis of finite convolutions of densities satisfying (1.4).

The conjugate densities, regarded as natural exponential families, form an object that is of great interest from the point of view of statistics. Here, the basic points of interest are:

- local asymptotic normality of the natural exponential families uniformly over an open parametric set;
- large deviations of the maximum likelihood estimator for the parameters of the natural exponential family;
- extension of the class of the loss functions admitting the convergence of the maximum likelihood estimator risk etc.

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