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PRINCIPAL EIGENVALUES FOR TIME CHANGED PROCESSES OF ONE-DIMENSIONAL α-STABLE PROCESSES

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Abstract. In this paper, we calculate the principal eigenvalues for time changed processes of Brownian motions and symmetric α-stable processes in one dimension.

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1. INTRODUCTION

Let $M^\alpha = (X_t^\alpha, P_x)$, $1 < \alpha \leqslant 2$, be a symmetric α-stable process on $\boldsymbol{R}$ and denote its Dirichlet form by $(\mathscr{E}^\alpha, \mathscr{F}^\alpha)$. Let D be an open set and M^D the absorbing α-stable process on D. Let μ be a measure in the Kato class and A_t^μ the positive continuous additive functional (PCAF) in the Revuz correspondence to μ. We now define

$$\lambda(\mu; D) = \inf\{\mathscr{E}^\alpha(f, f) \mid f \in C_0^\infty(D), \int_D f^2\, d\mu = 1\}.$$

Then $\lambda(\mu; D)$ is the principal eigenvalue for the time changed process of M^D by A_t^μ. It is difficult in general to obtain principal eigenvalues for symmetric α-stable processes because of the non-locality. We do not know the principal eigenvalue even for the absorbing process on an interval; a lower bound estimate was obtained in [3] (see also [2] and [10]).

A purpose of this paper is to calculate $\lambda(\mu; D)$ for special pairs of μ and D. For example, let δ_a be the Dirac measure at a. We can then calculate $\lambda(\delta_a + \delta_{-a}; \boldsymbol{R}\setminus\{0\})$, $a \neq 0$, by using the Green function of the absorbing process on $\boldsymbol{R}\setminus\{0\}$:

$$\lambda(\delta_a + \delta_{-a}; \boldsymbol{R}\setminus\{0\}) = -\frac{\Gamma(\alpha)\cos(\pi\alpha/2)}{(4 - 2^{\alpha-1})|a|^{\alpha-1}}$$

(Example 3.3). We also calculate principal eigenvalues for time changed processes of killed Brownian motions in one dimension.

Our motivation lies in the proof of the gaugeability: a measure μ is said to be *gaugeable* on D if

$$\sup_{x\in D} E_x[\exp(A^{\mu}_{\tau_D})] < \infty,$$

where τ_D is the exit time from D. It was shown in [5], [11] and [13] that for a Kato measure μ with compact support, μ is gaugeable on D if and only if $\lambda(\mu; D) > 1$. Hence, by the calculation of $\lambda(\delta_a+\delta_{-a}; \boldsymbol{R}\backslash\{0\})$, we can give a necessary and sufficient condition for $\delta_a+\delta_{-a}$ being gaugeable in terms of the index α and the point a.

2. PRELIMINARIES

Let $M^\alpha = (X^\alpha_t, P_x)$, $1 < \alpha \leqslant 2$, be the symmetric α-stable process on $\boldsymbol{R}$. Denote its Dirichlet form by $(\mathscr{E}^\alpha, \mathscr{F}^\alpha)$. In case of $\alpha = 2$, M^2 is the Brownian motion and $(\mathscr{E}^2, \mathscr{F}^2) = (\boldsymbol{D}/2, H^1(\boldsymbol{R}))$, where $H^1(\boldsymbol{R})$ is the Sobolev space of order one and

$$\boldsymbol{D}(f, f) = \int_{\boldsymbol{R}} \left(\frac{df}{dx}\right)^2 dx, \qquad f \in H^1(\boldsymbol{R}).$$

If $1 < \alpha < 2$, then M^α is a pure jump process and its Dirichlet form $(\mathscr{E}^\alpha, \mathscr{F}^\alpha)$ is as follows:

$$\mathscr{E}^\alpha(f, f) = \frac{1}{2}\mathscr{A}(\alpha) \iint_{\boldsymbol{R}\times\boldsymbol{R}} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}} dydx,$$

$$\mathscr{F}^\alpha = \left\{f \in L^2(\boldsymbol{R}) \mid \iint_{\boldsymbol{R}\times\boldsymbol{R}} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}} dydx < \infty\right\},$$

where

$$\mathscr{A}(\alpha) = \frac{\alpha 2^{\alpha-1}\Gamma((1+\alpha)/2)}{\pi^{1/2}\Gamma(1-\alpha/2)}, \qquad \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

Let ν be a smooth measure and A^ν_t the PCAF in the Revuz correspondence to ν ([6], Theorem 5.1.4). Let $M^\nu = (X^\nu_t, P^\nu_x)$ be the subprocess of M^α with respect to the multiplicative functional $\exp(-A^\nu_t)$ (see [6], Appendix A.2, for details):

$$E^\nu_x[f(X^\nu_t)] = E_x[\exp(-A^\nu_t) f(X^\alpha_t)].$$

Then the process M^ν generates the Dirichlet form $(\mathscr{E}^\nu, \mathscr{F}^\nu)$:

$$\mathscr{F}^\nu = \mathscr{F}^\alpha \cap L^2(\boldsymbol{R}; \nu), \qquad \mathscr{E}^\nu(f, f) = \mathscr{E}^\alpha(f, f) + \int_{\boldsymbol{R}} f^2 d\nu, \ f \in \mathscr{F}^\nu$$

([6], Theorem 6.1.1). Let $M^D = (X_t^D, P_x^D)$ be the *absorbing α-stable process* on D: denote by τ_D the exit time from D, $\tau_D = \inf\{t > 0 \mid X_t \notin D\}$. Let Δ be the cemetary point. We set

$$X_t^D = \begin{cases} X_t^\alpha, & 0 \leqslant t < \tau_D, \\ \Delta, & \tau_D \leqslant t, \end{cases}$$

and P_x^D satisfies

$$E_x^D[f(X_t^D)] = E_x[f(X_t^\alpha) \colon t < \tau_D].$$

Moreover, the Dirichlet form $(\mathscr{E}^D, \mathscr{F}^D)$ of M^D is the following:

$$\mathscr{F}^D = \{f \in \mathscr{F}^\alpha \mid f = 0 \text{ on } D^c\},$$

$$\mathscr{E}^D(f, f)$$

$$= \begin{cases} \frac{1}{2}\int_D \left(\frac{df}{dx}\right)^2 dx, & \alpha = 2, \\ \frac{1}{2}\mathscr{A}(\alpha) \iint_{D\times D} \frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}} dydx + \mathscr{A}(\alpha)\int_D f(x)^2 \int_{D^c} \frac{1}{|x-y|^{1+\alpha}} dydx, & 1 < \alpha < 2 \end{cases}$$

([6], Theorem 4.4.2, Example 4.4.1).

Now we review the notion of time changes. In general, let X be a locally compact separable metric space and m a positive Radon measure on X with full support. Let $M = (X_t, P_x, \zeta)$ be an m-symmetric transient Hunt process on X, where ζ is the lifetime of M, $\zeta = \inf\{t > 0 \mid X_t = \Delta\}$. We denote by $G(x, y)$ the Green function of M and by $G_\alpha(x, y)$ the α-resolvent density.

Definition 2.1. (i) A positive Radon measure μ on X is said to be in the *Kato class* $\mathscr{K}(G)$ if

$$\lim_{\alpha\to\infty} \sup_{x\in X} \int_X G_\alpha(x, y)\mu(dy) = 0.$$

(ii) A measure $\mu \in \mathscr{K}(G)$ is said to be in $\mathscr{K}_\infty(G)$ if for any $\varepsilon > 0$ there exists a compact set K and a constant $\delta > 0$ such that

$$\sup_{x\in X} \int_{K^c} G(x, y)\mu(dy) < \varepsilon,$$

and for all measurable sets $B \subset K$ with $\mu(B) < \delta$

$$\sup_{x\in X} \int_B G(x, y)\mu(dy) < \varepsilon.$$

Note that any finite measures in $\mathscr{K}(G)$ belong to $\mathscr{K}_\infty(G)$ (see [5]). Let $\mu \in \mathscr{K}(G)$. Then there exists a unique PCAF A_t^μ in the Revuz correspondence to μ (see [1] and [6]).

Let $\mu\in\mathscr{K}(G)$ and τ_t be the right continuous inverse of A_t^μ, $\tau_t=\inf\{s>0 \mid A_{s\wedge\zeta}^\mu>t\}$. Put $\check{X}_t=X_{\tau_t}$. Then $\check{M}=(\check{X}_t, P_x)$ is said to be the *time changed process* of M by A_t^μ. Denote by Y the topological support of μ and by $\tilde{Y}$ the quasi-support of μ. Then $\check{M}$ is a μ-symmetric Markov process on $\tilde{Y}$ and its lifetime is A_ζ^μ ([6], §6). Set

$$H_Y u(x)=E_x[u(X_{\sigma_Y}):\ \sigma_Y<\infty],$$

where σ_Y is the hitting time of Y, $\sigma_Y=\inf\{t>0 \mid X_t\in Y\}$. Let $(\mathscr{E},\mathscr{F})$ be the regular Dirichlet form of M. Then $\check{M}$ also generates the regular Dirichlet form $(\check{\mathscr{E}},\check{\mathscr{F}})$ on $L^2(Y;\mu)$ ([6], Theorem 6.2.1):

$$\check{\mathscr{F}}=\{\psi\in L^2(Y;\mu) \mid \psi=u\ \ \mu\text{-a.e. on } Y \text{ for some } u\in\mathscr{F}_e\},$$

$$\check{\mathscr{E}}(\psi,\psi)=\mathscr{E}(H_{\tilde{Y}}u, H_{\tilde{Y}}u),$$

where $(\mathscr{F}_e,\mathscr{E})$ is the extended Dirichlet space of $(\mathscr{F},\mathscr{E})$ ([6], p. 35). Moreover, $(\check{\mathscr{E}},\check{\mathscr{F}})$ satisfies

$$\check{\mathscr{E}}(u,u)=\inf\{\mathscr{E}(v,v) \mid v\in\mathscr{F},\ \tilde{v}=\tilde{u} \text{ q.e. on } Y\}, \tag{2.1}$$

where $\tilde{u}$ is a quasi-continuous version of u, and q.e. is the abbreviation for quasi-everywhere. The equation (2.1) is the so-called *Dirichlet principle.*

3. EXAMPLES

3.1. In case of $\alpha=2$. First we shall study the principal eigenvalues for time changed processes of killed Brownian motions in one dimension. Let μ be a Kato class measure with respect to M^ν. Define

$$\lambda(\mu;\nu)=\inf\left\{\frac{1}{2}D(f,f)+\int_R f^2\,d\nu \mid f\in\mathscr{F}^\nu,\ \int_R f^2\,d\mu=1\right\}.$$

Then the equation (2.1) implies that $\lambda(\mu;\nu)$ coincides with the principal eigenvalue for the time changed process of M^ν by A_t^μ.

EXAMPLE 3.1. Let $M^2=(B_t, P_x)$ be the one-dimensional Brownian motion. Set $\nu(dx)=\chi_{(a,b)}(x)\,dx$ for $a<b$. Then $A_t^{\alpha\nu}=\alpha\int_0^t\chi_{(a,b)}(B_s)\,ds$ for $\alpha>0$. Denote by $M^{\alpha\nu}=(B_t^{\alpha\nu}, P_x^{\alpha\nu})$ the killed Brownian motion with respect to $\exp(-A_t^{\alpha\nu})$. Then $M^{\alpha\nu}$ generates the Dirichlet form $(\mathscr{E}^{\alpha\nu}, H^1(R))$:

$$\mathscr{E}^{\alpha\nu}(f,f)=\frac{1}{2}D(f,f)+\alpha\int_a^b f^2\,dx,\qquad f\in H^1(R).$$

By definition,

$$\lambda(\beta\delta_z;\alpha\chi_{(a,b)})=\inf\{\mathscr{E}^{\alpha\nu}(f,f) \mid f\in H^1(R),\ \beta f^2(z)=1\}. \tag{3.1}$$

Let Cap be the 0-order capacity with respect to $M^{\alpha v}$. Since the right-hand side of (3.1) coincides with $\mathrm{Cap}(\{z\})/\beta$, its infimum is attained by

$$\frac{1}{\sqrt{\beta}} P_x^{\alpha v}(\sigma_z < \infty) = \frac{1}{\sqrt{\beta}} E_x\left[\exp\left(-\alpha \int_0^{\sigma_z} \chi_{(a,b)}(B_s)\,ds\right)\right].$$

Suppose first that $z < a$. Then we can see from [4], p. 167, 2.7.1, that

$$E_x\left[\exp\left(-\alpha \int_0^{\sigma_z} \chi_{(a,b)}(B_s)\,ds\right)\right]$$

$$= \begin{cases} 1, & x < z, \\ \dfrac{\sqrt{2\alpha}(a-x)\sinh(\sqrt{2\alpha}(b-a)) + \cosh(\sqrt{2\alpha}(b-a))}{\sqrt{2\alpha}(a-z)\sinh(\sqrt{2\alpha}(b-a)) + \cosh(\sqrt{2\alpha}(b-a))}, & z < x \leqslant a, \\ \dfrac{\cosh(\sqrt{2\alpha}(b-x))}{\sqrt{2\alpha}(a-z)\sinh(\sqrt{2\alpha}(b-a)) + \cosh(\sqrt{2\alpha}(b-a))}, & a < x \leqslant b, \\ \dfrac{1}{\sqrt{2\alpha}(a-z)\sinh(\sqrt{2\alpha}(b-a)) + \cosh(\sqrt{2\alpha}(b-a))}, & b < x. \end{cases}$$

Hence a direct calculation yields

$$\begin{aligned} \lambda(\beta\delta_z;\, \alpha\chi_{(a,b)}) &= \frac{1}{\beta}\mathscr{E}^{\alpha v}\left(P_\cdot^{\alpha v}(\sigma_z < \infty),\, P_\cdot^{\alpha v}(\sigma_z < \infty)\right) \\ &= \frac{1}{2\beta}\,\frac{\sqrt{2\alpha}\,\sinh(\sqrt{2\alpha}(b-a))}{\cosh(\sqrt{2\alpha}(b-a)) + \sqrt{2\alpha}(a-z)\sinh(\sqrt{2\alpha}(b-a))}. \end{aligned}$$

Next we assume that $a < z \leqslant b$. Then we can also see from [4], p. 167, 2.7.1, that

$$E_x\left[\exp\left(-\alpha \int_0^{\sigma_z} \chi_{(a,b)}(B_s)\,ds\right)\right] = \begin{cases} \dfrac{1}{\cosh(\sqrt{2\alpha}(z-a))}, & x \leqslant a, \\ \dfrac{\cosh(\sqrt{2\alpha}(x-a))}{\cosh(\sqrt{2\alpha}(z-a))}, & a < x < z, \\ \dfrac{\cosh(\sqrt{2\alpha}(b-x))}{\cosh(\sqrt{2\alpha}(b-z))}, & z < x \leqslant b, \\ \dfrac{1}{\cosh(\sqrt{2\alpha}(b-z))}, & b < x, \end{cases}$$

and thereby

$$\lambda(\beta\delta_z;\, \alpha\chi_{(a,b)}) = \frac{\sqrt{\alpha}}{4\sqrt{2}\beta}\left\{\frac{\sinh(2\sqrt{2\alpha}(z-a))}{\cosh^2(\sqrt{2\alpha}(z-a))} + \frac{\sinh(2\sqrt{2\alpha}(b-z))}{\cosh^2(\sqrt{2\alpha}(b-z))}\right\}.$$

EXAMPLE 3.2. For $n \in N$, let $\{a_i\}_{i=0}^n$ and $\{b_i\}_{i=1}^n$ be sequences which satisfy $a_0 < b_1 < a_1 < b_2 < \dots < b_n < a_n$. Here we take $\nu = \sum_{i=0}^n \alpha_i \delta_{a_i}$ for $\alpha_i \geqslant 0$. Then $A_t^\nu = \sum_{i=0}^n \alpha_i l_{a_i}(t)$, where $l_a(t)$ is the local time of the one-dimensional Brownian motion at a. Let $M^\nu = (B_t^\nu, P_x^\nu)$ be the killed Brownian motion with respect to $\exp(-A_t^\nu)$. Then its Dirichlet form $(\mathscr{E}^\nu, \mathscr{F}^\nu)$ is the following:

$$\mathscr{F}^\nu = \{f \in H^1(R) \mid \sum_{i=0}^n \alpha_i f(a_i)^2 < \infty\},$$

$$\mathscr{E}^\nu(f, f) = \frac{1}{2} D(f, f) + \sum_{i=0}^n \alpha_i f(a_i)^2, \quad f \in \mathscr{F}^\nu.$$

Put $\mu = \sum_{i=1}^n \beta_i \delta_{b_i}$ for $\beta_i > 0$. Then

$$\lambda\Big(\sum_{i=1}^n \beta_i \delta_{b_i}; \sum_{i=0}^n \alpha_i \delta_{a_i}\Big) = \inf\{\mathscr{E}^\nu(f, f) \mid f \in \mathscr{F}^\nu, \sum_{i=1}^n \beta_i f(b_i)^2 = 1\}.$$

Note that the infimum above is attained by the harmonic function u, which satisfies

$$u(x) = E_x[\exp(-A_{\sigma_B}^\nu) u(B_{\sigma_B})]$$

$$= \begin{cases} u(b_1) E_x[\exp(-\alpha_0 l_{a_0}(\sigma_1))], & x < b_1, \\ u(b_i) E_x[\exp(-\alpha_i l_{a_i}(\sigma_i)): \sigma_i < \sigma_{i+1}] + u(b_{i+1}) E_x[\exp(-\alpha_i l_{a_i}(\sigma_{i+1})): \sigma_{i+1} < \sigma_i], & \\ & b_i < x < b_{i+1}, \\ u(b_n) E_x[\exp(-\alpha_n l_{a_n}(\sigma_n))], & b_n < x, \end{cases}$$

where $B = \{b_i\}_{i=1}^n$ and σ_i is the hitting time of b_i. Then we see from [4], p. 164, 2.3.1, that

$$E_x[\exp(-\alpha_0 l_{a_0}(\sigma_1))] = \frac{1 + 2\alpha_0(x - a_0)}{1 + 2\alpha_0(b_1 - a_0)}, \quad a_0 \leqslant x < b_1,$$

$$E_x[\exp(-\alpha_n l_{a_n}(\sigma_n))] = \frac{1 + 2\alpha_n(a_n - x)}{1 + 2\alpha_n(a_n - b_n)}, \quad b_n < x \leqslant a_n.$$

We also see from [4], p. 174, 3.3.5, that

$$E_x[\exp(-\alpha_i l_{a_i}(\sigma_i)): \sigma_i < \sigma_{i+1}]$$

$$= \begin{cases} \dfrac{b_{i+1} - x + 2\alpha_i(b_{i+1} - a_i)(a_i - x)}{b_{i+1} - b_i + 2\alpha_i(b_{i+1} - a_i)(a_i - b_i)}, & b_i < x \leqslant a_i, \\ \dfrac{b_{i+1} - x}{b_{i+1} - b_i + 2\alpha_i(b_{i+1} - a_i)(a_i - b_i)}, & a_i < x < b_{i+1}, \end{cases}$$

and

$$E_x[\exp(-\alpha_i l_{a_i}(\sigma_{i+1})) \colon \sigma_{i+1} < \sigma_i]$$

$$= \begin{cases} \dfrac{x-b_i}{b_{i+1}-b_i+2\alpha_i(b_{i+1}-a_i)(a_i-b_i)}, & b_i < x \leqslant a_i, \\ \dfrac{x-b_i+2\alpha_i(a_i-b_i)(x-a_i)}{b_{i+1}-b_i+2\alpha_i(b_{i+1}-a_i)(a_i-b_i)}, & a_i < x < b_{i+1}. \end{cases}$$

Thus we have

$$\text{(3.2)} \quad \frac{1}{2}D(u, u)+\sum_{i=0}^{n} \alpha_i u(a_i)^2$$

$$= \frac{1}{2}\sum_{i=1}^{n-1} \frac{(1+2\alpha_i(b_{i+1}-a_i))u(b_i)^2-2u(b_i)u(b_{i+1})+(1+2\alpha_i(a_i-b_i))u(b_{i+1})^2}{b_{i+1}-b_i+2\alpha_i(b_{i+1}-a_i)(a_i-b_i)}$$

$$+\frac{\alpha_0}{1+2\alpha_0(b_1-a_0)}u(b_1)^2+\frac{\alpha_n}{1+2\alpha_n(a_n-b_n)}u(b_n)^2.$$

We shall find the minimum of the right-hand side of (3.2) under the assumption $\sum_{i=1}^{n} \beta_i u(b_i)^2 = 1$. Put

$$u(b_i) = x_i \quad \text{and} \quad A_i = \{b_{i+1}-b_i+2\alpha_i(b_{i+1}-a_i)(a_i-b_i)\}^{-1}.$$

Set

$$F(x_1, \ldots, x_n) = \frac{1}{2}\sum_{i=1}^{n-1} A_i\{(1+2\alpha_i(b_{i+1}-a_i))x_i^2-2x_i x_{i+1}+(1+2\alpha_i(a_i-b_i))x_{i+1}^2\}$$

$$+\frac{\alpha_0}{1+2\alpha_0(b_1-a_0)}x_1^2+\frac{\alpha_n}{1+2\alpha_n(a_n-b_n)}x_n^2$$

and

$$G(\kappa, x_1, \ldots, x_n) = F(x_1, \ldots, x_n)-\kappa\left(\sum_{i=1}^{n} \beta_i x_i^2-1\right).$$

As a direct calculation yields

$$\frac{1}{2}\sum_{k=1}^{n} x_k \frac{\partial G}{\partial x_k} = F(x_1, \ldots, x_n)-\kappa = 0,$$

it follows that

$$\lambda\left(\sum_{i=1}^{n} \beta_i \delta_{b_i}; \sum_{i=0}^{n} \alpha_i \delta_{a_i}\right) = \min\{\kappa \mid \det A(\kappa) = 0\},$$

where

$$A(\kappa)=\begin{pmatrix} B_1 & -A_1 & 0 & \dots & \dots & 0 \\ -A_1 & B_2 & -A_2 & \dots & \dots & \dots \\ 0 & -A_2 & B_3 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & -A_{n-2} & B_{n-1} & -A_{n-1} \\ 0 & \dots & \dots & 0 & -A_{n-1} & B_n \end{pmatrix},$$

$$B_1=\frac{2\alpha_0}{1+2\alpha_0(b_1-a_0)}+A_1(1+2\alpha_1(b_2-a_1))-2\kappa\beta_1,$$

$$B_k=A_{k-1}(1+2\alpha_{k-1}(a_{k-1}-b_{k-1}))+A_k(1+2\alpha_k(b_{k+1}-a_k))-2\kappa\beta_k, \quad 2\leqslant k\leqslant n-1,$$

$$B_n=\frac{2\alpha_n}{1+2\alpha_n(a_n-b_n)}+A_{n-1}(1+2\alpha_{n-1}(a_{n-1}-b_{n-1}))-2\kappa\beta_n.$$

When $n=1$, we get

$$\lambda(\beta_1\delta_{b_1};\ \alpha_0\delta_{a_0}+\alpha_1\delta_{a_1})=\frac{\alpha_0+\alpha_1+2\alpha_0\alpha_1(a_1-a_0)}{\beta_1(1+2\alpha_0(b_1-a_0))(1+2\alpha_1(a_1-b_1))}.$$

In particular, if $b_1-a_0=a_1-b_1=r$, then

$$\lambda(\beta_1\delta_{b_1};\ \alpha_0\delta_{a_0}+\alpha_1\delta_{a_1})=\frac{\alpha_0+\alpha_1+4\alpha_0\alpha_1 r}{\beta_1(1+2\alpha_0 r)(1+2\alpha_1 r)}.$$

When $n=2$ and $\alpha_0=\alpha_2=0$, we obtain

$$\lambda(\beta_1\delta_{b_1}+\beta_2\delta_{b_2};\ \alpha_1\delta_{a_1})=\frac{\beta_1(1+2\alpha_1(a_1-b_1))+\beta_2(1+2\alpha_1(b_2-a_1))}{4\beta_1\beta_2\{b_2-b_1+2\alpha_1(b_2-a_1)(a_1-b_1)\}}$$
$$-\frac{\sqrt{\{\beta_1(1+2\alpha_1(a_1-b_1))-\beta_2(1+2\alpha_1(b_2-a_1))\}^2+4\beta_1\beta_2}}{4\beta_1\beta_2\{b_2-b_1+2\alpha_1(b_2-a_1)(a_1-b_1)\}}.$$

Assume in addition that $\beta_1=\beta_2=\beta$ and $b_2-a_1=a_1-b_1=r$. Then

$$\lambda(\beta(\delta_{b_1}+\delta_{b_2});\ \alpha_1\delta_{a_1})=\frac{\alpha_1}{2\beta(1+\alpha_1 r)}.$$

3.2. In case of $1<\alpha\leqslant 2$. Next we shall consider the principal eigenvalues for time changed processes of absorbing α-stable processes with $1<\alpha\leqslant 2$. Let $\lambda(\mu;\ D)$ be the principal eigenvalue for $\check{M}^D$:

$$\lambda(\mu;\ D)=\inf\{\mathscr{E}^\alpha(f,f)\mid f\in C_0^\infty(D),\ \int_D f^2\,d\mu=1\}.$$

EXAMPLE 3.3. Let M^0 be the absorbing α-stable process on $\boldsymbol{R}\backslash\{0\}$ and G^0 its Green function. Then Getoor [7] showed that

$$G^0(x, y) = -\frac{1}{\Gamma(\alpha)\cos(\pi\alpha/2)}(|x|^{\alpha-1}+|y|^{\alpha-1}-|x-y|^{\alpha-1})$$

(see also [9], p. 379). By definition,

$$\lambda(\delta_a; \boldsymbol{R}\backslash\{0\}) = \inf\{\mathscr{E}^\alpha(f, f) \mid f\in C_0^\infty(\boldsymbol{R}\backslash\{0\}), f(a) = 1\}. \tag{3.3}$$

Then we see in a similar way to Example 3.1 that the infimum of (3.3) is attained by $G^0(\cdot, a)/G^0(a, a)$. Hence

$$\lambda(\delta_a; \boldsymbol{R}\backslash\{0\}) = \frac{1}{G^0(a, a)} = -\frac{\Gamma(\alpha)\cos(\pi\alpha/2)}{2|a|^{\alpha-1}}.$$

The following are three graphs of $\lambda(\delta_a; \boldsymbol{R}\backslash\{0\})$ with respect to $\alpha\in(1, 2]$. If $|a|$ is small, then $\lambda(\delta_a; \boldsymbol{R}\backslash\{0\})$ is increasing monotonously. However, $\lambda(\delta_a; \boldsymbol{R}\backslash\{0\})$ takes the maximal value for large $|a|$. We can guess that $\lambda(\delta_a; \boldsymbol{R}\backslash\{0\})$ takes the maximal value for $|a| > 1.5$.

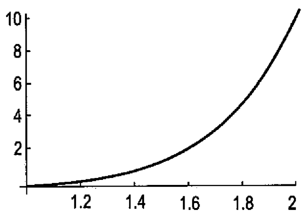

FIG. 1. $\lambda(\delta_{0.05}; \boldsymbol{R}\backslash\{0\})$

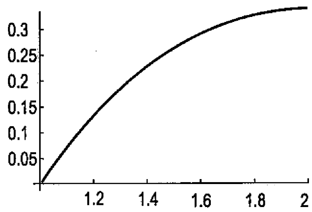

FIG. 2. $\lambda(\delta_{1.5}; \boldsymbol{R}\backslash\{0\})$

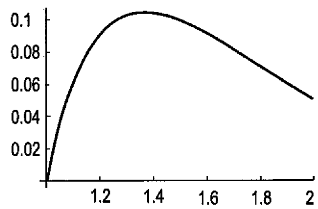

FIG. 3. $\lambda(\delta_{10}; \boldsymbol{R}\backslash\{0\})$

We can also show that

$$\lambda(\delta_a+\delta_{-a}; \boldsymbol{R}\backslash\{0\}) = \frac{1}{G^0(a, a)+G^0(a, -a)} = -\frac{\Gamma(\alpha)\cos(\pi\alpha/2)}{(4-2^{\alpha-1})|a|^{\alpha-1}}.$$

EXAMPLE 3.4. Let M^p be the absorbing α-stable process on $\boldsymbol{R}\backslash\{-p, p\}$. Denote by $G^p(x, y)$ the Green function of M^p. Then we see from (2.9) of [9] that

$$G^p(x, y) = L_p(x)+P_x(\sigma_p < \sigma_{-p})a(y-p)+P_x(\sigma_{-p} < \sigma_p)a(y+p)-a(y-x),$$

where

$$a(x) = -\frac{1}{\Gamma(\alpha)\cos(\pi\alpha/2)}|x|^{\alpha-1},$$

and L_p is some function. Noting that $G^p(x, p) = G^p(x, -p) = 0$, we obtain

$$L^p(x) = \tfrac{1}{2}(a(x-p)+a(x+p)-a(2p)).$$

Since it follows from Theorem 6.5 of [7] that

$$P_x(\sigma_{\pm p} < \sigma_{\mp p}) = \frac{1}{2} + \frac{1}{2a(2p)}\big(a(x \pm p) - a(x \mp p)\big),$$

we get

$$G^p(x, y) = \frac{1}{2}\big(a(x-p) + a(x+p) + a(y-p) + a(y+p) - a(2p)\big)$$

$$-\frac{1}{2a(2p)}\big(a(x-p) - a(x+p)\big)\big(a(y-p) - a(y+p)\big) - a(x-y).$$

Let $q \neq p$. Then we have

$$\lambda(\delta_q; \boldsymbol{R}\backslash\{-p, p\}) = \frac{1}{G^p(q, q)}$$

$$= \frac{-2\Gamma(\alpha)\cos(\pi\alpha/2)\,|2p|^{\alpha-1}}{4\,|p-q|^{\alpha-1}\,|p+q|^{\alpha-1} - (|p-q|^{\alpha-1} + |p+q|^{\alpha-1} - |2p|^{\alpha-1})^2}.$$

In particular,

$$\lambda(\delta_0; \boldsymbol{R}\backslash\{-p, p\}) = -\frac{\Gamma(\alpha)\cos(\pi\alpha/2)}{(2-2^{\alpha-2})\,|p|^{\alpha-1}}.$$

We can also prove the following:

$$\lambda(\delta_q + \delta_{-q}; \boldsymbol{R}\backslash\{-p, p\}) = \frac{1}{G^p(q, q) + G^p(q, -q)}$$

$$= \frac{-\Gamma(\alpha)\cos(\pi\alpha/2)}{2\,|p-q|^{\alpha-1} + 2\,|p+q|^{\alpha-1} - |2p|^{\alpha-1} - |2q|^{\alpha-1}}.$$

See [10], Section 3, and [13], Example 4.1, for more examples of principal eigenvalues for time changed processes of symmetric α-stable processes.

4. APPLICATION

In this section, we apply the results in the preceding section to the gaugeability. Recall first that $(\mathscr{E}, \mathscr{F})$ is the regular Dirichlet form associated with an m-symmetric transient Hunt process on X. Let us define

$$\lambda(\mu) = \inf\{\mathscr{E}(f, f) \mid f \in \mathscr{F}, \int_X f^2\, d\mu = 1\}, \qquad \mu \in \mathscr{K}_\infty(G).$$

Then Chen [5], Takeda [11] and Takeda and Uemura [13] proved the following:

THEOREM 4.1 ([5], Theorem 5.1; [11], Theorem 2.4; [13], Theorem 3.1). *For $\mu \in \mathscr{K}_\infty(G)$ with compact support it follows that*

$$\sup_{x \in X} E_x[\exp(A_\zeta^\mu)] < \infty \tag{4.1}$$

if and only if $\lambda(\mu) > 1$.

A measure $\mu \in \mathscr{K}_\infty(G)$ is said to be *gaugeable* if (4.1) holds. Applying Theorem 4.1 to the results in the preceding section, we can give conditions for some measures being gaugeable. For instance, let us consider Example 3.3. Denote by σ_0 the hitting time of 0. Since the strong Markov property implies

$$\sup_{x \in \boldsymbol{R} \setminus \{0\}} E_x\left[\exp\left(l_a(\sigma_0)\right)\right] = E_a\left[\exp\left(l_a(\sigma_0)\right)\right],$$

we have

$$E_a\left[\exp\left(l_a(\sigma_0)\right)\right] < \infty \Leftrightarrow 0 < |a| < \left(-\frac{\Gamma(\alpha)\cos(\pi\alpha/2)}{2}\right)^{1/(\alpha-1)}. \tag{4.2}$$

Let us make observations on (4.2). Fix $\alpha \in (1, 2]$. We first suppose that a is small. If a particle hits a, then it will hit 0 soon. We next suppose that a is large. Once a particle hits a, it will stay near a for a while and hit a many times by the time it arrives at 0.

Remark 4.2. Consider branching diffusion processes on a metric space. Then it is known that the expectation of the number of branches hitting a closed set coincides with the expectation of the Feynman–Kac functional (see [8]). Moreover, this relation also holds for branching symmetric α-stable processes on $\boldsymbol{R}^d$ ([12], Theorem 1.2). Combining this with Theorem 4.1 and our calculations of $\lambda(\mu, D)$, we can give a necessary and sufficient condition for the expectation of the number of branches hitting a closed set being finite.

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