PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 25, Fasc. 1 (2005), pp. 55–74

WEAK LIMITS AND INTEGRALS OF GAUSSIAN COVARIANCES IN BANACH SPACES

BY

J. M. A. M. VAN NEERVEN* (DELFT) AND L. WEIS* (KARLSRUHE)

Abstract. Let E be a separable real Banach space not containing an isomorphic copy of c_0. Let $\mathcal{Q}$ be a subset of $\mathscr{L}(E^*, E)$ with the property that each $Q \in \mathcal{Q}$ is the covariance of the centred Gaussian measure μ_Q on E. We show that the weak operator closure of $\mathcal{Q}$ consists of Gaussian covariances again, provided that

$$\sup_{Q \in \mathcal{Q}} \int_E \|x\|^2 \, d\mu_Q(x) < \infty.$$

If in addition E has type 2, the same conclusion holds for the weak operator closure of the convex hull of $\mathcal{Q}$. As an application, sufficient conditions are obtained for the integral of Gaussian covariance operators to be a Gaussian covariance. Analogues of these results are given for the class of γ-radonifying operators from a separable real Hilbert space H into E.

2000 Mathematics Subject Classification: Primary 28C20; Secondary: 35R15, 60B11, 60H05.

Key words and phrases: Gaussian Radon measure, covariance operator, γ-radonifying operator, Fatou lemma, type 2, cotype 2, weak operator topology.

1. INTRODUCTION

Let (μ_n) be a sequence of Gaussian Radon measures on a real Banach space E and let $(Q_n) \subseteq \mathscr{L}(E^*, E)$ be the associated sequence of their covariance operators. Assuming that the weak operator limit $\lim_{n\to\infty} Q_n = Q$ exists in $\mathscr{L}(E^*, E)$, it is natural to ask under what conditions Q is a Gaussian covar-

* The first-named author gratefully acknowledges support by 'VIDI subsidie' 639.032.201 in the 'Vernieuwingsimpuls' programme of the Netherlands Organization for Scientific Research (NWO) and by the Research Training Network HPRN-CT-2002-00281. The second-named author was supported by grants from the Volkswagenstiftung (I/78593) and the Deutsche Forschungsgemeinschaft (We 2847/1-1).

iance again. In this paper we show that this is the case if E does not contain an isomorphic copy of c_0 and the boundedness condition

$$\sup_n \int_E \|x\|^2 \, d\mu_n(x) < \infty$$

is satisfied. For separable E this implies that the weak operator closure of any family of Gaussian covariances $\mathcal{Q} \subseteq \mathcal{L}(E^*, E)$, which is bounded in second moment, consists of Gaussian covariances again, and if E has type 2, this result extends to the weak operator closure of the convex hull of $\mathcal{Q}$. As an application of this result we show that, in separable spaces with type 2, certain weak operator integrals of Gaussian covariances are Gaussian covariances again. These results are obtained in Sections 2 and 3.

Our motivation for studying these questions comes from the theory of stochastic equations. Let A be the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t \geq 0}$ on a real Banach space E and let $\{W(t)\}_{t \geq 0}$ be an E-valued Brownian motion. Denoting the law of $W(t)$ by ν_t, the following formula holds for the covariance of $W(t)$ in terms of the covariance operator R_t of ν_t:

$$E\langle W(t), x^*\rangle^2 = \langle R_t x^*, x^*\rangle = t\langle R_1 x^*, x^*\rangle, \qquad x^*, y^* \in E^*.$$

Extending well-known results for the case where E is a Hilbert space, it is shown in [3] and [8] that the stochastic differential equation

$$\begin{aligned} dU(t) &= AU(t)\,dt + dW(t), \qquad t \in [0, T], \\ U(0) &= 0 \end{aligned} \tag{1.1}$$

has a unique weak solution $\{U(t)\}_{t \in [0,T]}$ if and only if the operator $Q_T \in \mathcal{L}(E^*, E)$ defined by

$$Q_T x^* := \int_0^T S(t) R_1 S^*(t) x^* \, dt, \qquad x^* \in E^*,$$

is a Gaussian covariance operator. Since the operators $Q(t) := S(t) R_1 S^*(t)$ are Gaussian covariances, the abstract framework considered above applies. In this special situation our results show that if E has type 2, the operator Q_T is indeed a Gaussian covariance, and therefore the problem (1.1) has a weak solution.

The class of Gaussian covariance operators is closely related to that of γ-radonifying operators. Indeed, in Sections 4 and 5 we obtain analogues of our main results for this class of operators. In the final section we establish a converse of the main result of Section 5 for spaces with cotype 2.

2. WEAK LIMITS OF GAUSSIAN COVARIANCES

A Radon measure μ on a real Banach space E is called a *Gaussian measure* if for all $x^* \in E^*$ the image measure $\langle \mu, x^*\rangle$ is Gaussian. For such a measure μ on E there exists a unique vector $m \in E$, the *mean* of μ, and a unique bounded operator $Q \in \mathscr{L}(E^*, E)$, the *covariance* of μ, such that

$$(2.1) \qquad \langle Qx^*, y^*\rangle = \int_E \langle x-m, x^*\rangle \langle x-m, y^*\rangle \, d\mu(x) \quad \text{for all } x^*, y^* \in E^*.$$

Conversely, m and Q determine μ uniquely. A Gaussian measure μ is called *centred* if $m = 0$ or, equivalently, if the image measures $\langle \mu, x^*\rangle$ are centred for all $x^* \in E^*$. In this paper, all Gaussian measures will be centred. A necessary condition for a bounded operator $Q \in \mathscr{L}(E^*, E)$ to be a Gaussian covariance is that Q be positive and symmetric, i.e., $\langle Qx^*, x^*\rangle \geqslant 0$ for all $x^* \in E^*$ and $\langle Qx^*, y^*\rangle = \langle Qy^*, x^*\rangle$ for all $x^*, y^* \in E^*$. If E is a real Hilbert space, a positive symmetric operator $Q \in \mathscr{L}(E)$ (we identify E^* with E in the usual way) is the covariance of a Gaussian measure μ on E if and only if Q is of trace class. Taking μ to be centred, we have

$$\operatorname{tr}(Q) = \int_E \|x\|^2 \, d\mu(x).$$

In general Banach spaces, no simple explicit characterization of Gaussian covariances seems to be known. Our main tool for finding sufficient conditions on positive symmetric operators to be Gaussian covariances is the following Fatou type lemma:

LEMMA 2.1. *Let E be a real Banach space not containing an isomorphic copy of c_0 and let F be a norming subspace of E^*. Let $(Q_n) \subseteq \mathscr{L}(E^*, E)$ be a sequence of Gaussian covariances and assume that there exists a bounded operator $Q \in \mathscr{L}(E^*, E)$ such that*

$$(2.2) \qquad \lim_{n\to\infty} \langle Q_n x^*, x^*\rangle = \langle Qx^*, x^*\rangle \quad \textit{for all } x^* \in F.$$

If

$$(2.3) \qquad \sup_n \int_E \|x\|^2 \, d\mu_{Q_n}(x) < \infty,$$

then Q is a Gaussian covariance and

$$(2.4) \qquad \int_E \|x\|^2 \, d\mu_Q(x) \leqslant \liminf_{n\to\infty} \int_E \|x\|^2 \, d\mu_{Q_n}(x).$$

Here, for a given Gaussian covariance operator $Q \in \mathscr{L}(E^*, E)$ we write μ_Q for the unique centred Gaussian measure with covariance Q.

In the proof below we use freely the theory of Gaussian Radon measures on locally convex spaces presented in [2], Chapter 3, where also unexplained terminology can be found.

Proof. We begin with observing that (2.1) and (2.3) imply the uniform boundedness of the sequence (Q_n). Hence without loss of generality we may assume that F is norm closed in E^*.

Let $\nu_{Q_n} := j\mu_{Q_n}$ be the image measure under the canonical isometric embedding $j\colon E \subseteq F^*$. Each ν_{Q_n} is a centred Gaussian measure on F^*. Let $B(0, r)$ and $B_{F^*}(0, r)$ denote the closed balls of radius r in E and F^*, respectively. Combining (2.3), the weak*-compactness of $B_{F^*}(0, r)$, and the estimate

$$\nu_{Q_n}(\complement B_{F^*}(0, r)) \leqslant \frac{1}{r^2} \int_{B_{F^*}(0,r)} \|y^*\|^2 \, d\nu_{Q_n}(y^*) = \frac{1}{r^2} \int_{B(0,r)} \|x\|^2 \, d\mu_{Q_n}(x),$$

we can infer that the family (ν_{Q_n}) is uniformly tight as a family of Radon measures on $(F^*, \sigma(F^*, F))$; cf. [2], Example 3.8.13 (i). Let ν be any weak limit point. Then ν is a Gaussian Radon measure on $(F^*, \sigma(F^*, F))$. Let $R\colon F \to F^*$ be its covariance operator and let $y \in F$ be fixed. By a standard argument involving characteristic functions, (2.2) implies that $j \circ Q = R$. In particular, R takes its values in $j(E)$, and therefore we may identify Q and R as positive symmetric operators from $(F, \sigma(F, E))$ to $(E, \sigma(E, F))$. Let H be their common reproducing kernel Hilbert space. The canonical inclusion mapping $i\colon H \hookrightarrow E$ is weakly-to-$\sigma(E, F)$ continuous and its adjoint will be denoted by $i'\colon F \to H$; we have $R = i \circ i'$. By Theorem 3.2.7 in [2], H is separable, and we may choose a sequence (y_n) in F such that the sequence $(h_n) := (i' y_n)$ is an orthonormal basis for H. For all N and all $y \in F$,

$$\boldsymbol{E}\Big(\sum_{n \leqslant N} g_n \langle i h_n, y\rangle\Big)^2 = \sum_{n \leqslant N} \langle i h_n, y\rangle^2 \leqslant \|i' y\|_H^2 = \langle R y, y\rangle.$$

Hence, by Anderson's inequality [1], for all N we have

$$\boldsymbol{P}\Big(\Big\|\sum_{n \leqslant N} g_n i h_n\Big\|_{F^*}^2 \leqslant r^2\Big) \geqslant \nu\big(B_{F^*}(0, r)\big).$$

Since F is norming for E, this implies that the series $\sum_n g_n i h_n$ is bounded in probability in E. Since E does not contain an isomorphic copy of c_0, the Hoffmann-Jørgensen–Kwapień theorem ([6], Theorem 9.29) implies that $\sum_n g_n i h_n$ converges in E almost surely and in $L^2(\Omega; E)$. As a consequence, Q is the covariance of a Gaussian measure μ_Q on E. Note that from $R = j \circ Q$ we have $\nu = j \circ \mu_Q$.

It remains to prove (2.4). Let E_Q denote the closure of the linear support of μ_Q. Since μ_Q is Radon, E_Q is a separable closed subspace of E and we may choose a sequence of norm-one elements (y_n) in F such that $\|x\| = \sup_n |\langle x, y_n\rangle|$ for all $x \in E_Q$. For every $r > 0$ and N we have, by weak convergence,

$$\begin{aligned}
\int_{B(0,r)\cap E_Q} \sup_{n \leqslant N} |\langle x, y_n\rangle|^2 \, d\mu_Q(x) &\leqslant \int_{F^*} \sup_{n \leqslant N} |\langle y_n, y^*\rangle|^2 \wedge r^2 \, d\nu(y^*) \\
&= \lim_{n\to\infty} \int_{F^*} \sup_{n \leqslant N} |\langle x_n, y^*\rangle|^2 \wedge r^2 \, d\nu_{Q_n}(y^*) \leqslant \liminf_{n\to\infty} \int_{F^*} \|y^*\|^2 \wedge r^2 \, d\nu_{Q_n}(y^*) \\
&\leqslant \liminf_{n\to\infty} \int_E \|x\|^2 \wedge r^2 \, d\mu_{Q_n}(x) \leqslant \liminf_{n\to\infty} \int_E \|x\|^2 \, d\mu_{Q_n}(x).
\end{aligned}$$

By monotone convergence, (2.4) follows from this by first letting $N \to \infty$ and then $r \to \infty$. ∎

The following example shows that the lemma fails for $E = c_0$:

EXAMPLE 2.2. Let $T: l^2 \to c_0$ be the multiplication operator associated with the sequence $1/\sqrt{\ln 2}, 1/\sqrt{\ln 3}, \ldots$ For $n \geq 1$ let T_n denote the multiplication operator associated with the sequence

$$1/\sqrt{\ln 2}, 1/\sqrt{\ln 3}, \ldots, 1/\sqrt{\ln (n+1)}, 0, 0, \ldots$$

Then for every $n \geq 1$ the operator $Q_n := T_n \circ T_n^*$ is the covariance of a centred Gaussian measure μ_n on c_0. With $Q := T \circ T^*$ we have $\lim_{n\to\infty} \langle Q_n x^*, x^* \rangle = \langle Q x^*, x^* \rangle$ for all $x^* \in E^*$. As is shown in [7], Theorem 11, the assumptions of the lemma are satisfied but Q fails to be a Gaussian covariance operator.

Let us denote by $\mathcal{G}(E^*, E)$ the collection of all Gaussian covariances in $\mathcal{L}(E^*, E)$. A collection $\mathcal{Q} \subseteq \mathcal{G}(E^*, E)$ will be called *bounded in second moment* if

$$\sup_{Q\in\mathcal{Q}} \int_E \|x\|^2 \, d\mu_Q(x) < \infty .$$

Lemma 2.1 can be rephrased as follows: if $\mathcal{Q} \subseteq \mathcal{G}(E^*, E)$ is bounded in second moment, then its sequential weak operator closure is contained in $\mathcal{G}(E^*, E)$. For separable spaces E this may be strengthened as follows:

THEOREM 2.3. *Let E be a separable real Banach space not containing an isomorphic copy of c_0. Let $\mathcal{Q} \subseteq \mathcal{G}(E^*, E)$ be bounded in second moment and let $\overline{\mathcal{Q}}^{\mathrm{w}}$ denote the closure of $\mathcal{Q}$ in the weak operator topology of $\mathcal{L}(E^*, E)$. Then $\overline{\mathcal{Q}}^{\mathrm{w}} \subseteq \mathcal{G}(E^*, E)$, and for all $R \in \overline{\mathcal{Q}}^{\mathrm{w}}$ we have*

$$\int_E \|x\|^2 \, d\mu_R(x) \leq \sup_{Q\in\mathcal{Q}} \int_E \|x\|^2 \, d\mu_Q(x).$$

Proof. Since E is separable, we may pick a sequence $(x_n^*)_{n\geq 1}$ in E^* whose linear span F is a norming subspace for E. Fix an arbitrary $R \in \overline{\mathcal{Q}}^{\mathrm{w}}$. For each $n \geq 1$ let $Q_n \in \mathcal{Q}$ be an operator such that $|\langle (R - Q_n) x_j^*, x_k^* \rangle| \leq 1/n$ for $j, k = 1, \ldots, n$. Then $\lim_{n\to\infty} \langle Q_n x_j^*, x_j^* \rangle = \langle R x_j^*, x_k^* \rangle$ for all $j, k \geq 1$, and by polarization this implies $\lim_{n\to\infty} \langle Q_n x^*, x^* \rangle = \langle R x^*, x^* \rangle$ for all $x^* \in F$. The result now follows from Lemma 2.1 ∎

3. WEAK INTEGRALS OF GAUSSIAN COVARIANCES IN SPACES WITH TYPE 2

Recall that a Banach space E is said to have *type* 2 if there exists a constant $C \geq 0$ such that for all finite subsets $\{x_1, \ldots, x_N\}$ of E we have

$$E \Big\| \sum_{n=1}^{N} \varepsilon_n x_n \Big\|^2 \leq C^2 \sum_{n=1}^{N} \|x_n\|^2, \tag{3.1}$$

where $(\varepsilon_n)_{n=1}^N$ are independent Rademacher variables. The least possible constant C in (3.1) is called the *type* 2 *constant* of E and is denoted by C_2. Examples of spaces with type 2 are the Hilbert spaces and the L^p-spaces for $p \in [2, \infty)$.

LEMMA 3.1. *Let E be a real Banach space of type 2. Let $Q = \sum_{n=1}^N a_n Q_n$, where $a_n \geqslant 0$ and $Q_n \in \mathscr{G}(E^*, E)$ for all $n = 1, \ldots, N$. Then $Q \in \mathscr{G}(E^*, E)$ and*

$$\int_E \|x\|^2 \, d\mu_Q(x) \leqslant C_2^2 \sum_{n=1}^N a_n \int_E \|x\|^2 \, d\mu_{Q_n}(x).$$

Proof. Without loss of generality we may assume that $a_n > 0$ for all $n = 1, \ldots, N$. Let us denote by ν_n the centred Gaussian measure on E given by

$$\nu_n(B) = \mu_{Q_n}(B/\sqrt{a_n})$$

for Borel sets $B \subseteq E$. Then Q is the covariance of the centred Gaussian measure $\mu_Q = \nu_1 * \ldots * \nu_N$. For any choice of $(r_1, \ldots, r_N) \in \{-1, 1\}^N$ we have, using the symmetry of each of the ν_n,

$$\begin{aligned} \int_E \|x\|^2 \, d\mu_Q(x) &= \int_{E^N} \Big\| \sum_{n=1}^N x_n \Big\|^2 d\nu_1(x_1) \ldots d\nu_N(x_N) \\ &= \int_{E^N} \Big\| \sum_{n=1}^N r_n x_n \Big\|^2 d\nu_1(x_1) \ldots d\nu_N(x_N). \end{aligned}$$

Let $(\varepsilon_n)_{n=1}^N$ be independent Rademacher variables on a probability space $(\Omega, \boldsymbol{P})$. Putting $r_n := \varepsilon_n(\omega)$, taking expectations with respect to $\omega \in \Omega$ and applying Fubini's theorem, we obtain

$$\int_E \|x\|^2 \, d\mu_Q(x) = \int_{E^N} \boldsymbol{E} \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|^2 d\nu_1(x_1) \ldots d\nu_N(x_N).$$

Since E has type 2, the right-hand side can be estimated from above by

$$\begin{aligned} C_2^2 \int_{E^N} \sum_{n=1}^N \|x_n\|^2 \, d\nu_1(x_1) \ldots d\nu_N(x_N) \\ = C_2^2 \sum_{n=1}^N \int_E \|y\|^2 \, d\nu_n(y) = C_2^2 \sum_{n=1}^N a_n \int_E \|x\|^2 \, d\mu_{Q_n}(x). \quad \blacksquare \end{aligned}$$

A Banach space E with type 2 cannot contain an isomorphic copy of c_0. Hence we may combine Theorem 2.3 and Lemma 3.1 to obtain the following result, in which $\mathrm{co}\,\mathscr{Q}$ denotes the convex hull of $\mathscr{Q}$:

THEOREM 3.2. *Let E be a separable real Banach space with type 2. If $\mathscr{Q} \subseteq \mathscr{G}(E^*, E)$ is bounded in second moment, then $\overline{\mathrm{co}\,\mathscr{Q}}^{\mathrm{w}} \subseteq \mathscr{G}(E^*, E)$ and for all $R \in \overline{\mathrm{co}\,\mathscr{Q}}^{\mathrm{w}}$ we have*

$$\int_E \|x\|^2 \, d\mu_R(x) \leqslant C_2^2 \sup_{Q \in \mathscr{Q}} \int_E \|x\|^2 \, d\mu_Q(x).$$

This theorem will be applied next to show that in spaces E with type 2 the weak operator integral of a function with values in $\mathscr{G}(E^*, E)$ belongs to $\mathscr{G}(E^*, E)$. For this result we need the following elementary lemma.

LEMMA 3.3. *Let (X, λ) be a probability space and let $\eta > 0$ be arbitrary. For $m = 1, \ldots, M$ let $f_m\colon X \to \mathbf{R}$ be bounded. Then there exists a finite partition $P = A_1, \ldots, A_k$ of X into disjoint measurable sets with the following property: for every refinement $P' = B_1, \ldots, B_{k'}$ and every choice of points $\xi_j \in B_j$, $j = 1, \ldots, k'$, we have*

$$\int_X \Big|f_m(\xi) - \sum_{j=1}^{k} f_m(\xi_j) 1_{A_j}(\xi)\Big| \, d\lambda(\xi) < \eta, \qquad m = 1, \ldots, M.$$

Proof. Let $|f_m| \leqslant R$ for $m = 1, \ldots, M$. For N so large that $2R/N < \eta$, divide $[-R, R]$ into N disjoint intervals I_n of length $2R/N$ and let $B_{m,n} = f_m^{-1}(I_n)$, $m = 1, \ldots, M$, $n = 1, \ldots, N$. Consider the partition P of X generated by the $k = N^M$ sets $B_{1,n_1} \cap \ldots \cap B_{M,n_M}$ with $1 \leqslant n_1, \ldots, n_M \leqslant N$. If P' is a refinement of P and if $\xi_j \in B_j \in P'$, then $B_j \subseteq A_i$ for some $A_i \in P$, and therefore for all $\xi \in B_j$ and all m we have

$$|f_m(\xi) - f_m(\xi_j)| \leqslant 2R/N < \eta.$$

By integrating, we see that P has the required properties. ∎

THEOREM 3.4. *Let E be a separable real Banach space with type 2, let (X, λ) be a probability space, and let $Q\colon X \to \mathscr{L}(E^*, E)$ be a function with the following properties:*

(1) $Q(\xi) \in \mathscr{G}(E^*, E)$ for λ-almost all $\xi \in X$ and we have

$$\int_X \int_E \|x\|^2 \, d\mu_{Q(\xi)}(x) \, d\lambda(\xi) < \infty;$$

(2) $\xi \mapsto Q(\xi) x^*$ is Pettis integrable for all $x^* \in E^*$.

Then the operator $Q_X \in \mathscr{L}(E^*, E)$ defined by

$$Q_X x^* := \int_X Q(\xi) x^* \, d\lambda(\xi), \qquad x^* \in E^*,$$

belongs to $\mathscr{G}(E^*, E)$ and we have

$$\int_E \|x\|^2 \, d\mu_{Q_X}(x) \leqslant C_2^2 \int_X \int_E \|x\|^2 \, d\mu_{Q(\xi)}(x) \, d\lambda(\xi).$$

Remark 3.5. It is implicit in the formulation of the theorem that the function

$$\xi \mapsto \int_E \|x\|^2 \, d\mu_{Q(\xi)}(x)$$

is measurable. That this is indeed the case can be checked by an argument using approximation of $x \mapsto \|x\|^2$ by cylindrical functions. The details are somewhat tedious and are left to the reader.

Proof. Let X_0 be a set of full λ-measure in X with the property that $Q(\xi)$ is a Gaussian covariance for all $\xi \in X_0$. Without loss of generality we may assume that $X_0 = X$.

Step 1. We first prove the theorem under the additional assumption that the function Q is bounded in the operator norm of $\mathscr{L}(E^*, E)$. Let $\mathscr{R}$ be the set of all operators $R \in \operatorname{co} \mathscr{Q}$ of the form $R = \sum_{j=1}^{k} a_j Q(\xi_j)$, where

$$a_j = \mu(A_j) \quad \text{and} \quad \xi_j \in A_j \quad \text{for all } j = 1, \ldots, k \tag{3.2}$$

for some partition $P = A_1, \ldots, A_k$ of X.

For $\delta > 0$ arbitrary and fixed let $\mathscr{R}_\delta$ denote the collection of all $R \in \mathscr{R}$ for which P and the $\xi_j \in A_j$ in (3.2) satisfy the additional requirement

$$\Big|\int_X \int_E \|x\|^2 \, d\mu_{Q(\xi)}(x)\, d\lambda(\xi) - \int_X \sum_{j=1}^{k} 1_{A_j}(\xi) \int_E \|x\|^2 \, d\mu_{Q(\xi_j)}(x)\, d\lambda(\xi)\Big| < \delta. \tag{3.3}$$

Note that every partition $P = A_1, \ldots, A_k$ has a refinement $P' = B_1, \ldots, B_{k'}$ such that (3.3) holds for P' and a suitable choice of points $\xi_i \in B_i$.

We claim that $Q_X \in \overline{\mathscr{R}_\delta}^{\mathrm{w}}$. Suppose the contrary. Then some weakly open subset of $\mathscr{L}(E^*, E)$ containing Q_X is disjoint from $\mathscr{R}_\delta$. It follows that there exist an integer $M \geqslant 1$, elements $x_1^*, \ldots, x_M^*, y_1^*, \ldots, y_M^* \in E^*$, and an $\varepsilon > 0$ such that for all $R \in \mathscr{R}_\delta$ we have

$$|\langle (Q_X - R) x_l^*, y_m^* \rangle| \geqslant \varepsilon \quad \text{for some } l, m \in \{1, \ldots, M\}.$$

In particular, for all partitions P of X and all choices $\xi_j \in A_j$ subject to the condition (3.3) we have

$$\Big|\int_X \langle Q(\xi) x_l^*, y_m^* \rangle \, d\lambda(\xi) - \int_X \sum_{j=1}^{k} 1_{A_j}(\xi) \langle Q(\xi_j) x_l^*, y_m^* \rangle \, d\lambda(\xi)\Big| \geqslant \varepsilon$$

for some $l, m \in \{1, \ldots, M\}$. But this is impossible in view of Lemma 3.3. This proves the claim.

By Lemma 3.1 and (3.3), for any $R \in \mathscr{R}_\delta$ we have

$$\begin{aligned}
\int_E \|x\|^2 \, d\mu_R(x) &\leqslant C_2^2 \sum_{j=1}^{k} a_j \int_E \|x\|^2 \, d\mu_{Q(\xi_j)}(x) \\
&= C_2^2 \int_X \sum_{j=1}^{k} 1_{A_j}(\xi) \int_E \|x\|^2 \, d\mu_{Q(\xi_j)}(x)\, d\lambda(\xi) \leqslant C_2^2 (1+\delta) \int_X \int_E \|x\|^2 \, d\mu_{Q(\xi)}(x)\, d\lambda(\xi).
\end{aligned}$$

Moreover, by the claim and Theorem 3.2,

$$\int_E \|x\|^2 \, d\mu_{Q_X}(x) \leqslant \sup_{R \in \mathscr{R}_\delta} \int_E \|x\|^2 \, d\mu_R(x).$$

The theorem (for bounded Q) now follows by combining these estimates and noting that $\delta > 0$ was arbitrary.

Step 2. For general functions Q, let $Q^{(n)} := 1_{\{\|Q\| \leqslant n\}} Q$. Then $Q^{(n)}$ is bounded and satisfies the conditions (1) and (2). Hence, by Step 1, the operator $Q_n \in \mathscr{L}(E^*, E)$ defined by

$$Q_X^{(n)} x^* := \int_X Q^{(n)}(\xi) x^* \, d\lambda(\xi), \qquad x^* \in E^*,$$

is a Gaussian covariance. Clearly, we have $\langle Q_X^{(n)} x^*, x^* \rangle \uparrow \langle Q_X x^*, x^* \rangle$ as $n \to \infty$ for all $x^* \in E^*$. Therefore the condition (2.2) in Lemma 2.1 is satisfied and, by Anderson's inequality, the condition (2.3) is satisfied as well. The proof is concluded by an application of this lemma and noting that for $x^* \in E^*$ we have

$$\begin{aligned}\langle Q_X x^*, x^* \rangle &= \lim_{n \to \infty} \langle Q_X^{(n)} x^*, x^* \rangle \\ &= \lim_{n \to \infty} \int_X \langle Q^{(n)}(\xi) x^*, x^* \rangle \, d\lambda(\xi) = \int_X \langle Q(\xi) x^*, x^* \rangle \, d\lambda(\xi),\end{aligned}$$

where the last equality follows by monotone convergence. ∎

Corollary 3.6. *Let E be a separable real Banach space with type 2 and let (X, λ) be a probability space. Let $Q \in \mathscr{G}(E^*, E)$ be fixed and let $S\colon X \to \mathscr{L}(E)$ be a strongly measurable function satisfying*

$$\int_X \|S(\xi)\|^2 \, d\lambda(\xi) < \infty. \tag{3.4}$$

Then the operator $Q_X \in \mathscr{L}(E^*, E)$ defined by

$$Q_X x^* := \int_X S(\xi) Q S^*(\xi) x^* \, d\lambda(\xi)$$

belongs to $\mathscr{G}(E^*, E)$ and we have

$$\int_E \|x\|^2 \, d\mu_{Q_X}(x) \leqslant C_2^2 \Big(\int_X \|S(\xi)\|^2 \, d\lambda(\xi)\Big)\Big(\int_E \|x\|^2 \, d\mu_Q(x)\Big).$$

Proof. For all $x^* \in E^*$, the function $\xi \mapsto S(\xi) Q S^*(\xi) x^*$ is strongly measurable by Pettis's measurability theorem, and therefore Bochner integrable by (3.4). For all $\xi \in X$, $S(\xi) Q S^*(\xi)$ is the covariance operator of the image measure $S(\xi)\mu_Q =: \mu_\xi$, and therefore

$$\begin{aligned}\int_E \|x\|^2 \, d\mu_{Q_X}(x) &\leqslant C_2^2 \int_X \int_E \|x\|^2 \, d\mu_\xi(x) \, d\lambda(\xi) \\ &= C_2^2 \int_X \int_E \|S(\xi) y\|^2 \, d\mu_Q(y) \, d\lambda(\xi) \leqslant C_2^2 \Big(\int_X \|S(\xi)\|^2 \, d\lambda(\xi)\Big)\Big(\int_E \|y\|^2 \, d\mu_Q(y)\Big). \quad ∎\end{aligned}$$

An application of this result to stochastic evolution equations has been discussed in the Introduction.

4. WEAK LIMITS OF γ-RADONIFYING OPERATORS

Let H be a separable real Banach space. A bounded operator $T \in \mathscr{L}(H, E)$ is said to be *γ-radonifying* if $TT^* \in \mathscr{G}(E^*, E)$. Here we identify H and its dual in the usual way, which permits us to view TT^* as a bounded operator from E^* into E. It is well known that

$$\|T\|^2_{\gamma(H,E)} := \int_E \|x\|^2 \, d\mu_{TT^*}(x)$$

defines a norm $\|\cdot\|_{\gamma(H,E)}$ on the vector space $\gamma(H, E)$ of all γ-radonifying operators from H to E, and that $\gamma(H, E)$ is a Banach space with respect to this norm. If (h_n) is an orthonormal basis for H and (g_n) is a sequence of independent standard Gaussian variables, then

$$\|T\|^2_{\gamma(H,E)} = \mathbf{E}\Big\|\sum_n g_n T h_n\Big\|^2.$$

An overview of the theory of γ-radonifying operators is presented in [2]. We shall need the following ideal property, which is implied by Anderson's inequality: if $S_1: H_1 \to H$ and $S_2: E \to E_1$ are bounded and $T: H \to E$ is γ-radonifying, then $S_2 T S_1: H_1 \to E_1$ is γ-radonifying and

$$\|S_2 T S_1\|_{\gamma(H_1,E_1)} \leqslant \|S_2\| \, \|T\|_{\gamma(H,E)} \, \|S_1\|.$$

As an application of Lemma 2.1 we obtain the following Fatou lemma for γ-radonifying operators.

THEOREM 4.1. *Let H be a separable real Hilbert space, E a real Banach space not containing an isomorphic copy of c_0, and F a norming subspace of E^*. Let (T_n) be a bounded sequence in $\gamma(H, E)$ and let $T \in \mathscr{L}(H, E)$ be such that*

$$\lim_{n\to\infty} \langle T_n h, x^* \rangle = \langle Th, x^* \rangle \quad \textit{for all } h \in H,\ x^* \in F. \tag{4.1}$$

Then $T \in \gamma(H, E)$ and

$$\|T\|_{\gamma(H,E)} \leqslant \liminf_{n\to\infty} \|T_n\|_{\gamma(H,E)}. \tag{4.2}$$

Proof. Since the operators T_n and T have separable ranges, there is no loss of generality by assuming that E is separable.

Fix a sequence (x_j^*) of norm-one vectors in F such that $\|x\| = \sup_j |\langle x, x_j^* \rangle|$ for all $x \in E$. Also fix an integer $k \geqslant 1$. Noting that by (4.1) we have $\lim_{n\to\infty} T_n^* x_j^* = T^* x_j^*$ weakly in H for all $j \geqslant 1$, we choose a sequence of convex combinations of the form

$$S_n^{(k)} = \sum_{m=n}^{N_n^{(k)}} a_{m,n}^{(k)} T_m \tag{4.3}$$

such that

$$(4.4) \qquad \lim_{n\to\infty} \|S_n^{(k)*} x_j^* - T^* x_j^*\| = 0 \quad \text{for all } j = 1, \ldots, k.$$

By (4.3), the inequality $\|\cdot\| \leqslant \|\cdot\|_{\gamma(H,E)}$, and the boundedness of (T_n) in $\gamma(H, E)$,

$$(4.5) \qquad \sup_n \|S_n^{(k)}\| \leqslant \sup_n \|S_n^{(k)}\|_{\gamma(H,E)} \leqslant \sup_n (\sup_{m\geqslant n} \|T_m\|_{\gamma(H,E)}) < \infty.$$

From the estimates

$$\begin{aligned} |\langle (S_n^{(k)} - T)\, T^* x_j^*, x_j^*\rangle| &\leqslant \sum_{m=n}^{N_n^{(k)}} a_{m,n}^{(k)} |\langle (T_m - T)\, T^* x_j^*, x_j^*\rangle| \\ &\leqslant \sup_{m\geqslant n} |\langle (T_m - T)\, T^* x_j^*, x_j^*\rangle| \end{aligned}$$

and (4.1) it follows that

$$(4.6) \qquad \lim_{n\to\infty} \langle S_n^{(k)} T^* x_j^*, x_j^*\rangle = \langle TT^* x_j^*, x_j^*\rangle \quad \text{for all } j = 1, \ldots, k.$$

Therefore, by (4.4)–(4.6) we obtain

$$\begin{aligned} (4.7) \quad & \lim_{n\to\infty} \langle S_n^{(k)} S_n^{(k)*} x_j^*, x_j^*\rangle \\ &= \lim_{n\to\infty} \langle S_n^{(k)} T^* x_j^*, x_j^*\rangle + \lim_{n\to\infty} \langle S_n^{(k)} (S_n^{(k)*} x_j^* - T^* x_j^*), x_j^*\rangle = \langle TT^* x_j^*, x_j^*\rangle \end{aligned}$$

for all $j = 1, \ldots, k$.

For every $k \geqslant 1$ we use (4.7) to choose an index n_k such that

$$|\langle S_{n_k}^{(k)} S_{n_k}^{(k)*} x_j^*, x_j^*\rangle - \langle TT^* x_j^*, x_j^*\rangle| < 1/k \quad \text{for all } j = 1, \ldots, k.$$

Then

$$(4.8) \qquad \lim_{k\to\infty} \langle S_{n_k}^{(k)} S_{n_k}^{(k)*} x_j^*, x_j^*\rangle = \langle TT^* x_j^*, x_j^*\rangle \quad \text{for all } j \geqslant 1.$$

By polarization we infer from (4.8) that

$$(4.9) \qquad \lim_{k\to\infty} \langle S_{n_k}^{(k)} S_{n_k}^{(k)*} x^*, x^*\rangle = \langle TT^* x^*, x^*\rangle \quad \text{for all } x^* \in F_0,$$

where F_0 denotes the linear span of the sequence (x_j^*). It follows from (4.5), (4.9), and Lemma 2.1 that $T \in \gamma(H, E)$ and

$$(4.10) \qquad \|T\|_{\gamma(H,E)} \leqslant \limsup_{m\to\infty} \|T_m\|_{\gamma(H,E)}.$$

By applying (4.10) to suitable subsequences of (T_m) the estimate (4.2) follows. ∎

Under the stronger assumption that $\lim_{n\to\infty} T_n h = Th$ for all $h \in H$ this result is contained in [5], Proposition 4.10, where it is proved with the following concise argument. Fix an orthonormal basis (h_j) of H and an integer k.

Then, by the Fatou lemma,

$$E\Big\|\sum_{j\leqslant k} g_j Th_j\Big\|^2 \leqslant \liminf_{n\to\infty} E\Big\|\sum_{j\leqslant k} g_j T_n h_j\Big\|^2 \leqslant \liminf_{n\to\infty} \|T_n\|^2_{\gamma(H,E)}.$$

Hence

$$\sup_k E\Big\|\sum_{j\leqslant k} g_j Th_j\Big\|^2 \leqslant \liminf_{n\to\infty} \|T_n\|^2_{\gamma(H,E)}. \tag{4.11}$$

This means that T is almost summing in the sense of [4], Chapter 12. Since E does not contain c_0, the Hoffmann-Jørgensen–Kwapień theorem implies that $T\in\gamma(H, E)$ and (4.2) follows from (4.11).

Let (X, λ) be a separable σ-finite measure space. We call an operator $T\in\gamma(L^2(X), E)$ *representable* if there exists a function $\phi\colon X\to E$ such that for all $x^*\in E^*$ we have $T^*x^* = \langle\phi, x^*\rangle$. Here $\langle\phi, x^*\rangle\in L^2(X)$ is defined by $\langle\phi, x^*\rangle(\xi) := \langle\phi(\xi), x^*\rangle$ for $\xi\in X$. In this situation we say that ϕ *represents* T. We write $\gamma(X; E)$ for the vector space of all functions $\phi\colon X\to E$ representing an element T of $\gamma(L^2(X), E)$. For such a function we write $\|\phi\|_{\gamma(X;E)} := \|T\|_{\gamma(H,E)}$.

Our interest in the class $\gamma(X; E)$ is explained by the following result from [8]:

If $\phi\colon (0, T)\to E$ is a function such that $\langle\phi, x^*\rangle\in L^2(0, T)$ for all $x^*\in E^*$ and if $W=\{W(t)\}_{t\geqslant 0}$ is a real-valued Brownian motion, then ϕ is stochastically integrable with respect to W if and only if $\phi\in\gamma((0, T); E)$, and in this case we have

$$E\Big\|\int_0^T \phi(t)\,dW(t)\Big\|^2 = \|\phi\|^2_{\gamma((0,T);E)}.$$

Theorem 4.1 implies a Fatou lemma for functions in $\gamma(X; E)$. It generalizes Proposition 4.11 in [5], where stronger measurability and convergence assumptions were imposed. As in [5] the proof is based on Egoroff's theorem, but the details are more intricate. By the result from [8] just quoted, for $X = (0, T)$ it provides a sufficient condition for stochastic integrability of certain E-valued functions.

THEOREM 4.2. *Let (X, λ) be a separable σ-finite measure space and let E be a real Banach space not containing an isomorphic copy of c_0. Let (ϕ_n) be a sequence of functions in $\gamma(X; E)$ satisfying*

$$\sup_n \|\phi_n\|_{\gamma(X;E)} < \infty$$

and let $\phi\colon X\to E$ be a function such that

$$\lim_{n\to\infty} \langle\phi_n, x^*\rangle = \langle\phi, x^*\rangle \quad \mu\text{-almost everywhere}$$

for all $x^* \in E^*$. *If* ϕ *is Pettis integrable, then* $\phi \in \gamma(X; E)$ *and*

$$\|\phi\|_{\gamma(X;E)} \leqslant \liminf_{n\to\infty} \|\phi_n\|_{\gamma(X;E)}.$$

Proof. For all n and all $x^* \in E^*$ we have $\|\langle \phi_n, x^*\rangle\|_2 \leqslant \|\phi_n\|_{\gamma(X;E)} \|x^*\|$. Hence, by Fatou's lemma, $\langle \phi, x^*\rangle \in L^2(X)$ and

$$\|\langle \phi, x^*\rangle\|_2 \leqslant \liminf_{n\to\infty} \|\langle \phi_n, x^*\rangle\|_2 \leqslant \|x^*\| \liminf_{n\to\infty} \|\phi_n\|_{\gamma(X;E)} \quad \text{for all } x^* \in E^*.$$

Step 1. The separability of (X, λ) implies that $L^2(X)$ is separable. Let T_n: $L^2(X) \to E$ and T: $L^2(X) \to E$ be the operators represented by ϕ_n and ϕ, respectively. Note that T is well defined since ϕ is Pettis integrable. That the operators T_n are well defined follows immediately from the assumption that $\phi_n \in \gamma(X; E)$.

Let E_0 be a separable closed subspace of E containing the ranges of the operators T_n and T. Let (x_j^*) be a sequence of norm-one vectors in E^* such that $\|x\| = \sup_j |\langle x, x_j^*\rangle|$ for all $x \in E_0$.

We construct a sequence of measurable subsets $(X^{(k)})$ of X with the following properties:

(1) $\lambda(X^{(k)}) < \infty$ for all k;

(2) $\lambda\big(X \backslash \bigcup_k X^{(k)}\big) = 0$;

(3) $\lim_{n\to\infty} \langle \phi_n, x_j^*\rangle = \langle \phi, x_j^*\rangle$ uniformly on $X^{(k)}$ for all j and k.

We start by selecting a sequence of measurable subsets $(A^{(k)})$ of X with $\lambda(A^{(k)}) < \infty$ and $X = \bigcup_k A^{(k)}$; this is possible since (X, λ) is σ-finite. Next we use Egoroff's theorem to choose measurable subsets $A_m^{(k)} \subseteq A^{(k)}$ such that $\lambda(A^{(k)} \backslash A_m^{(k)}) \leqslant 2^{-m}$ and $\lim_{n\to\infty} \langle \phi_n, x_j^*\rangle = \langle \phi, x_j^*\rangle$ uniformly on $A_m^{(k)}$ for all $j = 1, \ldots, m$. Let

$$B_l^{(k)} := \bigcap_{m \geqslant l+1} A_m^{(k)}.$$

Then $B_l^{(k)} \subseteq A^{(k)}$, $\lambda(A^{(k)} \backslash B_l^{(k)}) \leqslant 2^{-l}$, and $\lim_{n\to\infty} \langle \phi_n, x_j^*\rangle = \langle \phi, x_j^*\rangle$ uniformly on $B_l^{(k)}$ for all $j \geqslant 1$. The sets

$$X^{(k)} := \bigcup_{i,l=1}^{k} B_l^{(i)}$$

have the desired properties.

Step 2. Put $\phi_n^{(k)} := 1_{X^{(k)}} \phi_n$ and $\phi^{(k)} := 1_{X^{(k)}} \phi$. Let $T_n^{(k)}$: $L^2(X) \to E$ and $T^{(k)}$: $L^2(X) \to E$ be the operators represented by $\phi_n^{(k)}$ and $\phi^{(k)}$, respectively. From $T_n^{(k)} f = T_n(1_{X^{(k)}} f)$ and $T^{(k)} f = T_n(1_{X^{(k)}} f)$ it follows that $T_n^{(k)}$ and $T^{(k)}$ take their values in E_0.

Let $f \in L^2(X)$ be fixed. From the estimate

$$|\langle T_n^{(k)} f - T^{(k)} f, x_j^*\rangle| \leqslant \int_{X^{(k)}} |f \langle \phi_n - \phi, x_j^*\rangle| \, d\lambda \leqslant \|1_{X^{(k)}} f\|_1 \|1_{X^{(k)}} \langle \phi_n - \phi, x_j^*\rangle\|_\infty$$

it follows that $\lim_{n\to\infty}\langle T_n^{(k)}f - T^{(k)}f, x_j^*\rangle = 0$ for all j and k. Theorem 4.1, applied to the Banach space E_0 and the norming subspace of E_0^* spanned by the restrictions of the x_j^* to E_0, implies that $\phi^{(k)}\in\gamma(X;E)$ and

$$\|\phi^{(k)}\|_{\gamma(X;E)} \leqslant \liminf_{n\to\infty}\|\phi_n^{(k)}\|_{\gamma(X;E)} \leqslant \liminf_{n\to\infty}\|\phi_n\|_{\gamma(X;E)},$$

where the second inequality is a consequence of Anderson's inequality. Next, for all $f\in L^2(X)$ we have, by Fatou's lemma,

$$\begin{aligned}\|T^{(k)}f - Tf\| &= \sup_j |\langle T^{(k)}f - Tf, x_j^*\rangle| \\ &\leqslant \sup_j \int_{\complement X^{(k)}} |f\langle\phi, x_j^*\rangle|\,d\lambda \leqslant \|1_{\complement X^{(k)}}f\|_2\|\langle\phi, x_j^*\rangle\|_2 \\ &\leqslant \|1_{\complement X^{(k)}}f\|_2 \liminf_{n\to\infty}\|\langle\phi_n, x_j^*\rangle\|_2 \leqslant \|1_{\complement X^{(k)}}f\|_2 \liminf_{n\to\infty}\|\phi_n\|_{\gamma(X;E)}.\end{aligned}$$

By dominated convergence it follows that $\lim_{k\to\infty}\|T^{(k)}f - Tf\| = 0$. Another application of Theorem 4.1 (or its special case discussed after the proof) implies that $\phi\in\gamma(X;E)$ and

$$\begin{aligned}\|\phi\|_{\gamma(X;E)} &\leqslant \liminf_{k\to\infty}\|\phi^{(k)}\|_{\gamma(X;E)} \\ &\leqslant \liminf_{k\to\infty}(\liminf_{n\to\infty}\|\phi_n\|_{\gamma(X;E)}) = \liminf_{n\to\infty}\|\phi_n\|_{\gamma(X;E)}. \quad \blacksquare\end{aligned}$$

5. WEAK INTEGRALS OF γ-RADONIFYING OPERATORS IN SPACES WITH TYPE 2

In this section we will prove an analogue of Theorem 3.4 for functions with values in $\gamma(H,E)$. Throughout we assume E to be separable.

Let (X,λ) be a probability space and let $T\colon X\to\mathscr{L}(H,E)$ be a function such that $T(\xi)\in\gamma(H,E)$ for λ-almost all $\xi\in X$, and for all $h\in H$ the function $\xi\mapsto T(\xi)h$ is strongly measurable. A standard argument involving the Pettis measurability theorem and the separability of $\gamma(H,E)$ (which follows from the separability of H and E) shows that T is strongly measurable as a $\gamma(H,E)$-valued function. If

$$\int_X \|T(\xi)\|^2_{\gamma(H,E)}\,d\lambda(\xi) < \infty,$$

then for all $f\in L^2(X;H)$ the integral

$$I_T f := \int_X T(\xi)f(\xi)\,d\lambda(\xi)$$

converges as a Bochner integral, and the resulting operator $I_T\colon L^2(X;H)\to E$ is bounded. To see this, note that from the inequality $\|T(\xi)\|\leqslant\|T(\xi)\|_{\gamma(H,E)}$ it follows that $\xi\mapsto T(\xi)f(\xi)$ is strongly measurable, and the Cauchy–Schwarz inequality then gives

$$\int_X \|T(\xi)f(\xi)\|\,d\lambda(\xi) \leqslant \|T\|_{L^2(X;\gamma(H,E))}\|f\|_{L^2(X;H)} < \infty.$$

It follows that the function $\xi \mapsto T(\xi)f(\xi)$ is Bochner integrable. Hence the operator I_T is well defined and satisfies $\|I_T\| \leqslant \|T\|_{L^2(X;\gamma(H,E))}$. For later considerations we note that $I_T^* x^* = T^*(\cdot)x^*$ for all $x^* \in E^*$.

THEOREM 5.1. *If E has type 2, then under the above assumptions the operator $I_T\colon L^2(X; H) \to E$ is γ-radonifying and*

$$\|I_T\|^2_{\gamma(L^2(X;H),E)} \leqslant C_2^2 \int_X \|T(\xi)\|^2_{\gamma(H,E)}\,d\lambda(\xi).$$

Proof. Since $T\colon X \to \gamma(H, E)$ is strongly measurable, there exists a sequence (T_n) of $\gamma(H, E)$-valued step functions with the following properties:

(i) for all $n \geqslant 1$ we have $\int_X \|T_n\|^2_{\gamma(H,E)}\,dm \leqslant \int_X \|T\|^2_{\gamma(H,E)}\,d\lambda$;

(ii) $\lim_{n\to\infty} \int_X \|T_n - T\|^2_{\gamma(H,E)}\,d\lambda = 0$.

Let us write $T_n = \sum_{k=1}^{N_n} 1_{B_{k,n}} \otimes T_{k,n}$ with the $B_{k,n}$ measurable and disjoint and with $T_{k,n} \in \gamma(H, E)$. For each $n \geqslant 1$ let

$$Q_n = \sum_{k=1}^{N_n} \lambda(B_{k,n})\,Q_{k,n},$$

where $Q_{k,n} = T_{k,n} \circ T_{k,n}^*$. Let $i_n\colon H_n \hookrightarrow E$ denote the reproducing kernel Hilbert space associated with Q_n. Then $Q_n = i_n \circ i_n^*$. By Lemma 3.1, i_n is γ-radonifying and

$$\|i_n\|^2_{\gamma(H_n,E)} \leqslant C_2^2 \sum_{k=1}^{N_n} \lambda(B_{k,n})\,\|T_{k,n}\|^2_{\gamma(H,E)} \tag{5.1}$$
$$= C_2^2 \int_X \|T_n\|^2_{\gamma(H,E)}\,d\lambda \leqslant C_2^2 \int_X \|T\|^2_{\gamma(H,E)}\,d\lambda.$$

Let $Q := I_T \circ I_T^*$ and note that

$$\langle Qx^*, y^*\rangle = [I_T^* x^*, I_T^* y^*]_{L^2(X;H)}$$
$$= \int_X [T^*(\xi)x^*, T^*(\xi)y^*]_H\,d\lambda(\xi) \qquad \text{for all } x^*, y^* \in E^*.$$

Therefore for all $x^*, y^* \in E^*$ we have

$$|\langle Q_n x^*, y^*\rangle - \langle Qx^*, y^*\rangle| \leqslant \int_X |[T_n^*(\xi)x^* - T^*(\xi)x^*, T_n^*(\xi)y^*]_H|\,d\lambda(\xi) \tag{5.2}$$
$$+ \int_X |[T^*(\xi)x^*, T_n^*(\xi)y^* - T^*(\xi)y^*]_H|\,d\lambda(\xi),$$

which tends to 0 as $n \to \infty$ by (i), (ii), the Cauchy–Schwarz inequality, and the inequality $\|\cdot\| \leqslant \|\cdot\|_{\gamma(H,E)}$.

By (5.1) and (5.2) we may apply Theorem 3.2 to the operators Q_n and Q and infer that I_T is γ-radonifying with

$$\|I_T\|^2_{\gamma(L^2(X;H),E)} \leqslant C_2^2 \int_X \|T\|^2_{\gamma(H,E)}\, d\lambda. \blacksquare$$

COROLLARY 5.2. *Under the assumptions of Theorem* 5.1, *the operator* $T\colon H \to E$ *defined by*

$$Th := \int_X T(\xi) h d\lambda(\xi)$$

is γ*-radonifying and*

$$\|T\|^2_{\gamma(H,E)} \leqslant C_2^2 \int_X \|T(\xi)\|^2_{\gamma(H,E)}\, d\lambda(\xi).$$

Proof. This follows by restricting the operator I_T of Theorem 5.1 to the closed subspace of $L^2(X; H)$ consisting of all functions of the form $1 \otimes h$ with $h \in H$ and noting that $\|T\|_{\gamma(H,E)} \leqslant \|I_T\|_{\gamma(H,E)}$. $\blacksquare$

6. THE SPACES $L^2(X; \gamma(H, E))$ AND $\gamma(L^2(X; H), E)$

In this section we take an operator-theoretical look at Theorem 5.1. With the notation of the previous section, for simple functions $T\colon X \to \gamma(H, E)$ we have

$$I_T f = \int_X T(\xi) f(\xi)\, d\lambda(\xi), \tag{6.1}$$

where the right-hand side can be defined in an elementary way. We claim that the operators I_T belong to $\gamma\big(L^2(X; H), E\big)$ *regardless whether E has type* 2 *or not.* By linearity it is enough to prove this for simple functions of the form $T = 1_B \otimes S$, where $B \subseteq X$ has finite measure and $S \in \gamma(H, E)$. But then we have $I_T = S \circ i_B$, where $i_B\colon L^2(X; H) \to H$ is defined by

$$i_B f := \int_X 1_B(\xi) f(\xi)\, d\lambda(\xi).$$

Hence I_T is γ-radonifying by the right ideal property. The contents of Theorem 5.1 may be summarized by saying that if E has type 2, the mapping $I\colon T \mapsto I_T$ has a unique extension to a bounded operator

$$I\colon L^2\big(X; \gamma(H, E)\big) \hookrightarrow \gamma\big(L^2(X; H), E\big)$$

of norm $\|I_T\| \leqslant C_2$. In line with the development so far, we derived this result from the Fatou lemma for Gaussian covariances. We proceed with an independent and considerably more elementary proof of this result. The reason for including this argument will become apparent in the sequel when we prove a converse for spaces with cotype 2.

LEMMA 6.1. *If E has type 2, the mapping $I\colon T \mapsto I_T$ defined by* (6.1) *has a unique extension to a continuous embedding*

$$I\colon L^2(X; \gamma(H, E)) \hookrightarrow \gamma(L^2(X; H), E)$$

of norm $\|I\| \leqslant C_2$.

Proof. Consider a simple function $T = \sum_{m=1}^{M} 1_{B_m} \otimes T_m$, where the $B_m \subseteq X$ are disjoint and have finite positive measure, and $T_m \in \gamma(H, E)$ for all $m = 1, \ldots, M$. Choose an orthonormal basis $(h_n)_{n\geqslant 1}$ for H. By the separability of (X, λ), the space $L^2(X)$ is separable and we may choose an orthonormal basis $(f_m)_{m\geqslant 1}$ for $L^2(X)$, the first M elements of which are given by $f_m := 1/\sqrt{\lambda(B_m)}\, 1_{B_m}$. Then the doubly indexed sequence $(f_m \otimes h_n)_{m,n\geqslant 1}$ is an orthonormal basis in $L^2(X; H)$. Finally, choose a doubly indexed orthogaussian sequence $(g_{mn})_{m,n\geqslant 1}$ and an independent Rademacher sequence $(\tilde{\varepsilon}_m)_{m=1}^{M}$. Then, using orthogonality, the symmetry of the g_{mn}, Fubini's theorem, and the type 2 property, we estimate:

$$
\begin{aligned}
\text{(6.2)}\qquad \|I_T\|^2_{\gamma(L^2(X;H),E)} &= \mathbf{E}\Big\| \sum_{m,n\geqslant 1} g_{mn} I_T(f_m \otimes h_n)\Big\|^2 \\
&= \mathbf{E}\Big\| \sum_{m,n\geqslant 1} g_{mn} \int_X f_m(\xi)\, T(\xi)\, h_n\, d\lambda(\xi)\Big\|^2 \\
&= \mathbf{E}\Big\| \sum_{m,n\geqslant 1} g_{mn} \sum_{k=1}^{M} \int_X f_m(\xi)\, 1_{B_k}(\xi)\, T_k h_n\, d\lambda(\xi)\Big\|^2 \\
&= \mathbf{E}\Big\| \sum_{m=1}^{M} g_{mn} \sqrt{\lambda(B_m)} \sum_{n\geqslant 1} T_m h_n\Big\|^2 \\
&= \tilde{\mathbf{E}}\mathbf{E}\Big\| \sum_{m=1}^{M} \tilde{\varepsilon}_m g_{mn} \sqrt{\lambda(B_m)} \sum_{n\geqslant 1} T_m h_n\Big\|^2 \\
&\leqslant C_2^2\, \mathbf{E} \sum_{m=1}^{M} \lambda(B_m) \Big\| \sum_{n\geqslant 1} g_{mn} T_m h_n\Big\|^2 \\
&= C_2^2 \sum_{m=1}^{M} \lambda(B_m)\, \|T_m\|^2_{\gamma(H,E)} = C_2^2\, \|T\|^2_{L^2(X;\gamma(H,E))}.
\end{aligned}
$$

This proves that $I\colon T \mapsto I_T$ is bounded of norm $\|I\| \leqslant C_2$ on the dense subspace of all simple functions in $L^2(X; \gamma(H, E))$, and the unique extendability follows.

To check that I is an embedding, suppose that $I_{T_1} = I_{T_2}$ for certain $T_1, T_2 \in L^2(X; \gamma(H, E))$. Then from $\langle I_{T_1} f, x^*\rangle = \langle I_{T_2} f, x^*\rangle$ for all $f \in L^2(X; H)$ and $x^* \in E^*$ it follows that $T_1^* x^* = T_2^* x^*$ in $L^2(X; H)$ for all $x^* \in E^*$, and hence $\langle T_1 h, x^*\rangle = \langle T_2 h, x^*\rangle$ in $L^2(X)$ for all $h \in H$ and $x^* \in E^*$. By strong measurability this implies $T_1 h = T_2 h$ in $L^2(X; E)$ for all $h \in H$. Since H is separable, we obtain $T_1 = T_2$ λ-almost everywhere. ■

We proceed with an analogue of Lemma 6.1 for spaces with cotype 2. Recalling that $I\colon T \mapsto I_T$ is injective on the simple functions, we can infer that the inverse mapping $I^{-1}\colon I_T \mapsto T$ is well defined on the subspace $\gamma_0(L^2(X; H), E)$ of all operators $I \in \gamma(L^2(X; H), E)$ of the form $I = I_T$ with T simple.

Lemma 6.2. *If E has cotype 2, the mapping I^{-1} has a unique extension to a continuous embedding*

$$I^{-1}\colon \gamma(L^2(X; H), E) \hookrightarrow L^2(X; \gamma(H, E))$$

of norm $\|I^{-1}\| \leqslant c_2$, where c_2 is the cotype 2 constant of E.

Proof. By reversing the estimates in (6.2) we see that the operator I^{-1} is bounded from $\gamma_0(L^2(X; H), E)$ into $L^2(X; \gamma(H, E))$ of norm $\|I^{-1}\| \leqslant c_2(E)$. By an easy approximation argument, $\gamma_0(L^2(X; H), E)$ is dense in $\gamma(L^2(X; H), E)$ and the unique extendability follows.

To see that I^{-1} is injective, define $J\colon L^2(X; \gamma(H, E)) \to \mathscr{L}(L^2(X; H), E)$ by

$$(JT)f := \int_X T(\xi) f(\xi)\, d\lambda(\xi)$$

and let $j\colon \gamma(L^2(X; H), E) \hookrightarrow \mathscr{L}(L^2(X; H), E)$ be the natural inclusion mapping. On $\gamma_0(L^2(X; H), E)$ we have $J \circ I^{-1} = j$ and by continuity this identity extends to all of $\gamma(L^2(X; H), E)$. Hence if $I^{-1}S_1 = I^{-1}S_2$ for certain $S_1, S_2 \in \gamma(L^2(X; H), E)$, then $jS_1 = jS_2$ as elements of $\mathscr{L}(L^2(X; H), E)$, and therefore $S_1 = S_2$. ∎

By Theorems 6.1 and 6.2 one expects that the inclusion

$$L^2(X; \gamma(H, E)) \hookrightarrow \gamma(L^2(X; H), E)$$

is proper when E is a space with type 2 but not with cotype 2, and similarly that the inclusion

$$\gamma(L^2(X; H), E) \hookrightarrow L^2(X; \gamma(H, E))$$

is proper when E is a space with cotype 2 but not with type 2. The following examples confirm this for the spaces l^p in the appropriate ranges of p.

In fact, the first example shows that in case of type 2 it may even happen that I_T is γ-radonifying while none of the integrated operators $T(\xi)$ has this property.

Example 6.3. Let $H = l^2$ and $E = l^p$ with $2 < p < \infty$. For $k = 1, 2, \ldots$ we choose sets $A_k \subseteq [0, 1]$ of Lebesgue measure $1/k$ in such a way that for all $t \in [0, 1]$ we have

$$\#\{k \geqslant 1\colon t \in A_k\} = \infty. \tag{6.3}$$

Define the operators $T(t)\colon l^2 \to l^p$ as coordinatewise multiplication with the sequence $(a_1(t), a_2(t), \ldots)$, where

$$a_k(t) = \begin{cases} 1 & \text{if } t \in A_k, \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

Then $\|T(t)\| = 1$ for all $t \in [0, 1]$ and none of the operators $T(t)$ is γ-radonifying. Indeed, by Theorem V.5.6 in [9], $Q(t) := T(t) \circ T^*(t)$ is a Gaussian covariance operator if and only if

$$\sum_{k=1}^{\infty} \langle Q(t) u_k^*, u_k^* \rangle^{p/2} < \infty,$$

where u_k^* denotes the k-th unit vector of l^q $(1/p + 1/q = 1)$. From $Q(t) u_k^* = a_k^2(t) u_k$, where u_k is the k-th unit vector of l^p, and from (6.3) and (6.4) we see that this sum diverges for all $t \in [0, 1]$.

The operator $I_T\colon L^2([0, 1]; l^2) \to l^p$, $I_T f := \int_0^1 T(t) f(t)\, dt$, is well defined and bounded. Putting $Q_T := I_T \circ I_T^*$, we have

$$\langle Q_T u_k^*, u_k^* \rangle = \int_0^1 \langle Q(t) u_k^*, u_k^* \rangle\, dt = \int_0^1 a_k^2(t)\, dt = |A_k| = 1/k. \tag{6.5}$$

Consequently,

$$\sum_{k \geqslant 1} \langle Q_T u_k^*, u_k^* \rangle^{p/2} = \sum_{k \geqslant 1} \frac{1}{k^{p/2}} < \infty.$$

It follows that Q_T is a Gaussian covariance operator and I_T is γ-radonifying. Note that by the first identity in (6.5) and polarization we have

$$Q_T u^* = \int_0^1 Q(t) u^*\, dt$$

for all $u^* \in l^q$, i.e., Q_T is the integral of the function $t \mapsto Q(t)$.

The next example shows that in case of cotype 2 there exist functions in $L^\infty([0, 1]; \gamma(l^2, l^p))$ which do not represent an element of $\gamma(L^2([0, 1]; l^2), l^p)$:

EXAMPLE 6.4. Let $H = l^2$ and $E = l^p$ with $1 < p < 2$. For $k = 1, 2, \ldots$ we now choose sets $A_k \subseteq [0, 1]$ of Lebesgue measure $1/k^{2/p}$ in such a way that for all $t \in [0, 1]$ we have

$$\#\{k \geqslant 1 : t \in A_k\} \leqslant N,$$

where N is an arbitrary fixed integer greater than $\sum_{k \geqslant 1} 1/k^{2/p}$. As before we define the operators $T(t)\colon l^2 \to l^p$ as coordinatewise multiplication with the sequence $(a_1(t), a_2(t), \ldots)$ defined as in (6.4). For all $t \in [0, 1]$, $T(t)$ is γ-radonifying and

$$\|T(t)\|_{\gamma(l^2, l^p)} \leqslant C_p N^{2/p},$$

where the constant C_p depends on p only. For each $h \in l^2$ the function $t \mapsto T(t)h$ is strongly measurable, and by the separability of $\gamma(l^2, l^p)$ this easily implies the strong measurability of $t \mapsto T(t)$. Consequently we obtain $T \in L^\infty([0, 1]; \gamma(l^2, l^p))$. However, the corresponding operator $I_T \in \mathscr{L}(L^2([0, 1]; l^2), l^p)$ fails to be γ-radonifying. Indeed, with the notation of the previous example we have

$$\sum_{k \geqslant 1} \langle Qu_k^*, u_k^* \rangle^{p/2} = \sum_{k \geqslant 1} \frac{1}{(k^{2/p})^{p/2}} = \sum_{k \geqslant 1} \frac{1}{k} = \infty.$$

REFERENCES

[1] T. W. Anderson, *The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities*, Proc. Amer. Math. Soc. 6 (1955), pp. 170–176.

[2] V. I. Bogachev, *Gaussian Measures*, Math. Surveys and Monographs, Vol. 62, Amer. Math. Soc., Providence, RI, 1998.

[3] Z. Brzeźniak and J. M. A. M. van Neerven, *Stochastic convolution in separable Banach spaces and the stochastic linear Cauchy problem*, Studia Math. 143 (2000), pp. 43–74.

[4] J. Diestel, H. Jarchow and A. Tonge, *Absolutely Summing Operators*, Cambridge Studies in Adv. Math., Vol. 34, Cambridge 1995.

[5] N. Kalton and L. Weis, *The H^∞-functional calculus and square function estimates*, in preparation.

[6] M. Ledoux and M. Talagrand, *Probability in Banach Spaces*, Ergebnisse d. Math. u. ihre Grenzgebiete, Vol. 23, Springer, 1991.

[7] W. Linde and A. Pietsch, *Mappings of Gaussian cylindrical measures in Banach spaces*, Theory Probab. Appl. 19 (1974), pp. 445–460.

[8] J. M. A. M. van Neerven and L. Weis, *Stochastic integration of functions with values in a Banach space*, Studia Math. 166 (2005), pp. 131–170.

[9] N. N. Vakhania, V. I. Tarieladze, and S. A. Chobanyan, *Probability Distributions in Banach Spaces*, D. Reidel Publishing Company, Dordrecht 1987.

Delft Institute of Applied Mathematics
Technical University of Delft
P.O. Box 5031
2600 GA Delft, The Netherlands
E-mail: J.vanNeerven@math.tudelft.nl

Mathematisches Institut I
Technische Universität Karlsruhe
D-76128 Karlsruhe, Germany
E-mail: Lutz.Weis@math.uni-karlsruhe.de

Received on 5.11.2004;
revised version on 4.2.2005