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PROBABILITY OF FAILURE WITH DISCRETE CLAIM DISTRIBUTION

BY

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Abstract. A concept of the probability of failure is introduced. Some popular methods of exact computation of ruin probability are adopted to compute failure probability. Based on the formula presented in [5] a generalization of a ruin probability algorithm is proposed that can also be used for failure probability. The algorithm's computational complexity is studied and it is proved to be more effective for failure probability than for ruin probability. Finally, some numerical examples for failure probability computations are given.

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1. INTRODUCTION

Let us consider the standard risk process

$$U(t) = u + ct - \sum_{i=1}^{N_t} X_i, \tag{1}$$

where u is the initial capital of an insurance company, ct represents the premium income, and N_t is a Poisson random variable with mean $t\lambda$. Let N_t, X_1, $X_2, \ldots$ be independent and $X_1, X_2, \ldots$ be identically distributed positive insurance claims. u is the initial capital and c is the income rate. The sum $\sum_{i=1}^{N_t} X_i$ will be further denoted by S_t.

One of the fundamental problems in both theoretical and practical approaches in actuarial literature (e.g. [2], [3]) is the problem of the time of ruin of the company whose capital is described by the risk process. Let the time of ruin be denoted by

$$R = \begin{cases} \inf\{t\colon\ t > 0 \ \wedge\ U(t) < 0\} & \text{if the ruin occurs,} \\ +\infty & \text{otherwise.} \end{cases}$$

Let us consider ruin probability on the finite time horizon $[0, T]$, i.e.

$$\psi(u, T) = P(R \leqslant T).$$

Although the ruin probability problem plays a central role in insurance mathematics, another problem can be of equal importance for an insurance company. The company usually wishes not only to survive the next year, but also expects a reasonable rate of return. Let us introduce the probability of failure.

DEFINITION. For $0 \leqslant w$ the *probability of failure* in time T is determined by

$$\psi(u, T, w) = P\big(R < T \vee (R \geqslant T \wedge U(T) < w)\big). \tag{2}$$

For convenience we will denote the *probability of non-failure* by

$$\delta(u, T, w) = 1 - \psi(u, T, w).$$

The reason why this problem is worth considering can be illustrated by a question asked by the investor with initial capital u: *what is the probability that the company will not go bankrupt and will bring interest not smaller than the risk-free financial instruments during time T?* This probability can be mathematically expressed as $1 - \psi\big(u, T, (1+i)^T u\big)$, where i is the risk-free interest rate. Figure 1 illustrates the investor's dilemma.

The probability of failure can be also viewed as a natural generalization of the probability of ruin in finite time while

$$\psi(u, T) = \psi(u, T, -\infty).$$

Note that – unlike ruin probability – the concept of failure probability makes sense only in the finite time case.

In this paper, we will consider the discrete claim distribution, i.e. $P(X \in N) = 1$. Every continuous claim distribution with probability density function $p(x)$ can be approximated by a discrete probability function $P(x)$, so

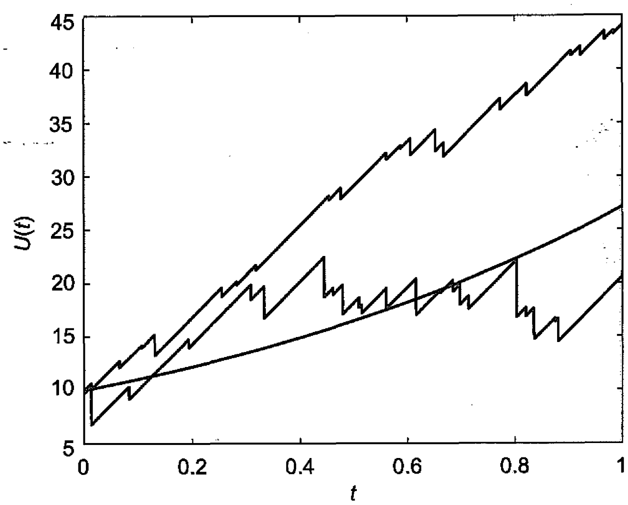

FIGURE 1. None of the above realizations of the risk process causes a ruin, however one of them causes the failure for $w = 27$ (or $i = 1.7$).

that it is not a strong limitation. One way to do this is to put $P(X = n) = \int_n^{n+1} p(u)\,du$ for $n \in N$, but there are many other approximation possibilities of course. For convenience we will also assume that $u + cT \in N$.

The remainder of this paper is organized as follows: in Section 2, generalizations of two ruin probability algorithms for discrete claims are presented. These generalizations allow to calculate failure probabilities. A brief study of computational complexity of one of them is provided. In Section 3, a similar generalization is proposed for the discrete time model. Finally, Section 4 contains two numerical examples of the application of the probability of failure.

2. CONTINUOUS TIME MODELS

2.1. Failure probability based on conditional probabilities. An important approach to ruin probability was presented by Ignatov and Kaishev in [5]. Let $x = (x_1, x_2, \ldots)$ be subsequent discrete claims. Let the function $b_i(c_1, \ldots, c_i)$ be defined as follows: $b_0 = 1$, $b_1 = c_1$ and

$$b_i(c_1, \ldots, c_i) = \det \begin{pmatrix} \frac{c_1}{1!} & 1 & 0 & \ldots & 0 \\ \frac{c_2^2}{2!} & \frac{c_2}{1!} & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{c_{i-1}^{i-1}}{(i-1)!} & \frac{c_{i-1}^{i-2}}{(i-2)!} & \frac{c_{i-1}^{i-3}}{(i-3)!} & \ldots & 1 \\ \frac{c_i^i}{i!} & \frac{c_i^{i-1}}{(i-1)!} & \frac{c_i^{i-2}}{(i-2)!} & \ldots & \frac{c_i}{1!} \end{pmatrix}.$$

Equation (33) from [5] states that if the vector of claims $\bar{x} = (x_i)_i$ is given, then the non-ruin probability is

$$\delta(u, T \mid \bar{X} = \bar{x}) = e^{-T} K_{\bar{x}}, \tag{3}$$

where

$$K_{\bar{x}} = \sum_{j=0}^{k_{\bar{x}}-1} (-1)^j b_j\left(\frac{x_1 - u}{c}, \ldots, \frac{x_1 + \ldots + x_j - u}{c}\right) \sum_{m=0}^{k_{\bar{x}}-1-j} \frac{T^m}{m!}$$

and $k_{\bar{x}}$ denotes a value such that for $n = cT + u + 1$

$$x_1 + \ldots + x_{k_{\bar{x}}-1} \leqslant n - 1 < x_1 + \ldots + x_{k_{\bar{x}}}.$$

The non-conditional ruin probability can be now obtained as a sum over all possible claims' vectors, i.e.

$$\delta(u, T) = e^{-T} \sum_{1 \leqslant x_1, \ldots, 1 \leqslant x_n} P(X_1 = x_1, \ldots, X_n = x_n) K_{\bar{x}}.$$

Remark. In the recursive calculation of the $K_{\bar{x}}$ only the determinant of the largest matrix, i.e. $B_{\max} = b_{k_{\bar{x}}-1}$, is critical, as the other determinants are side effects of the recursive calculation of $B_{\max}$.

The following claim shows how the Ignatov–Kaishev method can be generalized in order to be used to calculate the failure probability.

CLAIM 2.1. *Let $n' = cT+u+1-w$ and let $k'_{\bar{x}}$ denote a value such that*

$$x_1 + \ldots + x_{k'_{\bar{x}}-1} \leqslant n'-1 < x_1 + \ldots + x_{k'_{\bar{x}}}$$

and let $K'_{\bar{x}}$ be defined like $K_{\bar{x}}$ but with $k_{\bar{x}}$ replaced by $k'_{\bar{x}}$. Then the probability of non-failure as defined in (2) can be expressed as

$$\delta(u, T, w \mid \bar{X} = \bar{x}) = e^{-T} K'_{\bar{x}}.$$

Proof. Let the claims $\bar{X} = \bar{x}$ be given. Let τ_1 denote the moment when the first claim occurs and τ_i be the waiting time between the $(i-1)$-st and i-th claim for $i > 1$. Then the following holds:

$$P(R > T \wedge U(T) \geqslant w \mid \bar{X} = \bar{x}) = P\left(\bigcap_{i=1}^{k'_{\bar{x}}}\left(\tau_1 + \ldots + \tau_i \geqslant \min\left(T, \frac{x_1 + \ldots + x_i - u}{c}\right)\right)\right).$$

Since the equality

$$P\left(\bigcap_{i=1}^{k}\left(\tau_1 + \ldots + \tau_i \geqslant \min\left(T, \frac{x_1 + \ldots + x_i - u}{c}\right)\right)\right) = e^{-T}\sum_{j=0}^{k-1}(-1)^j b_j\left(\frac{x_1-u}{c}, \ldots, \frac{x_1 + \ldots + x_j - u}{c}\right)\sum_{m=0}^{k-1-j}\frac{T^m}{m!}$$

was proved in [5] without any specific assumptions about k, exactly the same procedure can be used to prove the claim where k is replaced by $k'_{\bar{x}}$. ∎

Now we obtain

COROLLARY 2.2. *The result of Ignatov and Kaishev can be generalized to deliver the probability of non-failure*:

$$\delta(u, T, w) = e^{-T}\sum_{1 \leqslant x_1, \ldots, 1 \leqslant x_{n'}} P(X_1 = x_1, \ldots, X_{n'} = x_{n'}) K'_{\bar{x}}. \tag{4}$$

The problem with the above is that it contains an infinite sum. Due to this sum, the equality cannot be applied in a numerical algorithm. Therefore, a finite equivalent of formula (4) is needed. It is provided by the following

THEOREM 2.3. *Let $n' = cT+u+1-w$. Furthermore, let the singleton $C_1^{n'} = \{(n', n', \ldots)\}$ contain an infinite sequence and for $m > 1$ let $C_m^{n'}$ be a set of sequences such that for each element $\bar{x} \in C_m^{n'}$ the following holds*:

(i) *for all i, $x_i \in \{1, 2, \ldots\}$*;

(ii) $\sum_{i=1}^{m-1} x_i \leqslant n'-1$;

(iii) *for all* $i \geqslant m$, $x_i = n'$.

Then

$$(5)\qquad \delta(u, T, w) = e^{-T} \sum_{i=1}^{n'} \sum_{\bar{x} \in C_i^{n'}} P(X_1 = x_1, \ldots, X_{i-1} = x_{i-1}, X_i \geqslant n') K'_{\bar{x}}.$$

Proof. Let $k'_{\bar{x}}$ be defined as in Claim 2.1 and let D_i be a set such that $\bar{x} \in D_i \Leftrightarrow k'_{\bar{x}} = i$. Since $1 \leqslant k'(x) \leqslant n'$, it is obvious that

$$\begin{aligned}
\delta(u, T, w) &= \sum_{i=1}^{n'} \sum_{\bar{x} \in D_i} P(\bar{X} = \bar{x}) \delta(u, T, w \mid \bar{X} = \bar{x}) \\
&= \sum_{i=1}^{n'} \sum_{\bar{x} \in D_i} P(\bar{X} = \bar{x}) \delta(u, T, w \mid X_1 = x_1, \ldots, X_i = x_i) \\
&= \sum_{i=1}^{n'} \sum_{\bar{x} \in D_i} P(\bar{X} = \bar{x}) \\
&\qquad \times \delta(u, T, w \mid X_1 = x_1, \ldots, X_{i-1} = x_{i-1}, X_i = n', X_{i+1} = n', \ldots).
\end{aligned}$$

If two different vectors $\bar{x}$ and $\bar{y}$ are members of D_i and $x_j = y_j$ for $j < i$, then

$$\delta(u, T, w \mid \bar{X} = \bar{x}) = \delta(u, T, w \mid \bar{X} = \bar{y}).$$

Hence

$$\begin{aligned}
&\sum_{i=1}^{n'} \sum_{\bar{x} \in D_i} P(\bar{X} = \bar{x}) \\
&\qquad \times \delta(u, T, w \mid X_1 = x_1, \ldots, X_{i-1} = x_{i-1}, X_i = n', X_{i+1} = n', \ldots) \\
&= \sum_{i=1}^{n'} \sum_{\bar{x} \in C_i^{n'}} P(X_1 = x_1, \ldots, X_{i-1} = x_{i-1}, X_i = n') \delta(u, T, w \mid \bar{X} = \bar{x}).
\end{aligned}$$

Now, applying Claim 2.1, we have

$$\begin{aligned}
&\sum_{i=1}^{n'} \sum_{\bar{x} \in C_i^{n'}} P(X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) \delta(u, T, w \mid \bar{X} = \bar{x}) \\
&\qquad = e^{-T} \sum_{i=1}^{n'} \sum_{\bar{x} \in C_i^{n'}} P(X_1 = x_1, \ldots, X_{i-1} = x_{i-1}, X_i = n') K'_{\bar{x}}. \quad \blacksquare
\end{aligned}$$

2.2. Computational complexity of the Ignatov–Kaishev method. Since the infinite sum (4) was reduced to a finite sum in (5), this formula can be now used in practical failure probability applications. Now, it would be interesting to determine the numerical complexity of calculating (5). Because the determination of the sum $K'_{\bar{x}}$ for each claim vector is the critical numerical problem, we will consider the computational complexity of the algorithm in terms of the required number of computations of $K'_{\bar{x}}$.

THEOREM 2.4. *The computational complexity of the naive algorithm for determining* $1-\psi(u, T, w)$ *is* $O(2^{n'})$, *where* $n' = u+cT+1-w$.

Proof. Let us first consider $\# C_i^{n'}$ — the size of the set $C_i^{n'}$. We know that $\# C_i^{n'}$ equals the number of all possible ways of packing n' undistinguishable balls into $(i-1)+1 = i$ distinguishable boxes (the extra box is added to allow the $i-1$ 'real' boxes to contain less than n balls) in such a way that each of the $i-1$ boxes contains at least one ball. This is equal to the number of possibilities of packing $n'-(i-1)$ balls into i boxes, i.e.

$$\binom{n'-(i-1)+i-1}{i-1} = \binom{n'}{i-1}.$$

We are now interested in

$$\sum_{i=1}^{n'} \# C_i^{n'} = \sum_{i=1}^{n'} \binom{n'}{i-1}.$$

The above equals the number of all possible proper subsets of a set consisting of n' elements. Hence

$$\sum_{i=1}^{n'} \binom{n'}{i-1} = 2^{n'}-1. \blacksquare$$

It is clearly seen that the parameter $n'-1$, the maximal allowed total claim, plays a critical role in the efficiency of the algorithm. In case of the ruin probability $n'-1$ is chosen largest possible, namely $n'-1 = u+cT$. Hence the complexity of $O(2^{n'})$ for this algorithm is not satisfying in case of ruin probability. However, it is clear that we can use the same algorithm in a far more effective way if we are interested in computing failure probability instead of ruin probability.

2.3. Failure probability based on Appel polynomials. In [6] Lefevre and Picard solved the classical ruin problem using the generalized Appel polynomials. For the sake of simplicity we will assume that $\lambda = 1$. Let the auxiliary polynomial $e_n(x)$ be defined as

$$e_n(x) = \begin{cases} 1 & \text{if } n = 0, \\ \sum_{i=0}^{n} \frac{x^i}{i!} P\left(\sum_{j=1}^{i} X_j = n\right) & \text{if } n > 0. \end{cases}$$

The generalized Appel polynomial can be now defined as

$$A_n(x) = \begin{cases} e_n(x) & \text{if } 0 \leqslant n \leqslant u, \\ \sum_{j=0}^{u} \frac{cx-n+u}{cx-j+u} e_j\left(\frac{j-u}{c}\right) e_{n-j}\left(x+\frac{u-j}{c}\right) & \text{if } n > u. \end{cases}$$

Then, according to [6], the probability of non-ruin in the finite time T can be expressed as

$$P(R > T \wedge S_n = n) = e^{-T} A_n(T) \mathbf{1}_{\{T \geqslant v_n\}}, \tag{6}$$

where $v_n = (n-u)/c$. Namely,

$$1-\psi(u, T) = e^{-T} \sum_{n=0}^{\infty} A_n(T) \mathbf{1}_{\{T \geqslant v_n\}}.$$

Again, we can see that this non-ruin probability construction can be generalized to provide the probability of non-failure in a very intuitive way.

CLAIM 2.5. *Non-failure probability can be expressed using the generalized Appel polynomials by*

$$1-\psi(u, T, w) = e^{-T} \sum_{n=0}^{u+cT-w} A_n(T).$$

Proof. The equality is a simple consequence of (6). ∎

The problem with this elegant result is that the Appel polynomials provide numerical complexity, and it is not a trivial task to use them efficiently. We will not study the complexity of this approach here. Some ideas of how the Appel polynomials can be handled numerically and effectively can be found in [1].

3. FAILURE PROBABILITY IN DISCRETE TIME

In this section we will consider a discrete time model, e.g. $t = 1, 2, \ldots, T$. Without loss of generality we assume that the premium revenue per time unit (say a year) is one. In this model, the ruin may occur only at the beginning of a year, i.e. for $t = 1, 2, \ldots, T$. The claims are i.i.d. and the number of claims is independent on their sizes as in the previous model. Let Y_i be the aggregated claim in the i-th year. Y_i's are also i.i.d. and we denote the aggregate probability function by $f(x) = P(Y_1 = x)$.

The model presented in this section is a modification of the one proposed by De Vylder and Goovaerts in [7] and recalled by Dickson in [4]. The failure in one step is simply expressed as:

$$\psi(u, 1, w) = 1 - \sum_{i=0}^{u-w+1} f(i).$$

If we assume that the failure occurs, then either the ruin occurs in the first step or the ruin does not occur in the first step, but the failure occurs during the next $T-1$ steps. This can be expressed as the recursive equation:

$$\psi(u, T, w) = \psi(u, 1, 0) + \sum_{j=0}^{u+1} f(j)\psi(u+1-j, T-1, w).$$

To improve the numerical efficiency of this recursive algorithm, a truncation procedure similar to the one introduced in [7] can be used. The idea is to use a function $f^{\varepsilon}(x)$ instead of the original $f(x)$. Let $\varepsilon > 0$ be small and k be the largest natural number such that $\sum_{i=0}^{k} f(x) \leqslant 1-\varepsilon$. We have

$$f^{\varepsilon}(x) = \begin{cases} f(x) & \text{if } x \leqslant k, \\ 0 & \text{otherwise.} \end{cases}$$

Let now $\psi^{\varepsilon}(u, T, w)$ denote the modified failure probability calculated recursively using the modified function $f^{\varepsilon}(x)$ as follows:

$$\psi^{\varepsilon}(u, 1, w) = \begin{cases} \psi(u, 1, w) & \text{if } u \leqslant k, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\psi^{\varepsilon}(u, T, w) = \psi^{\varepsilon}(u, 1, 0) + \sum_{j=0}^{u+1} f(j)\, \psi^{\varepsilon}(u+1-j, T-1, w). \tag{7}$$

This improvement is justified by the following

THEOREM 3.1. *If* $k > T+u$, *then*

$$\psi^{\varepsilon}(u, T, w) \leqslant \psi(u, T, w) \leqslant \psi^{\varepsilon}(u, T, w) + T\varepsilon.$$

Proof. The first inequality is obvious. Let $U^{\varepsilon}(t)$ denote the modified risk process which is a copy of the original process but with the only difference that if a claim of size larger than k happens in the $U(t)$, then a claim of size ∞ happens in the $U^{\varepsilon}(t)$. While the aggregated claims are independent in each time unit, the probability that $U(t) = U^{\varepsilon}(t)$ equals $(1-\varepsilon)^T$. We have

$$(1-\varepsilon)^T \geqslant 1 - T\varepsilon.$$

The above inequality is clear for $T = 0$. Assuming that it is true for T, let us prove it for $T+1$:

$$\begin{aligned}(1-\varepsilon)^{T+1} &= (1-\varepsilon)(1-\varepsilon)^T \geqslant (1-\varepsilon)(1-T\varepsilon) = 1 - T\varepsilon - \varepsilon + T\varepsilon^2 \\ &\geqslant 1 - T\varepsilon - \varepsilon = 1-(T+1)\varepsilon.\end{aligned}$$

Hence, the probability that $U(t) > U^{\varepsilon}(t)$ does not exceed $T\varepsilon$. Assuming that the failure happened each time $U(t) > U^{\varepsilon}(t)$, the probability of failure cannot exceed $\psi^{\varepsilon}(u, t, w) + T\varepsilon$. ∎

4. NUMERICAL EXAMPLES

We present numerical examples for the calculation of the failure probabilities in the discrete time model from the previous section. We choose one heavy-tailed truncated Pareto distribution and one light-tailed truncated exponential distribution as single claim distributions. The same claim distributions were considered in [4].

The computations were performed using the recursive algorithm expressed by formula (7). They were performed for different initial capitals u and different final capitals w. The results are presented as functions depending on w. The time horizon for all calculations was set to 10. Figure 2 presents the failure probabilities for the risk processes and the time required to compute them for different initial and final capitals. It is not surprising that the failure probability grows with w and falls with the initial capital u and that for a small initial capital and large w the failure is sure.

More interesting is the behavior of the CPU time required to compute probability as a function of w. As could have been expected, the CPU time falls

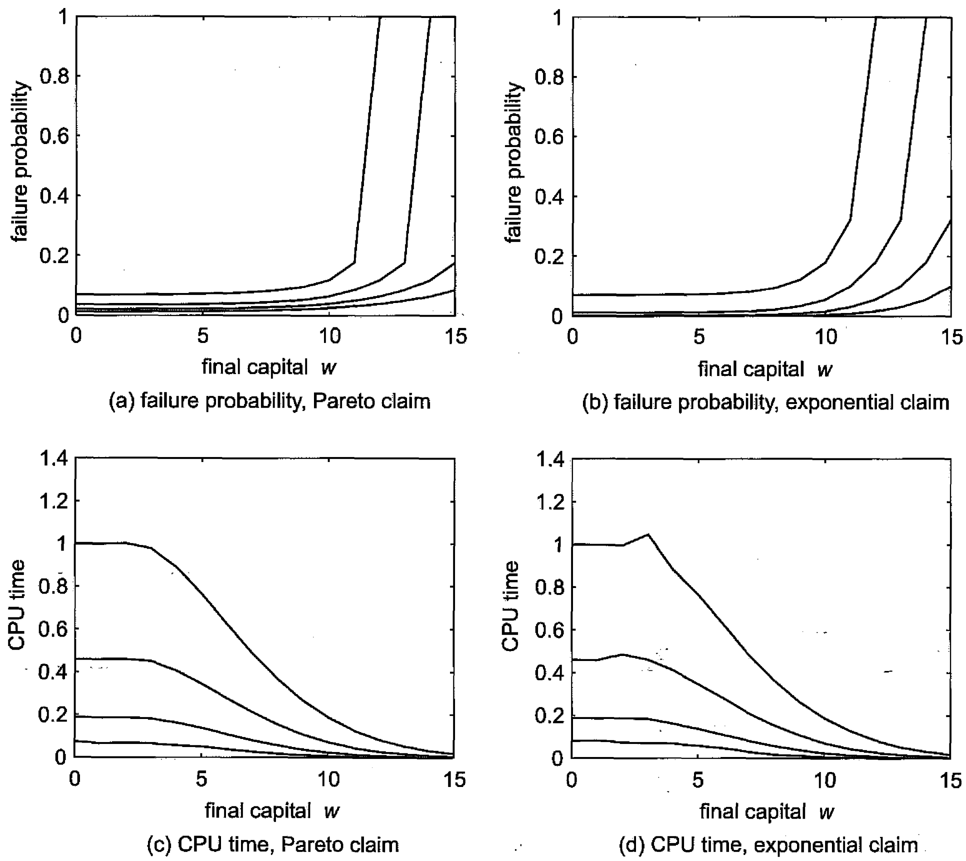

FIGURE 2. Failure probabilities and CPU computation times for the risk processes starting from the initial capital of one, three, five and seven, respectively (from top to bottom in the upper panel and from bottom to top in the lower panel). The left panel was obtained for the Pareto claim distribution, the right panel – for exponential claim distribution.

rapidly with w. In fact, the slope is largest for large initial capital. The empirical results show that the discrete time failure probability is computationally less expensive than the ruin probability. These results accord with analogous result obtained in Theorem 2.4 for continuous time. This can be a strong motivation for using failure probability instead of pure ruin probabilities in some practical applications.

5. CONCLUSION

The failure probability problem is a natural generalization of the ruin probability in a finite time horizon. In many practical cases it can be even more important to know the failure probability than just to be able to determine only ruin probability.

Many popular methods of solving the ruin probability problem can be adopted to solve the failure probability problem as well. Moreover, in many cases the modified methods have better computational complexity and are less time-consuming. The above facts confirm that failure probability is an interesting and valid subject of study.

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