PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 25, Fasc. 2 (2005), pp. 363–383

DISTRIBUTIONAL ANALYSIS OF THE STOCKS COMPRISING THE DAX 30

BY

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Abstract. In this paper, we analyze the returns of stocks comprising the German stock index DAX with respect to the α-stable distribution. We apply nonparametric estimation methods such as the Hill estimator as well as parametric estimation methods conditional on the α-stable distribution. We find for both the nonparametric and parametric estimation methods that the α-stable hypothesis cannot be rejected for the return distribution. We then employ the GARCH model; the fit of innovations modeled with an underlying α-stable distribution is compared to the fit obtained from modeling the innovations with the skew-t distribution. The α-stable distribution is found to outperform the skew-t distribution.

AMS Subject Classification: 91B28, 62M10, 62M15.

Key words and phrases: Stable distributions, heavy-tails, tail estimation, ARMA-GARCH, DAX 30.

1. INTRODUCTION

As shown in [23] and [7], stock returns have an underlying heavy-tailed distribution. In other words, they are leptokurtic. This can also be found in [5] and [3]. What followed these initial findings was a vast amount of monographs and articles covering the stock price behavior with emphasis on the U.S. capital market. An exhaustive account of these studies is provided in [31]. Research with respect to this issue for the German equity market is not so extensive. Studies since the 1980s that focus on German stocks include [1], [17], [20]–[22], [33], [35], [37], and [38]. Studies in the 1970s are described in [27].

In this paper, we investigate the distribution behavior of daily logarithmic stock returns for German blue chip companies. While the distribution that is assumed in major theories in finance and risk management is the Gaussian distribution, we show that the α-stable distribution offers a reasonable improvement if not the best choice among the alternative distributions that have been proposed in the literature over the past four decades.

The most important equity index in Germany is the DAX® index which contains the 30 most liquid German blue chip stocks. Prices used to compute the return were obtained from the Frankfurt Stock Exchange.[1] The return for each stock includes cash dividends and is adjusted for stock splits and capital adjustments. The period investigated is January 1, 1988, through September 30, 2002.

Inclusion in the DAX depends on requirements such as market capitalization and trading volume. As a result, some of the 30 constituent stocks are periodically replaced by others. During the period of investigation, there were 55 stocks that had been included in the DAX. To assure that the statistics estimated were generated from sufficient data, we restricted the sample to stocks with a minimum of 1,000 observations. This reduces the original number of candidate stocks from 55 to 35.

The problems related to the correct assessment of the empirical distribution of the returns are with respect to the overall shape, tail estimation, and determination of existing moments. Particularly in the context of finite sample observations, the last can easily lead one to mistakenly conclude in favor of distributions with lighter tails. To exemplify, the moments of a Gaussian distribution exist to all orders. This is not the case, for example, with the Pareto or Student's t-distributions even though sample moments of those distributions exist since data samples are finite. It can be shown that even these can grow quickly with increasing order which is usually the case with financial data.

The paper is organized as follows. In the next section, the basic notion of α-stable random variables is reviewed. In Section 3, we present the results based on nonparametric estimation methods for the return distribution. Section 4 provides methods and results of the parametric estimation techniques conditional on the α-stable class of distributions. Section 5 models volatility clustering based on different error distributions and reports the results of the alternative GARCH models. A summary of our findings is presented in the final section.

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[1] In addition, the automated quotations for the same stocks from the Xetra® were analyzed to determine whether there are deviations in the results caused by the slightly different regulatory procedures offered by the two exchanges. However, since significant differences between the prices were not observed, results for the automated quotation are omitted. The stock prices from both sources were provided by the capital market database Karlsruher Kapitalmarkt Datenbank (KKMDB) at the University of Karlsruhe.

2. DEFINITION OF α-STABLE DISTRIBUTIONS

For an exhaustive treatment of the topic of α-stable randomness, the monograph [34] should be consulted which has become a standard in this field. Here, a brief idea is given as to what the meaning of α-stable distributions implies, in the definition below.

DEFINITION 2.1. If for any $a, b > 0$ and independent copies X_1, X_2 of X there exist $c > 0$ and $d \in \boldsymbol{R}$ such that

$$aX_1 + bX_2 \stackrel{d}{=} cX + d, \tag{1}$$

where $\stackrel{d}{=}$ denotes equality in distribution, then X is a *stable random variable.*

Generally, α-stable random quantities are described by the quadruple $(\alpha, \beta, \sigma, \mu)$ or, with the notation of [34], $S_\alpha(\sigma, \beta, \mu)$, where the index of stability, α, is the characteristic parameter of the tail as well as the peak at the median. Scaling is described by σ, β indicates the degree of skewness whereas μ is the location parameter which is not necessarily the mean.

An important property of the α-stable random variables is that they can be looked upon as the distributional limit of a standardized sum of an increasing number of i.i.d. random variables. They are said to have a *domain of attraction* (DA). This is a generalization of the central limit theorem known for the Gaussian distribution. Note that the normal distribution is a special case of the α-stable distributions. In that case, $\alpha = 2$, β is meaningless, μ is the mean, and the variance is $2\sigma^2$.

Even though an analytical form of the probability density function (pdf) does not exist for most combinations of the four parameters, the distribution can be identified by the unique characteristic functions which are given to be as in the following

DEFINITION 2.2. X is said to be *stable* if there exist $0 < \alpha \leqslant 2$, $\sigma \geqslant 0$, $\beta \in [-1, 1]$, and $\mu \in \boldsymbol{R}$ such that

$$\Phi(\theta) \equiv \begin{cases} \exp\{-\sigma^\alpha |\theta|^\alpha (1 - i\beta (\operatorname{sign}\theta) \tan((\pi\alpha)/2)) + i\mu\theta\}, & \alpha \neq 1, \\ \exp\{-\sigma|\theta| (1 + i\beta \cdot 2\pi^{-1} (\operatorname{sign}\theta) \ln|\theta|) + i\mu\theta\}, & \alpha = 1. \end{cases} \tag{2}$$

In general, α-stable distributions are favorable for modeling financial returns because of their ability to display skewness often observed in reality. The possibly more important feature, however, is that they can capture the leptokurtosis of financial returns. In the tails, α-stable distributions decay like a Pareto distribution, and hence they are also referred to as Pareto-stable. As is often the case, large price movements are more frequent than indicated by the normal distribution which can be particularly harmful if price changes are negative.

3. NONPARAMETRIC ESTIMATION OF RETURN DISTRIBUTION

In this section, we report the results of three nonparametric tests for the return distribution: kurtosis, Kolmogorov–Smirnov, and Hill tail.

3.1. Kurtosis. An initial statistic of interest to reveal information as to whether a sample can be considered normal or heavy-tailed is the kurtosis

TABLE 1. Nonparametric estimates: kurtosis and Kolmogorov–Smirnov test
Column (2): Kurtosis measurements of the returns with over 1,000 trading days (Jan. 1988–Sep. 2002)
Columns (3)–(6): Kolmogorov–Smirnov test results. $H = 0$: normal hypothesis not rejected. $H = 1$: normal hypothesis rejected. P is the significance level, KSSTAT is the value of the KS statistic, and CV is the critical value

(1) WKN	(2) Kurtosis	(3) H	(4) P	(5) KSSTAT	(6) CV
500340	5.9	1	$7.25 \cdot 10^{-6}$	7.57	4.11
515100	6.3	1	$3.37 \cdot 10^{-14}$	6.53	2.23
519000	9.0	1	$1.43 \cdot 10^{-20}$	7.90	2.23
543900	6.2	1	$2.74 \cdot 10^{-5}$	5.05	2.90
550000	11.0	1	$6.37 \cdot 10^{-9}$	5.98	2.60
550700	40.5	1	$2.25 \cdot 10^{-12}$	8.42	3.08
551200	15.2	1	$4.17 \cdot 10^{-15}$	7.78	2.57
575200	11.8	1	$1.35 \cdot 10^{-16}$	7.07	2.23
575800	8.1	1	$2.99 \cdot 10^{-11}$	6.50	2.50
593700	14.2	1	$1.38 \cdot 10^{-11}$	5.88	2.23
604843	11.7	1	$9.42 \cdot 10^{-19}$	7.53	2.23
627500	12.7	1	$8.44 \cdot 10^{-13}$	6.54	2.35
648300	9.8	1	$3.28 \cdot 10^{-20}$	7.83	2.23
656000	11.1	1	$1.68 \cdot 10^{-12}$	6.87	2.50
660200	21.5	1	$3.27 \cdot 10^{-18}$	1.14	3.43
695200	9.1	1	$3.96 \cdot 10^{-11}$	6.36	2.46
703700	11.6	1	$3.46 \cdot 10^{-20}$	8.37	2.38
716463	8.4	1	$2.63 \cdot 10^{-5}$	6.22	3.56
717200	6.1	1	$1.99 \cdot 10^{-14}$	6.58	2.23
723600	11.0	1	$3.49 \cdot 10^{-13}$	7.09	2.51
725750	5.4	1	$4.35 \cdot 10^{-3}$	4.49	3.48
748500	7.2	1	$1.11 \cdot 10^{-6}$	5.05	2.56
761440	7.7	1	$1.46 \cdot 10^{-20}$	7.34	2.07
762620	15.1	1	$2.55 \cdot 10^{-21}$	8.76	2.42
766400	8.1	1	$4.06 \cdot 10^{-8}$	4.88	2.23
781900	13.7	1	$4.74 \cdot 10^{-6}$	5.48	2.93
802000	18.7	1	$1.59 \cdot 10^{-10}$	6.65	2.65
802200	12.5	1	$3.27 \cdot 10^{-23}$	8.40	2.23
803200	10.4	1	$2.03 \cdot 10^{-17}$	7.25	2.23
804010	12.2	1	$7.30 \cdot 10^{-14}$	7.26	2.51
804610	14.7	1	$3.41 \cdot 10^{-23}$	9.43	2.50
823210	9.1	1	$7.10 \cdot 10^{-8}$	5.93	2.75
823212	5.6	0	$9.37 \cdot 10^{-2}$	3.45	3.79
840400	9.9	1	$1.01 \cdot 10^{-18}$	7.53	2.23
843002	6.8	1	$7.20 \cdot 10^{-6}$	6.41	3.48

defined as

$$\mathfrak{K} = \frac{E(X-\mu)^4}{\sigma^4}.$$

In the normal case, this statistic takes on the value 3 whereas in the case of heavy-tails the values are higher.

As can be seen from column (2) of Table 1, for the stocks in this study kurtosis is significantly greater than 3, indicating leptokurtosis for all 35 returns series.[2] This finding agrees with the findings of other researchers who have investigated the German equity market. See, for example, [17] and [38]. For financial data, kurtosis is usually greater than 3 as stated in [11].

3.2. Kolmogorov–Smirnov test. As a test for Gaussianity, we apply the two-sided Kolmogorov–Smirnov test with its well-known test statistic

$$K_n = \sup_{x \in \boldsymbol{R}} |F_0(x) - F_n(x)|,$$

where F_0 is the theoretical cumulative distribution function (cdf) tested for, and F_n is the sample distribution. For all but one stock in our study, the Gaussian distribution could be safely rejected at the 95% confidence level. The values for the Kolmogorov–Smirnov test are given in columns (3)–(6) of Table 1.

3.3. Hill tail-estimator. The following approach uses the semiparametric Hill estimation of the tail index as a proxy for the extreme Pareto part of the tail if it should exist. The tail estimator was first introduced by Hill [15] to infer the Pareto-type behavior for the sample data. The estimator applies if the tails of the underlying cdf follow the Pareto law with tail index α_P. The Pareto cdf is in the DA of the α-stable Paretian law for $0 < \alpha < 2$[3] with tail probability in the limit

$$P(Y \geqslant y) = 1 - F(y) \approx L y^{-\alpha}, \qquad y \to \infty,$$

with slowly varying L. For $X_{(n)} < X_{(n-1)} < \ldots$, the estimator is defined as[4]

(3) $$\hat{\alpha}_{\text{Hill}} = \frac{r}{\sum_{i=1}^{r} \ln X_{(i)} - r \cdot \ln X_{(r+1)}}$$

which under certain conditions is consistent.[5]

A problem arises with respect to the determination of the proper threshold index r indicating the beginning of the Pareto tail of the underlying cdf. This

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[2] In Table 1, WKN is the abbreviation of the German word "Wertpapierkennummer" which means security code number.

[3] For different values of α, the characteristic exponent of the α-stable parametrization and the Pareto tail parameter do not correspond.

[4] Indices in parentheses denote the ordered sample.

[5] See [32].

may suffice to hint at the questionable quality of the estimator.[6] Annaert et al. [2] investigated the reliability of the Hill estimator. Based on Monte Carlo simulation, they find that the Hill estimator retrieves the heavy-tailed characteristic or tail parameter with sufficient exactness whenever the true underlying Pareto-stable distribution is in the realm of non-Gaussianity. However, the parameter space in the simulation of [2] was very limited in that β and μ were set to 0, and γ was restricted to 0.01. On the other hand, we conducted a different Monte Carlo simulation with a more flexible parameter space. As a result, we cannot confirm their support for the Hill estimator. Instead, our findings cast serious doubt on the Hill estimator's reliability because it systematically overestimates the tail parameter. Even for fairly low α, we find that the estimator trespasses the border-line value 2 with a high probability.

As just mentioned, a problem arises with respect to the determination of the proper threshold index r indicating the beginning of the Pareto tail of the underlying cdf when computing the Hill estimator. As r increases, α_H gradually descends to cross the conditional value of the estimated α-stable parameter. Beyond certain values of r, α_H falls to approach the value of 1. The Hill estimator estimates the α-stable characteristic parameter correctly, in some instances, at tail lengths of between 10% and 15%. But no common threshold value can be determined for all the stocks analyzed in this study.[7]

With this ambiguity existing as to where the tail of the underlying sample distribution begins, Lux [21] still rejects the hypothesis of tails stemming from an α-stable distribution for German blue chip stocks as a result of Hill estimation based on varying tail lengths of 2.5%, 5%, 10%, and 15%. Covering an earlier period, Akgiray et al. [1] performed a test for the tail indices of the most liquid German stocks based on maximum likelihood estimation of the generalized Pareto distribution, $1-(1+\gamma x\omega^{-1})^{1/\gamma}$, rather than the Hill estimator. They also rejected the α-stable hypothesis for the tails even though they cannot deny the overall good fit this class of distributions provides, and suggest a universal 10% tail area optimal.

Results of the Hill estimation of the tail index for our sample stocks are reported in Table 2 with standard errors and 95% confidence bounds, respectively. The instability of the estimator for varying tail lengths becomes strikingly obvious. The plots (not displayed here) reveal that the tail corresponds to the characteristic stable parameter for tail sizes roughly within 10% and 15%. As can be seen by the lower bounds, when the respective tail lengths represent the extreme 15% of the returns, in 31 out of 35 cases, we cannot reject a stable distribution at the 95% confidence level. Still, we find that the Hill estimator is inappropriate to serve as a reliable estimator for the tail index.

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[6] Admittedly, there have been attempts to find methodologies for assessing the appropriate tail sizes. See, for example, [22].

[7] Problems of this sort are also mentioned in [32].

TABLE 2. Hill estimates of log-returns (with over 1,000 observations). Standard errors and 95% confidence bounds in parentheses, respectively

WKN	$\hat{\alpha}_{Hill}$ 15%	$\hat{\alpha}_{Hill}$ 10%	$\hat{\alpha}_{Hill}$ 5%	$\hat{\alpha}_{Hill}$ 2.5%
500340	1.8616	2.4965	3.3593	4.7388
	−0.1481	−0.2447	−0.4746	−0.9842
	(1.5749; 2.1483)	(2.257; 2.9673)	(2.4633; 4.2552)	(2.9513; 6.5262)
515100	2.2716	2.5888	3.2018	3.6164
	−0.0968	−0.1353	−0.2380	−0.3854
	(2.826; 2.4606)	(2.3250; 2.8526)	(2.7404; 3.6632)	(2.8774; 4.3554)
519000	1.8129	2.5960	2.7820	2.9701
	−0.0772	−0.1077	−0.2068	−0.3165
	(1.6620; 1.9637)	(1.8498; 2.2695)	(2.3811; 3.1829)	(2.3632; 3.5770)
543900	2.1379	2.5170	3.7595	3.8560
	−0.1190	−0.1720	−0.3668	−0.5448
	(1.9062; 2.3697)	(2.1829; 2.8511)	(3.538; 4.4653)	(2.8276; 4.8845)
550000	2.8910	2.4401	2.9580	3.4210
	−0.1039	−0.1490	−0.2574	−0.4274
	(1.8864; 2.2918)	(2.1501; 2.7300)	(2.4609; 3.4552)	(2.6079; 4.2341)
550700	1.8320	2.1401	2.9854	3.2290
	−0.1085	−0.1557	−0.3112	−0.4862
	(1.6208; 2.0433)	(1.8382; 2.4420)	(2.3882; 3.5826)	(2.3155; 4.1425)
551200	1.8241	2.2767	2.8059	3.6056
	−0.0899	−0.1378	−0.2424	−0.4470
	(1.6488; 1.9994)	(2.85; 2.5448)	(2.3377; 3.2740)	(2.7549; 4.4564)
575200	2.1572	2.5713	3.1801	3.3505
	−0.0919	−0.1344	−0.2364	−0.3571
	(1.9777; 2.3367)	(2.3093; 2.8333)	(2.7219; 3.6384)	(2.6659; 4.0351)
575800	2.1258	2.2923	2.8145	3.5725
	−0.1018	−0.1348	−0.2362	−0.4299
	(1.9272; 2.3244)	(2.298; 2.5548)	(2.3579; 3.2710)	(2.7530; 4.3920)
593700	2.1192	2.4859	2.9345	3.2762
	−0.0903	−0.1299	−0.2181	−0.3491
	(1.9429; 2.2955)	(2.2326; 2.7392)	(2.5116; 3.3573)	(2.6067; 3.9456)
604843	1.8024	2.1875	2.9855	3.3556
	−0.0768	−0.1143	−0.2219	−0.3576
	(1.6524; 1.9524)	(1.9647; 2.4104)	(2.5553; 3.4158)	(2.6699; 4.0413)
627500	1.9455	2.4141	3.8200	3.4590
	−0.0876	−0.1335	−0.2429	−0.3915
	(1.7745; 2.1166)	(2.1540; 2.6742)	(2.6117; 3.5522)	(2.7103; 4.2077)
648300	1.8024	2.3913	2.7990	3.6715
	−0.0768	−0.1250	−0.2080	−0.3913
	(1.6525; 1.9524)	(2.1477; 2.6350)	(2.3957; 3.2024)	(2.9212; 4.4217)
656000	2.0611	2.3120	2.6687	3.1270
	−0.0988	−0.1362	−0.2239	−0.3763
	(1.8683; 2.2539)	(2.0468; 2.5772)	(2.2358; 3.1016)	(2.4097; 3.8444)
660200	1.6200	1.8110	2.1767	2.2613
	−0.1070	−0.1474	−0.2547	−0.3871
	(1.4120; 1.8280)	(1.5259; 2.0961)	(1.6905; 2.6629)	(1.5423; 2.9802)

Table 2 ctd.

WKN	$\hat{\alpha}_{\text{Hill}}$ 15%	$\hat{\alpha}_{\text{Hill}}$ 10%	$\hat{\alpha}_{\text{Hill}}$ 5%	$\hat{\alpha}_{\text{Hill}}$ 2.5%
695200	2.0248 −0.0954 (1.8385; 2.2110)	2.3918 −0.1383 (2.1225; 2.6611)	3.0154 −0.2487 (2.5344; 3.4963)	3.2034 −0.3800 (2.4785; 3.9284)
703700	1.8443 −0.0841 (1.6801; 2.0084)	2.2362 −0.1252 (1.9924; 2.4801)	2.5567 −0.2040 (2.1618; 2.9517)	3.0567 −0.3505 (2.3869; 3.7265)
716463	2.2053 −0.1515 (1.9112; 2.4994)	2.7453 −0.2320 (2.2969; 3.1936)	2.9038 −0.3520 (2.2331; 3.5745)	3.2880 −0.5800 (2.2139; 4.3620)
717200	1.9554 −0.0833 (1.7927; 2.1181)	2.3835 −0.1246 (2.1407; 2.6264)	3.1413 −0.2335 (2.6886; 3.5939)	3.2798 −0.3495 (2.6096; 3.9500)
723600	1.9545 −0.0940 (1.7710; 2.1379)	2.2557 −0.1334 (1.9961; 2.5154)	2.5070 −0.2111 (2.0990; 2.9151)	3.0554 −0.3703 (2.3497; 3.7612)
725750	2.4456 −0.1641 (2.1268; 2.7645)	2.8363 −0.2339 (2.3840; 3.2887)	3.2696 −0.3878 (2.5296; 4.0095)	3.8904 −0.6759 (2.6368; 5.1439)
748500	2.3886 −0.1170 (2.1604; 2.6167)	2.6807 −0.1611 (2.3672; 2.9941)	2.9423 −0.2523 (2.4549; 3.4297)	3.3119 −0.4074 (2.5360; 4.0877)
761440	1.8189 −0.0720 (1.6782; 1.9596)	2.2911 −0.1113 (2.0741; 2.5082)	2.5757 −0.1777 (2.2306; 2.9208)	3.1915 −0.3144 (2.5868; 3.7962)
762620	1.7494 −0.0812 (1.5909; 1.9079)	2.0657 −0.1177 (1.8365; 2.2949)	2.3389 −0.1897 (1.9719; 2.7060)	2.7752 −0.3225 (2.1593; 3.3910)
766400	2.1288 −0.0907 (1.9517; 2.3059)	2.3991 −0.1254 (2.1546; 2.6435)	2.8568 −0.2123 (2.4451; 3.2684)	3.6100 −0.3847 (2.8724; 4.3477)
781900	2.1301 −0.1196 (1.8971; 2.3631)	2.4259 −0.1674 (2.1009; 2.7509)	3.6467 −0.3592 (2.9557; 4.3377)	4.9462 −0.7059 (3.6145; 6.2778)
802000	2.0424 −0.1037 (1.8402; 2.2446)	2.2791 −0.1422 (2.0026; 2.5556)	2.7775 −0.2474 (2.3000; 3.2549)	3.0806 −0.3942 (2.3317; 3.8295)
802200	1.8774 −0.0800 (1.7212; 2.0336)	2.1146 −0.1105 (1.8992; 2.3301)	2.7013 −0.2008 (2.3120; 3.0905)	3.2240 −0.3436 (2.5652; 3.8828)
803200	2.0268 −0.0863 (1.8582; 2.1954)	2.3021 −0.1203 (2.0675; 2.5367)	2.9800 −0.2215 (2.5506; 3.4094)	3.5650 −0.3799 (2.8365; 4.2934)
804010	1.9459 −0.0935 (1.7635; 2.1284)	2.2807 −0.1346 (2.0186; 2.5427)	2.8516 −0.2401 (2.3874; 3.3157)	2.9843 −0.3617 (2.2949; 3.6736)
804610	1.8021 −0.0863 (1.6337; 1.9705)	2.2054 −0.1297 (1.9529; 2.4580)	2.7932 −0.2344 (2.3401; 3.2462)	3.4286 −0.4125 (2.6421; 4.2151)

WKN	$\hat{\alpha}_{\text{Hill}}$ 15%	$\hat{\alpha}_{\text{Hill}}$ 10%	$\hat{\alpha}_{\text{Hill}}$ 5%	$\hat{\alpha}_{\text{Hill}}$ 2.5%
823210	1.9243	2.5392	3.2903	4.5295
	−0.1016	−0.1646	−0.3041	−0.6048
	(1.7264; 2.1223)	(2.2193; 2.8591)	(2.7040; 3.8766)	(3.3834; 5.6756)
823212	2.3372	2.6067	3.6152	4.6111
	−0.1709	−0.2350	−0.4703	−0.8848
	(2.0057; 2.6686)	(2.1534; 3.0601)	(2.7225; 4.5079)	(2.9879; 6.2343)
840400	2.0676	2.4309	2.7247	2.8750
	−0.0881	−0.1271	−0.2025	−0.3064
	(1.8955; 2.2396)	(2.1832; 2.6786)	(2.3321; 3.1173)	(2.2875; 3.4625)
843002	2.2079	2.6067	3.1357	3.9076
	−0.1478	−0.2150	−0.3720	−0.6789
	(1.9207; 2.4951)	(2.1910; 3.0225)	(2.4260; 3.8454)	(2.6485; 5.1667)

4. PARAMETRIC ESTIMATION CONDITIONAL ON THE α-STABLE DISTRIBUTION

So far, we have rejected the hypothesis of Gaussian returns. Additionally, we concluded that the Hill estimator does not suffice to determine the tail index. Hence the hypothesis of Pareto-type tails in the realm of α-stability could not be rejected. Now, conditional on the assumption that the α-stable distribution is correct, we set about to estimate the four stable parameters based on three different techniques: maximum likelihood estimation (MLE), quantile estimation, and characteristic function based estimation. All estimation results can be found in Table 3.

4.1. Maximum likelihood estimation. In the following, parameter estimates are obtained conditional on the α-stable distribution function. For conducting MLE of the parameters with the likelihood $f(x \mid \alpha, \gamma, \beta, \mu_1)$, two methods have been suggested. The first method, suggested by [29],[8] minimizes the information matrix which is known to be the negative inverse of the Hessian matrix of the likelihood function. This is done by some numerically efficient gradient search. The second method is based on a computationally efficient Fast Fourier Transformation (FFT) introduced by Mittnik et al. [25]. We will refer to the first and second methods as the Nolan method and FFT method, respectively.

The FORTRAN program code of the Nolan method used in this study is incorporated in an executable program offered on Nolan's internet web page. Applying some constraints concerning the boundaries, etc., values obtained for α for our sample of stocks are between 1.4605 and 1.9117. The values of β are significantly different from no skewness, i.e. $\beta = 0$, with a majority indicating

[8] The reader can find a vast resource of α-stable MLE on Nolan's web site at American University including his executable program codes.

TABLE 3. Parametric estimation results: maximum likelihood estimate MLE – columns (2)–(5); McCulloch quantile estimation results (MCE) – columns (6)–(9); and characteristic function based estimation results (CFE) – columns (10)–(13)

(1) WKN	(2) α	(3) γ	(4) β	(5) μ	(6) α	(7) γ	(8) β	(9) μ	(10) α	(11) γ	(12) β	(13) μ
	MLE				MCE				CFE			
500340	1.6182	0.0145	0.0817	−0.0010	1.4871	0.0137	0.0250	0.0003	1.7577	0.0150	0.4577	0.0011
515100	1.6811	0.0091	−0.0590	0.0007	1.5301	0.0086	−0.0556	0.0001	1.7849	0.0093	0.0185	0.0007
519000	1.5444	0.0098	0.0470	0.0003	1.4471	0.0094	0.0687	0.0006	1.6623	0.0101	0.1269	0.0010
543900	1.7370	0.0100	0.1411	−0.0002	1.5511	0.0094	0.0810	0.0005	1.8236	0.0102	0.4648	0.0009
550000	1.7014	0.0093	0.0951	0.0002	1.5907	0.0088	0.1129	0.0005	1.7796	0.0094	0.1920	0.0007
550700	1.6590	0.0105	0.1502	−0.0005	1.5079	0.0098	0.1109	0.0008	1.7347	0.0107	0.2990	0.0008
551200	1.6113	0.0089	0.0826	0.0001	1.5016	0.0086	0.0760	0.0005	1.7085	0.0092	0.2673	0.0011
575200	1.6770	0.0093	−0.1332	0.0008	1.5225	0.0087	−0.0740	0	1.7673	0.0095	−0.1633	0.0002
575800	1.6429	0.0089	0.0983	0.0002	1.5758	0.0087	0.0732	0.0007	1.7482	0.0092	0.2151	0.0010
578580	1.7085	0.0139	−0.1553	−0.0003	1.5886	0.0130	−0.1739	−0.0012	1.6986	0.0051	−0.0032	0.0024
593700	1.7214	0.0109	0.0195	0.0002	1.5727	0.0103	−0.0012	0	1.7986	0.0111	−0.0143	0.0002
604843	1.9117	0.0288	0.3086	−0.0054	1.4607	0.0080	0.0830	0.0006	1.7145	0.0088	0.1391	0.0006
627500	1.6642	0.0098	0.1014	−0.0001	1.5298	0.0094	0.0829	0.0005	1.7533	0.0101	0.2704	0.0008
648300	1.5633	0.0079	0.0259	0.0002	1.44	0.0074	0.0010	0	1.6877	0.0082	0.1071	0.0006
656000	1.6542	0.0103	0.0302	0.0011	1.5759	0.0100	0.0488	0.0011	1.7342	0.0106	0.0002	0.0011
660200	1.4605	0.0104	0.0024	−0.0006	1.3975	0.0100	−0.0444	−0.0005	1.5377	0.0107	0.0859	0.0001
695200	1.6738	0.0102	−0.0125	0.0001	1.535	0.0097	−0.0149	−0.0001	1.7740	0.0105	−0.0715	−0.0001
703700	1.5466	0.0081	0.1107	0.0000	1.4195	0.0076	0.0434	0.0003	1.6731	0.0085	0.1602	0.0007
716463	1.6716	0.0181	−0.0954	0.0020	1.5149	0.0168	−0.1392	0	1.7723	0.0185	−0.1897	0.0008
717200	1.6366	0.0089	0.0408	0.0005	1.4674	0.0083	0.0211	0.0004	1.7561	0.0092	0.0657	0.0007
723600	1.6416	0.0080	0.0370	0.0005	1.5781	0.0077	0.0421	0.0006	1.7118	0.0081	0.0326	0.0006
725750	1.8217	0.0143	−0.2349	0.0006	1.6935	0.0136	−0.1593	−0.0008	1.8623	0.0143	−0.2041	−0.0001

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
748500	1.7807	0.0101	0.0884	0.0004	1.6827	0.0096	0.1061	0.0004	1.8327	0.0102	0.1927	0.0008
761440	1.5851	0.0088	−0.0260	0.0004	1.4715	0.0084	−0.0566	−0.0001	1.7084	0.0091	−0.0786	0.0001
762620	1.5657	0.0083	0.0817	0.0002	1.4622	0.0078	0.0764	0.0005	1.6652	0.0086	0.0322	0.0004
766400	1.7218	0.0117	−0.0793	0.0008	1.6684	0.0114	−0.0574	0.0003	1.8055	0.0119	−0.1804	0.0002
781900	1.7502	0.0093	0.1419	0.0000	1.5893	0.0089	0.1238	0.0006	1.8280	0.0095	0.4307	0.0008
802000	1.6891	0.0079	0.1689	0.0001	1.5673	0.0074	0.1477	0.0006	1.7618	0.0080	0.2513	0.0008
802200	1.5478	0.0099	−0.0058	0.0003	1.4277	0.0093	0.0002	0	1.6495	0.0102	0.0742	0.0006
803200	1.6271	0.0092	−0.0135	0.0002	1.4822	0.0085	0.0228	0.0002	1.7225	0.0094	0.0272	0.0003
804010	1.6105	0.0085	0.0372	0.0005	1.5283	0.0083	0.0488	0.0006	1.7137	0.0088	0.1020	0.0009
804610	1.5021	0.0080	0.1207	0.0002	1.4138	0.0076	0.1188	0.0009	1.6198	0.0083	0.2581	0.0015
823210	1.7218	0.0109	0.1965	−0.0002	1.569	0.0102	0.1777	0.0010	1.8090	0.0111	0.3514	0.0009
823212	1.8506	0.0159	−0.0166	−0.0002	1.8019	0.0157	0.1195	0.0004	1.8828	0.0159	0.3694	0.0003
840400	1.6281	0.0098	−0.0338	0.0004	1.5274	0.0094	−0.0376	−0.0002	1.7120	0.0100	−0.0236	0.0003
843002	1.6863	0.0146	−0.0223	0.0004	1.5244	0.0136	−0.0348	−0.0003	1.7871	0.0148	0.0721	0.0006

slight positive skewness.[9] The FFT method[10] applies an FFT approximation of the pdf to conduct the computation of the likelihood. The benefit of the FFT method is the reduction in computation time.[11] The estimates for α corresponded to those obtained from the Nolan method, ranging from 1.44617 to 1.8168. For the FFT method, too, the values for β generally suggest skewness for most stocks.

The minimum value obtained for α was identical for both the Nolan and FFT methods. It was also found for the same stock. This is in contrast to the maxima which were, additionally, obtained for different stocks. Interestingly, though, the maximum value estimated by the Nolan method matched the α value estimated for that very same stock by the FFT method.

4.2. Quantile estimation. While Fama and Roll [8] provided the foundation for the quantile estimator, it was McCulloch [24] who modified the estimator, providing estimation of parameters for skewed α-stable pdf's. The estimator matches sample quantiles and theoretical quantiles tabulated for different values of the parameter tuple.[12]

The values for α for our sample of stocks range from 1.3975 to 1.8019. It is somewhat striking that the values seem to be slightly lower than those obtained from the MLE using both the Nolan and FFT methods.[13]

4.3. Characteristic function based estimation. The last of the three estimators we used in this study is the characteristic function based estimator. Its existence is not surprising since the theoretical characteristic functions of the α-stable distribution are known. Hence, one only needs to fit the sample characteristic function (SCF) and retrieve the parameters. Generally, this approach is based on [19]. Let the SCF be

$$\hat{\Phi}(\theta) = \frac{1}{n} \sum_{t=1}^{n} \exp\{i\theta \tilde{y}_t\}. \tag{4}$$

Ordinary Least Squares (OLS) estimates for the stable parameters are obtained from the natural logarithm of equation (4).

[9] For further complications inherent in the program code as to the computational results, the reader is referred to the manual given by the program's author.

[10] Since estimates from the FFT method do not significantly deviate from the Nolan method, they are not listed here.

[11] The code in MATLAB was provided by Stoyan Stoyanov, FinAnalytica Inc.

[12] The implementation of the McCulloch estimator in MATLAB was enabled through the translation of the original FORTRAN code by Stoyan Stoyanov, FinAnalytica Inc.

[13] This type of downward bias was found, for example, as a result of Monte Carlo studies by Blattberg and Gonedes [3] using the quantile estimator by Fama and Roll [8] and should be less likely when applying the estimator by McCulloch [24] due to the fact that it is a consistent estimator.

The problem with the numerical method as proposed by Koutrouvelis [19] is that the frequencies θ_k, most suitable for the respective regression, must be looked up in tables indexed by sample size and initial parameter estimates. This leads to a large computational effort. Kogon and Williams [18] remedied this shortcoming by using a common, finite interval for the θ_k with fixed grid size for all parameters and samples. This procedure, called the *Fixed-Interval* (FI) estimator, results in a substantial computational improvement. They suggested that the best interval would be [0.1, 1] with up to 50 equally spaced grid points. While with respect to precision the FI estimator is slightly inferior to the original one by [19] for some parameter tuples, this is more than offset by its speed.

For the characteristic function based estimator, an implementation in MATLAB of the FI estimator has been used.[14] The fixed interval was set as suggested by [18] with 10 scalar frequency points and step size 0.1. Estimation results are reported in Table 3. The values obtained for α for the stocks in our sample are between 1.5377 and 1.8828. Values for $|\beta| > 0.1$ can be observed in two cases. The majority of the values indicates slight positive skewness. Computation time was significantly reduced compared to the previous alternatives.

It is evident from all estimation results that the parameters indicate non-Gaussian distributions of the returns, i.e. values α are well below 2. Results are reasonably close throughout the different methods despite theoretical descrepancies of the three estimators.

5. MODELING THE RETURNS AS GARCH

Our last set of empirical results, and possibly the most interesting, are those obtained from an analysis of the autoregressive moving average (ARMA) innovations with respect to generalized autoregressive conditional heteroscedasticity (GARCH). The ARMA-GARCH model used in this study is

$$r_t = \sum_{i=1}^{p} \phi_i r_{t-1} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j},$$

$$h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-1}^2 + \sum_{i=j}^{p} \beta_j h_{t-j},$$

where $\varepsilon_t \mid \mathfrak{F}_t \sim N(0, h_t)$, and $\mathfrak{F}_t$ is the filtration at time t. Empirically, it has been observed in [4] that a simple GARCH(1,1) performs at least as well as a long-lagged ARCH(8) process. An attribute of the special GARCH(1,1) process for modeling financial data series is its capability to capture leptokurtosis. While the results from fitting the returns series to ARMA structures

[14] This has been implemented by Stoyan Stoyanov, FinAnalytica Inc.

are not displayed here,[15] in most cases the preferred model was an MA(1). In some cases, the AR(1) model was found to provide the best fit. The selection criterion considered first-differencing appropriate for one stock, only, but in that instance it was found that the suggested model was not invertible. Consequently, we imposed a no-first-differencing restriction and obtained an MA(1) model with similar value for the selection criterion. In all cases, diagnostic and significance checks suggested that the returns are best modeled as white noise.

Francq and Zakoïan [10] prove that under quite general conditions for pure GARCH as well as ARMA-GARCH Quasi-MLE such as Berndt, Hall, Hall, and Hausmann (BHHH) produces asymptotically normal estimators when innovations satisfy second moment conditions. In the case of infinite variance processes such as α-stable innovations, convergence occurs even faster.

For a valuable and exhaustive account of various sorts of ARCH processes and their respective estimation methods, the reader is strongly advised to consult [6].

An interesting generalization of GARCH processes, the R-GARCH(r,p,q), is presented in [30]. It incorporates an additional component into the dynamics of the conditional variance, namely, the sum of r i.i.d. lagged positive random variables η_{t-i}.[16] For strictly α-stable innovations η_i totally skewed to the right, Nowicka-Zagrajek and Weron [30] show that the sum of the underlying process is unconditionally symmetric stable. This is of particular importance when considering aggregated returns.

The introduction of a Student's t-distribution permitting skewness or, simply, skew-t distribution, serves as a reasonable competitor to the α-stable distribution in this context. The univariate pdf about mean or location zero, first introduced as a multivariate version in [14], takes the following form according to Fernandez and Steel [9]:

$$(5) \qquad p(\varepsilon_t \mid \tau, \nu, \lambda) = 2\frac{\Gamma((\nu+1)/2)\,\tau}{\Gamma(\nu/2)(\pi\nu)^{1/2}(\gamma+1/\gamma)} \times\left[1+\frac{\tau^2}{\nu}\varepsilon_t^2\left\{\frac{1}{\gamma^2}I_{[0,\infty)}(\varepsilon_t)+\gamma^2 I_{(-\infty,0)}(\varepsilon_t)\right\}\right]^{-(\nu+1)/2}.$$

The parameter ν indicates the degrees of freedom as with the t-distribution and the parameter γ corresponds to skewness, with $\gamma = 1$ indicating symmetry. Any other value for γ indicates skewness of some degree. The parameter τ^2 is interpreted as precision. It is inversely proportional to the scaling parameter σ^2, which, in turn, is a real multiple of the variance if it exists. In applications

[15] Results are available upon request.

[16] The process is assumed to, conditionally, follow a Gaussian law with variance governed by the GARCH structure.

in the literature, τ is very often set equal to 1.[17] Equation (5) reduces to the regular Student's t pdf when $\beta = 0$, $\lambda = 1$, and $\tau = 1$.[18]

The model we suggest is a GARCH(1,1) structure of the generalized form

$$c_t^\delta = \alpha_0 + \alpha\,|\varepsilon_{t-1} - \mu|^\delta + \beta c_{t-1}^\delta,$$

where we know $\delta = 2$ from the original set-up with Gaussian innovations. In the Gaussian case, $c_t = h_T^{1/2}$. This is impossible, however, if the distribution under consideration does not have finite moments of order greater than δ for some $\delta < 2$. Since first absolute moments exist for the theoretical distributions fitted to the log-return series as well as innovations, $\delta = 1$ is chosen as in [32].[19] The GARCH(1,1) structure is parsimonious regarding a parameter use and still enjoys popularity for its great flexibility in financial applications as noted for example by Nelson [28] and several of his later articles.

In our paper, the skew-t and α-stable distributions were tested against each other as alternative distributions for the ARMA residuals $\{\varepsilon_t\}$, virtually being the log-returns in many cases. In fact, the normal distribution was also analyzed; however, since it performed poorly, we did not consider it any further.

Depending on the distribution, the notation $S_{\alpha,\beta}^\delta\,\mathrm{GARCH}(r,s)$ and $t_{\nu,\lambda}^\delta\,\mathrm{GARCH}(r,s)$ can be used to indicate α-stable or skew-t innovations, respectively. Preference is based on the maximized logarithmic likelihood[20] functions of the i.i.d. $\{r_t\}$. In the α-stable case, the likelihood equals

$$\prod_{t=1}^{n} \frac{1}{c_t} S_{\alpha,\beta}\left(\frac{\varepsilon_t - \mu}{c_t}\right),$$

which in contrast to the normal and skew-t distributions is known not to have an analytical solution. Consequently, it has to be approximated numerically.

For a numerical approximation of the α-stable likelihoods, MATLAB encoded numerical FFT approximations were performed. The skew-t likelihoods are analytically solvable.[21] Results show that for some stocks, the skew-t and α-stable alternatives behave alike according to the log-likelihood values.

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[17] See, for example, [12].

[18] Alternative representations of the skew-t pdf can be found, for example, in [16].

[19] Mittnik and Padella [26] leave more room to play in the sense that δ enters as a variable parameter for each distribution, respectively.

[20] Conditioning starting values are set equal to their expected values. However, as argued in [26], these values have little to no impact on the outcome of the estimation.

[21] Basic GARCH estimation programs in MATLAB provided by Kevin Sheppard from the University of California at San Diego were altered by us to allow for the α-stable distribution. The current internet location is http://www.kevinsheppard.com/research/ucsd_garch/ucsd_garch.aspx.

TABLE 4. Values of AD-statistic of $S^{\delta}_{\alpha,\beta}$ GARCH and $t^{\delta}_{\nu,\lambda}$ GARCH innovations, respectively, and positions of the AD-statistic in parentheses

(1) WKN	(2) AD normal	(3) AD skew-*t*	(4) AD stable	(5) WKN	(6) AD normal	(7) AD skew-*t*	(8) AD stable
500340	$2.91 \cdot 10^{01}$	$6.21 \cdot 10^{-02}$	$1.48 \cdot 10^{-01}$	717200	$8.02 \cdot 10^{02}$	$6.00 \cdot 10^{-02}$	$6.19 \cdot 10^{-02}$
	(right end)	(right end)	(left end)		(left end)	(median)	(right end)
515100	$7.24 \cdot 10^{47}$	$6.49 \cdot 10^{-01}$	$5.59 \cdot 10^{-02}$	723600	$3.32 \cdot 10^{10}$	$3.37 \cdot 10^{-01}$	$7.43 \cdot 10^{-02}$
	(left end)	(left end)	(right end)		(left end)	(left end)	(right end)
519000	$8.50 \cdot 10^{10}$	$1.13 \cdot 10^{-01}$	$8.59 \cdot 10^{-02}$	725750	$3.28 \cdot 10^{00}$	$4.62 \cdot 10^{-02}$	$5.01 \cdot 10^{-02}$
	(right end)	(left end)	(right end)		(left end)	(left end)	(right end)
543900	$1.71 \cdot 10^{02}$	$5.06 \cdot 10^{-02}$	$7.81 \cdot 10^{-02}$	748500	$6.28 \cdot 10^{04}$	$1.21 \cdot 10^{-01}$	$4.62 \cdot 10^{-02}$
	(left end)	(right end)	(left end)		(right end)	(left end)	(left quartile)
550000	$1.67 \cdot 10^{02}$	$5.04 \cdot 10^{-02}$	$7.71 \cdot 10^{-02}$	761440	$9.24 \cdot 10^{13}$	$8.65 \cdot 10^{-02}$	$6.62 \cdot 10^{-02}$
	(left end)	(right end)	(left end)		(right end)	(left end)	(left end)
550700	$7.00 \cdot 10^{22}$	$3.87 \cdot 10^{-01}$	$9.12 \cdot 10^{-02}$	762620	∞	$1.36 \cdot 10^{-01}$	$6.91 \cdot 10^{-02}$
	(right end)	(left end)	(median)		(right end)	(left end)	(left end)
551200	$5.45 \cdot 10^{19}$	$1.26 \cdot 10^{-01}$	$5.95 \cdot 10^{-02}$	766400	$8.57 \cdot 10^{11}$	$3.63 \cdot 10^{-01}$	$5.98 \cdot 10^{-02}$
	(right end)	(left end)	(right end)		(left end)	(left end)	(median)
575200	$1.17 \cdot 10^{19}$	$2.39 \cdot 10^{-01}$	$6.36 \cdot 10^{-02}$	781900	$1.26 \cdot 10^{08}$	$1.23 \cdot 10^{-01}$	$8.19 \cdot 10^{-02}$
	(left end)	(left end)	(median/left)		(left end)	(left end)	(right quartile)
575800	$2.07 \cdot 10^{09}$	$2.46 \cdot 10^{-01}$	$7.21 \cdot 10^{-02}$	802000	∞	$2.04 \cdot 10^{-01}$	$5.01 \cdot 10^{-02}$
	(left end)	(left end)	(right end)		(right end)	(left end)	(right end)
593700	$1.39 \cdot 10^{18}$	$2.98 \cdot 10^{-01}$	$4.87 \cdot 10^{-02}$	802200	$1.27 \cdot 10^{09}$	$1.12 \cdot 10^{-01}$	$8.69 \cdot 10^{-02}$
	(right end)	(left end)	(right end)		(right end)	(left end)	(left end)
604843	$1.67 \cdot 10^{24}$	$2.68 \cdot 10^{-01}$	$6.99 \cdot 10^{-02}$	803200	$1.99 \cdot 10^{16}$	$1.57 \cdot 10^{-01}$	$6.80 \cdot 10^{-02}$
	(left end)	(left end)	(right end)		(left end)	(left end)	(right end)
627500	$4.30 \cdot 10^{13}$	$1.35 \cdot 10^{-01}$	$7.45 \cdot 10^{-02}$	804010	$1.73 \cdot 10^{12}$	$3.48 \cdot 10^{-01}$	$7.58 \cdot 10^{-02}$
	(left end)	(left end)	(right end)		(left end)	(left end)	(right end)

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
648300	$3.63 \cdot 10^{14}$ (left end)	$1.46 \cdot 10^{-01}$ (left end)	$6.73 \cdot 10^{-02}$ (right end)	804610	$7.36 \cdot 10^{13}$ (left end)	$1.92 \cdot 10^{-01}$ (left end)	$9.88 \cdot 10^{-02}$ (right end)
656000	$1.33 \cdot 10^{14}$ (left end)	$2.78 \cdot 10^{-01}$ (left end)	$6.97 \cdot 10^{-02}$ (median)	823210	$1.26 \cdot 10^{07}$ (right end)	$9.74 \cdot 10^{-02}$ (median)	$6.44 \cdot 10^{-02}$ (right quartile)
660200	$1.33 \cdot 10^{14}$ (left end)	$2.78 \cdot 10^{-01}$ (left end)	$6.97 \cdot 10^{-02}$ (median)	823212	$2.35 \cdot 10^{01}$ (left end)	$1.28 \cdot 10^{-01}$ (left end)	$7.43 \cdot 10^{-02}$ (left end)
695200	$6.08 \cdot 10^{07}$ (right end)	$1.35 \cdot 10^{-01}$ (left end)	$5.35 \cdot 10^{-02}$ (right end)	840400	∞ (left end)	$7.97 \cdot 10^{-02}$ (left end)	$6.05 \cdot 10^{-02}$ (left end)
703700	∞ (left/right end)	$1.19 \cdot 10^{-01}$ (left end)	$6.37 \cdot 10^{-02}$ (left/right end)	843002	$3.61 \cdot 10^{04}$ (left end)	$1.34 \cdot 10^{-01}$ (right end)	$7.53 \cdot 10^{-02}$ (left end)
716463	$1.33 \cdot 10^{11}$ (left end)	$3.15 \cdot 10^{-01}$ (left end)	$1.04 \cdot 10^{-01}$ (median)				

The fit was also compared using the Anderson–Darling (AD) goodness-of-fit test,

$$AD = \sup_{x \in \mathbf{R}} \frac{|F_s(x) - \hat{F}(x)|}{\sqrt{\hat{F}(x)(1 - \hat{F}(x))}},$$

where $\hat{F}(\cdot)$ denotes the estimated parametric pdf, $F_s(\cdot)$ is the empirical sample pdf computed as

$$F_s(x) = \frac{1}{n} \sum_{t=1}^{n} I_{(-\infty, x]}\left(\frac{\varepsilon_t - \hat{\mu}}{\hat{h}_t^{1/2}}\right),$$

and $I(\cdot)$ is the indicator function.

The AD-statistic is well suited for detecting poorness of fit, particularly in the tails of the cdf. As can be seen in Table 4, the α-stable distribution outperforms the skew-t alternative in most instances.[22] Even though we tested lag structures of up to ($r = 5$, $s = 5$),[23] our preference was with a lag structure of (1,1) justifying the GARCH(1,1) model for the reasons commonly cited in the literature.

6. CONCLUSION

All of the tests performed in this study reject the Gaussian hypothesis for the logarithmic returns of the German blue chip stocks we analyzed. The nonparametric estimation results indicate that the rejection of the stable hypothesis by other researchers is not based on a reliable empirical test. The modeling of returns using α-stable distributions we report seems promising in spite of the lack of an analytic form of the probability distribution function. This is due to the tight fit of the approximated α-stable cdf to the empirical cdf combined with dependable estimation of the stable parameters.

As a negative aspect mentioned by several researchers, for example Lux [21], the α-stable alternative sometimes slightly overemphasizes the mass in the extreme parts of the tails compared to finite empirical data vectors. This is in contrast to our findings. We discovered that the tail shape of the α-stable class is extremely suitable for the returns we considered, particularly in the context of GARCH modeling. The alternatives in our study provided by the normal and skew-t distributions could not systematically outperform the α-stable dis-

[22] Analyzing foreign exchange data of U.S. dollar versus several important international currencies, Mittnik and Paolella [26] found comparable results. But one has to keep in mind that their counterpart distribution is the Student's t-distribution with less flexibility than the skew-t distribution we use in this study. So, our results might be considered even more striking in this context.

[23] Tabulated results of lags up to five are available upon request.

tribution. Instead, they produced equivalent results at best. Particularly, with respect to fitting the empirical tails, they performed poorly.

Theoretically, using the α-stable distribution is reasonable because it is the distributional limit of a series of standardized random variables in the domain of attraction. Thus, the α-stable class is a natural candidate for modeling the return distribution. Practically, when protecting portfolios against extreme losses, it becomes particularly important to assess the extreme parts of the lower tails adequately. Hence the stable Paretian distribution ought to be favored due to its very good overall fit of the distribution function in addition to the superior tail fit.

Acknowledgements. Rachev gratefully acknowledges research support by grants from the Division of Mathematical, Life and Physical Sciences, College of Letters and Science, University of California, Santa Barbara, the Deutsche Forschungsgemeinschaft, and the Deutscher Akademischer Austauschdienst. When not designed by the authors, the programs encoded in MATLAB were generously provided by Stoyan Stoyanov, FinAnalytica Inc or by the originator referenced accordingly.

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Received on 24.10.2005;
revised version on 12.12.2005