PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 26, Fasc. 1 (2006), pp. 63–76

CONVERGENCE RATES IN THE LAW OF LARGE NUMBERS FOR ARRAYS

BY

VICTOR KRUGLOV (Moscow) AND ANDREI VOLODIN (Regina)

In memory of Professor Urbanik

Abstract. In this paper we present new sufficient conditions for complete convergence for sums of arrays of rowwise independent random variables. These conditions appear to be necessary and sufficient in the case of partial sums of independent identically distributed random variables. Many known results on complete convergence can be obtained as corollaries to theorems proved in this paper.

2000 Mathematics Subject Classification: 60F15, 60G50.

Key words and phrases: Arrays of rowwise independent random variables, complete convergence, rate of convergence in the law of large numbers.

1. INTRODUCTION

The paper by Hsu and Robbins (1947) initiated numerous explorations of the complete convergence of sums of independent random variables. Their research was continued by Erdös (1949), (1950), Spitzer (1956), and Baum and Katz (1965).

The paper by Kruglov et al. (2006) contains two general theorems that provide sufficient conditions for complete convergence for sums of arrays of rowwise independent random variables. One of them is presented below as Theorem A. The purpose of this paper is especially to show the strength of this theorem. In fact, we propose an approach with the help of which we are able to prove a number of new results, and in a unified form to reprove many known theorems, on complete convergence for sums of independent random variables. Specifically, three theorems proved below contain, as particular cases, Spitzer's theorem (1956), a number of theorems of Baum and Katz (1965), the basic theorem of Bai and Su (1985), Maejima's theorem (1977), the theorem of Hu, Moricz and Taylor (1989), Theorems 2.1, 4.1, 5.1, 7.1–7.4 of Gut (1992), and the sufficient part of Gut's (1985) theorem.

In the following we assume that all random variables under consideration are defined on a probability space $(\Omega, \mathscr{F}, P)$. We use standard notation, in particular: $I[A]$ denotes the indicator function of a set $A \subseteq \Omega$. The proofs of some auxiliary statements are presented in the last part of the paper.

THEOREM A. *Let $\{(X_{nk}, 1 \leqslant k \leqslant m_n), n \geqslant 1\}$ be a sequence of rowwise independent random variables and let $\{c_n, n \geqslant 1\}$ be a sequence of non-negative numbers. Suppose that*

(i) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{m_n} P\{|X_{nk}| > \varepsilon\} < \infty$ *for all* $\varepsilon > 0$;

(ii) *there exist $j > 0$, $\delta > 0$ and $p \geqslant 1$ such that*

$$\sum_{n=1}^{\infty} c_n \big(E\,|\sum_{k=1}^{m_n} [X_{nk} I[|X_{nk}| \leqslant \delta] - E(X_{nk} I[|X_{nk}| \leqslant \delta])]|^p\big)^j < \infty.$$

Then

$$\sum_{n=1}^{\infty} c_n P\{\max_{1 \leqslant m \leqslant m_n} |\sum_{k=1}^{m} [X_{nk} - E(X_{nk} I[|X_{nk}| \leqslant \delta])]| > \varepsilon\} < \infty \quad \textit{for any } \varepsilon > 0.$$

2. MAIN RESULTS

The next definition emphasizes the class of random variables that we consider in this paper.

DEFINITION 1. The array $\{(X_{nk}, 1 \leqslant k \leqslant m_n), n \geqslant 1\}$ of random variables is *stochastically dominated in mean of order α by a random variable X with respect to a sequence* $\{b_n, n \geqslant 1\}$ of positive numbers if

$$\frac{1}{m_n} \sum_{k=1}^{m_n} P\{|X_{nk}| > x\} \leqslant DP\{b_n^{\alpha} |X| > x\} \tag{1}$$

for some $D > 0$, $\alpha \in (-\infty, \infty)$, and for all $x > 0$, $n \geqslant 1$.

For $\alpha = 0$ this notion reduces to the notion of stochastic domination in the Cesàro sense which became common in the literature devoted to complete convergence of sums of random variables.

In Definition 1 a sequence $\{b_n, n \geqslant 1\}$ of positive constants is presented. We will deal with non-decreasing sequences of positive numbers that satisfy one or both of the following conditions:

$$\sum_{n=1}^{\infty} b_n^{-\gamma} < \infty \quad \text{for any } \gamma > 1, \tag{2}$$

$$b_{n+1} \leqslant b b_n \quad \text{for some } b > 0 \text{ and for all } n \geqslant 1. \tag{3}$$

The class of sequences with these assumptions is sufficiently wide. It contains, for example, the sequences $b_n = n$, $b_n = \delta^n$, $b_n = n(\ln(1+n))^{\beta}$, $n \geqslant 1$, where $\delta > 1$ and β is an arbitrary real number.

DEFINITION 2. A non-negative function $h(x)$, $x \in [0, \infty)$, belongs to the *class* $\mathscr{H}_p$ for some $p \in (0, 2)$ if it is non-decreasing, does not equal zero identically,

$$\limsup_{x \to \infty} h(\gamma x)/h(x) < \infty \qquad \text{for any } \gamma > 0,$$

and

$$\limsup_{y \to \infty} \int_y^{\infty} h(x) x^{-2/p} dx / \big(h(y) y^{1-2/p}\big) < \infty .$$

The class $\mathscr{H}_p$ contains all slowly varying at infinity non-decreasing non-negative functions which are not equal to zero identically, and in particular $\ln^{\beta}(1+x)$ with $\beta > 0$. The functions x^{α} and $x^{\alpha} \ln^{\beta}(1+x)$ with $\alpha \in [0, 2/p-1)$ and $\beta > 0$ are also in $\mathscr{H}_p$.

THEOREM 1. *Let* $\{(X_{nk}, 1 \leqslant k \leqslant m_n), n \geqslant 1\}$ *be an array of rowwise independent random variables, with means zero when they exist, which is stochastically dominated in mean of order* α *by a random variable* X *with respect to a non-decreasing sequence* $\{b_n, n \geqslant 1\}$ *of positive numbers with properties* (2) *and* (3), $b_0 = 0$, $m_n \leqslant c b_n$, $n \geqslant 1$, $E\big(|X|^{rq/(1-\alpha q)} h(|X|^{q/(1-\alpha q)})\big) < \infty$ *for some* $0 < q < 2$, $r \geqslant 1$, $\alpha < 1/q - 1/2$, $c > 0$, $h \in \mathscr{H}_p$ *with* $p = q/(1-\alpha q)$. *Then*

$$\sum_{n=1}^{\infty} b_n^{r-2} h(b_n)(b_n - b_{n-1}) P\Big\{\max_{1 \leqslant m \leqslant m_n} \Big|\sum_{k=1}^{m} X_{nk}\Big| > \varepsilon b_n^{1/q}\Big\} < \infty \tag{4}$$

for all $\varepsilon > 0$.

Proof. We make use of Theorem A with $c_n = b_n^{r-2} h(b_n)(b_n - b_{n-1})$ and $X_{nk}/b_n^{1/q}$ for all X_{nk}. The exact values of constants c and D in (1) do not play any role in our proof. In the following we consider them equal to one. Assumption (i) of Theorem A follows from (1) and Lemma 2 with $\xi = |X/\varepsilon|$ and $p = q/(1-\alpha q)$. Indeed,

$$\begin{aligned}\sum_{n=1}^{\infty} b_n^{r-2} h(b_n)(b_n - b_{n-1}) \sum_{k=1}^{m_n} P\{|X_{nk}| > \varepsilon b_n^{1/q}\} \\ \leqslant \sum_{n=1}^{\infty} b_n^{r-1} h(b_n)(b_n - b_{n-1}) P\{|X| > \varepsilon b_n^{1/q-\alpha}\} \\ \leqslant E\big(|X/\varepsilon|^{rq/(1-\alpha q)} h(|X/\varepsilon|^{q/(1-\alpha q)})\big) < \infty .\end{aligned}$$

We used the inequality $m_n \leqslant b_n$. It can be easily seen that the last expectation is finite by the condition $E\big(|X|^{rq/(1-\alpha q)} h(|X|^{q/(1-\alpha q)})\big) < \infty$ and the properties of the function $h \in \mathscr{H}_p$. By (1) and Lemma 1 we have

$$\begin{aligned}\sum_{k=1}^{m_n} E(|X_{nk}|^2 I[|X_{nk}| \leqslant b_n^{1/q}]) \leqslant b_n^{2/q} m_n P\{|X| > b_n^{1/q-\alpha}\} \\ + b_n^{2\alpha} m_n E(X^2 I[|X| \leqslant b_n^{1/q-\alpha}]).\end{aligned} \tag{5}$$

In order to prove assumption (ii) of Theorem A we note that

$$\sum_{n=1}^{\infty} c_n \left(E \left| \frac{1}{b_n^{1/q}} \sum_{k=1}^{m_n} (X_{nk} I[|X_{nk}| \leqslant b_n^{1/q}] - E(X_{nk} I[|X_{nk}| \leqslant b_n^{1/q}])) \right|^2 \right)^j$$
$$\leqslant \sum_{n=1}^{\infty} \frac{c_n}{b_n^{2j/q}} \Big(\sum_{k=1}^{m_n} E(X_{nk}^2 I[|X_{nk}| \leqslant b_m^{1/q}]) \Big)^j$$
$$\leqslant \sum_{n=1}^{\infty} \frac{c_n}{b_n^{2j/q}} (b_n^{2/q+1} P\{|X| > b_n^{1/q-\alpha}\} + b_n^{2\alpha+1} E(|X|^2 I[|X| \leqslant b_n^{1/q-\alpha}]))^j$$
$$\leqslant 2^j \sum_{n=1}^{\infty} \frac{c_n}{b_n^{2j/q}} (b_n^{2/q+1} P\{|X|^{q/(1-\alpha q)} > b_n\})^j$$
$$+ 2^j \sum_{n=1}^{\infty} \frac{c_n}{b_n^{2j/q}} (b_n^{2\alpha+1} E(|X|^2 I[|X| \leqslant b_n^{1/q-\alpha}]))^j.$$

We used the notation $c_n = b_n^{r-2} h(b_n)(b_n - b_{n-1})$ and inequalities (5), $m_n \leqslant b_n$ and $(a+b)^j \leqslant 2^j(a^j+b^j)$ for all $a \geqslant 0$, $b \geqslant 0$ and $j > 0$. By Markov's inequality we obtain

$$P\{|X|^{q/(1-\alpha q)} > b_n\} = P\{|X|^{rq/(1-\alpha q)} > b_n^r\} \leqslant b_n^{-r} E|X|^{rq/(1-\alpha q)},$$

and hence

$$(6) \qquad \sum_{n=1}^{\infty} \frac{c_n}{b_n^{2j/q}} (b_n^{2/q+1} P\{|X|^{q/(1-\alpha q)} > b_n\})^j \leqslant \sum_{n=1}^{\infty} h(b_n) \frac{(E|X|^{rq/(1-\alpha q)})^j}{b_n^{(r-1)j-r+1}}.$$

Note that $p = q/(1-\alpha q) \in (0, 2)$ by the condition $\alpha < 1/q - 1/2$. Since $h \in \mathscr{H}_p$, for $b_n \geqslant 1$ we have

$$\infty > \int_1^{\infty} h(x) x^{-2/p} dx \geqslant \int_{b_n}^{\infty} h(x) x^{-2/p} dx \geqslant \frac{p}{2-p} h(b_n) b_n^{(p-2)/p}.$$

Consequently, by (2), the series (6) converges for $j > 1 + 2(1-\alpha q)/(r-1)$ and $r > 1$. Next, if $rq/(1-\alpha q) \geqslant 2$, then $E|X|^2 < \infty$ and

$$\sum_{n=1}^{\infty} \frac{c_n}{b_n^{2j/q}} (b_n^{2\alpha+1} E|X|^2 I[|X| \leqslant b_n^{1/q-\alpha}])^j \leqslant \sum_{n=1}^{\infty} \frac{h(b_n)(E|X|^2)^j}{b_n^{(2/q-2\alpha-1)j-r+1}}.$$

The last series converges for $j > 1 + rq/(2 - 2\alpha q - q)$ by the condition (2). Note that $2 - 2\alpha q - q > 0$ by the assumption $\alpha < 1/q - 1/2$. If $rq/(1-\alpha q) < 2$, then

$$E(|X|^2 I[|X| \leqslant b_n^{1/q-\alpha}]) \leqslant b_n^{(2-2\alpha q - rq)/q} E|X|^{rq/(1-\alpha q)}.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{c_n}{b_n^{2j/q}} (b_n^{2\alpha+1} E|X|^2 I[|X| \leqslant b_n^{1/q-\alpha}])^j \leqslant \sum_{n=1}^{\infty} \frac{h(b_n)(E|X|^{rq/(1-\alpha q)})^j}{b_n^{(r-1)j-r+1}}.$$

The last series converges for $j > 1+2(1-\alpha q)/(r-1)$ and $r > 1$. Hence the assumption (ii) of Theorem A is fulfilled with $j > 1+\max\{2(1-\alpha q)/(r-1), rq/(2-2\alpha q-q)\}$ and $r > 1$. Consider the case $r = 1$ separately. Let us put $c_n' = b_n^{-1} h(b_n)(b_n-b_{n-1})$. For $j = 1$ we have

$$\sum_{n=1}^{\infty} c_n' E\left|\frac{1}{b_n^{1/q}} \sum_{k=1}^{m_n} \left(X_{nk} I[|X_{nk}| \leqslant b_n^{1/q}] - E(|X_{nk}| I[|X_{nk}| \leqslant b_n^{1/q}])\right)\right|^2$$

$$\leqslant \sum_{n=1}^{\infty} \frac{c_n'}{b_n^{2/q}} \sum_{k=1}^{m_n} E(|X_{nk}|^2 I[|X_{nk}| \leqslant b_n^{1/q}])$$

$$\leqslant \sum_{n=1}^{\infty} \frac{c_n'}{b_n^{2/q}} (b_n^{2/q+1} P\{|X| > b_n^{1/q-\alpha}\} + b_n^{2\alpha+1} E(|X|^2 I[|X| \leqslant b_n^{1/q-\alpha}]))$$

$$= \sum_{n=1}^{\infty} h(b_n)(b_n-b_{n-1}) P\{|X|^{q/(1-\alpha q)} > b_n\}$$

$$+ \sum_{n=1}^{\infty} \frac{h(b_n)(b_n-b_{n-1})}{b_n^{2(1-\alpha q)/q}} E(|X|^2 I[|X| \leqslant b_n^{(1-\alpha q)/q}]).$$

The last two series converge by Lemmas 2 and 4 (see Section 3) with $\xi = |X|$ and $p = q/(1-\alpha q)$ for $q/(1-\alpha q) < 2$. The case $q/(1-\alpha q) \geqslant 2$ is impossible by the assumption $\alpha < 1/q-1/2$. By Theorem A,

$$\sum_{n=1}^{\infty} b_n^{r-2} h(b_n)^{\beta} (b_n-b_{n-1}) P\{\max_{1 \leqslant m \leqslant m_n} |Y_m| > \varepsilon b_n^{1/q}\} < \infty \qquad \text{for all } \varepsilon > 0,$$

where $Y_m = \sum_{k=1}^{m} (X_{nk} - E(X_{nk} I[|X_{nk}| \leqslant b_n^{1/q}]))$. Consequently, by Lemma 5 we obtain (4). ∎

If the constants b_n have a special behaviour, then, as the following theorem shows, the increment b_n-b_{n-1} in (4) may be substituted by b_n.

THEOREM 2. *Let $\{(X_{nk}, 1 \leqslant k \leqslant m_n), m_n \geqslant 1, n \geqslant 1\}$ be an array of row-wise independent random variables, with means zero when they exist, which is stochastically dominated in mean of order α by a random variable X with respect to the sequence $b_n = m_1 + \ldots + m_n$, $n \geqslant 1$, with the property (3), $E(|X|^{rq/(1-\alpha q)} h(|X|^{q/(1-\alpha q)})) < \infty$ for some $0 < q < 2$, $r \geqslant 1$, $\alpha < 1/q-1/2$, $h \in \mathscr{H}_p$ with $p = q/(1-\alpha q)$. Then*

$$(7) \qquad \sum_{n=1}^{\infty} b_n^{r-1} h(b_n) P\{\max_{1 \leqslant m \leqslant m_n} |\sum_{k=1}^{m} X_{nk}| > \varepsilon b_n^{1/q}\} < \infty \qquad \textit{for all } \varepsilon > 0.$$

Proof. The sequence of constants $b_n = m_1 + \ldots + m_n$, $n \geqslant 1$, increases and $b_{n+1}-b_n = m_n \geqslant 1$. It satisfies (2) because $b_n \geqslant n$. Now we may proceed as in the proof of the previous theorem with minor modifications. ∎

For identically distributed random variables it is possible to provide not only sufficient, but also necessary conditions.

THEOREM 3. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent identically distributed random variables, $S_n = X_1 + \ldots + X_n$, $0 < q < 2$, $r \geqslant 1$, $h \in \mathscr{H}_q$. The following conditions are equivalent*:

$$E(|X_1|^{rq} h(|X_1|^q)) < \infty \quad \text{and} \quad EX_1 = 0 \quad \text{for } q \geqslant 1, \tag{8}$$

$$\sum_{n=1}^{\infty} n^{r-2} h(n) P\{\max_{1 \leqslant m \leqslant n} |S_m| > \varepsilon n^{1/q}\} < \infty \quad \text{for all } \varepsilon > 0, \tag{9}$$

$$\sum_{n=1}^{\infty} n^{r-2} h(n) P\{|S_n| > \varepsilon n^{1/q}\} < \infty \quad \text{for all } \varepsilon > 0. \tag{10}$$

If $r > 1$, then each of the conditions (8)–(10) *is equivalent to*

$$\sum_{n=1}^{\infty} n^{r-2} h(n) P\{\sup_{m \geqslant n} |m^{-1/q} S_m| > \varepsilon\} < \infty \quad \text{for all } \varepsilon > 0. \tag{11}$$

Proof. Assume that (8) holds. In contrast to Theorem 1 now the case when $rq \geqslant 1$, $0 < q < 1$, $EX_1 \neq 0$ may occur. Suppose that these conditions are satisfied. By Theorem 1 we have

$$\sum_{n=1}^{\infty} n^{r-2} h(n) P\{\max_{1 \leqslant m \leqslant n} |\sum_{k=1}^{m} (X_k - EX_k)| > \varepsilon n^{1/q}\} < \infty \quad \text{for all } \varepsilon > 0.$$

Note that

$$\lim_{n \to \infty} \frac{1}{n^{1/q}} \max_{1 \leqslant m \leqslant n} |\sum_{k=1}^{m} EX_k| \leqslant \lim_{n \to \infty} \frac{E|X_1|}{n^{1/q-1}} = 0.$$

It is proved that (8) implies (9), and hence also (10). Assume now that (10) is true. Let $\{X'_n, n \geqslant 1\}$ be a sequence of random variables which are independent among themselves and of the sequence $\{X_n, n \geqslant 1\}$ such that X'_n and X_n have the same distribution for all $n \geqslant 1$. For independent identically distributed symmetric random variables $X_n^{(s)} = X_n - X'_n$, $n \geqslant 1$, the following inequality holds:

$$\sum_{n=1}^{\infty} n^{r-2} h(n) P\{|S_n^{(s)}| > \varepsilon n^{1/q}\} < \infty \quad \text{for all } \varepsilon > 0, \tag{12}$$

where $S_n^{(s)} = X_1^{(s)} + \ldots + X_n^{(s)}$. Now we prove that

$$\lim_{n \to \infty} P\{|S_n^{(s)}| > \varepsilon n^{1/q}\} = 0 \quad \text{for all } \varepsilon > 0. \tag{13}$$

This is obvious for $r \geqslant 2$. Let $1 \leqslant r < 2$. Assume the contrary. Then there exist numbers $\varepsilon > 0$, $\gamma > 0$ and a sequence $\{m_n, n \geqslant 1\}$ of natural numbers such that

$$P\{|S_{m_n}^{(s)}| > \varepsilon m_n^{1/q}\} > \gamma, \quad n \geqslant 1.$$

We may assume that $m_{n+1} \geqslant 2m_n$ for all $n \geqslant 1$. If this is not true, then the sequence $\{m_n, n \geqslant 1\}$ contains a subsequence with this property. Since

$$P\{\sum_{k=m_n+1}^{m} X_k^{(s)} \geqslant 0\} \geqslant 1/2 \quad \text{for all } m = m_n+1, \ldots, 2m_n,$$

we have

$$2P\{|S_m^{(s)}| > 2^{-1/q}\varepsilon m^{1/q}\} \geqslant 2P\{|S_{m_n}^{(s)}| > 2^{-1/q}\varepsilon m^{1/q}, \sum_{k=m_n+1}^{m} X_k^{(s)} \geqslant 0\}$$
$$\geqslant P\{|S_{m_n}^{(s)}| > 2^{-1/q}\varepsilon m^{1/q}\} \geqslant P\{|S_{m_n}^{(s)}| > \varepsilon m_n^{1/q}\} \geqslant \gamma,$$

which together with (12) implies that

$$\infty > \sum_{n=1}^{\infty} \sum_{m=m_n+1}^{2m_n} m^{r-2} h(m) P\{|S_m^{(s)}| > 2^{-1/q}\varepsilon m^{1/q}\} \geqslant \frac{\gamma}{2} \sum_{n=1}^{\infty} m_n^{r-1} h(m_n) = \infty.$$

We obtain a contradiction which proves (13).

Let us put $a_n = P\{|X_1^{(s)}| > \varepsilon n^{1/q}\}$. Note that

$$1-(1-a_n)^n = P\{\max_{1 \leqslant k \leqslant n} |X_k^{(s)}| > \varepsilon n^{1/q}\}$$
$$\leqslant P\{\max_{1 \leqslant k \leqslant n} |S_k^{(s)}| > \tfrac{1}{2}\varepsilon n^{1/q}\} \leqslant 2P\{|S_n^{(s)}| > \tfrac{1}{2}\varepsilon n^{1/q}\}.$$

The last inequality is Lévy's maximal inequality for sums of independent symmetrically distributed random variables (Loève (1977), Part III, Chapter V, Section 18.1 C). By (13) we have $\lim_{n\to\infty} n\ln(1-a_n) = 0$. By the inequalities $1-e^x \geqslant e^x|x|$ for $x \leqslant 0$ and $\ln(1-y) \leqslant -y$ for $y \in [0, 1)$, we obtain

$$\tfrac{1}{2} na_n \leqslant \tfrac{1}{2}|n\ln(1-a_n)| \leqslant 1-\exp\bigl(n\ln(1-a_n)\bigr) \leqslant 2P\{\max_{1 \leqslant m \leqslant n} |S_m^{(s)}| > \tfrac{1}{2}\varepsilon n^{1/q}\}$$

for all n greater than some n_0. This and (12) imply that

$$\sum_{n=1}^{\infty} n^{r-1} h(n) P\{|X_1^{(s)}| > \varepsilon n^{1/q}/2\} < \infty \quad \text{for all } \varepsilon > 0.$$

By Lemma 3 we have $E\bigl(|X_1^{(s)}|^{rq} h(|X_1^{(s)}|^q)\bigr) < \infty$, and hence $E\bigl(|X_1|^{rq} h(|X_1|^q)\bigr) < \infty$.

Now we prove that $a = EX_1 = 0$ for $1 \leqslant q < 2$. Assume that this is not true, that is, $a \neq 0$. Since $|a|n \leqslant |S_n - an| + |S_n|$, we have

$$1 = P\{|S_n - an| + |S_n| > |a|n^{1/q}/2\} \leqslant P\{|S_n - an| > |a|n^{1/q}/4\} + P\{|S_n| > |a|n^{1/q}/4\},$$

and hence

$$\infty = \sum_{n=n_0}^{\infty} n^{r-2} h(n) \leqslant \sum_{n=1}^{\infty} n^{r-2} h(n) P\{|S_n - an| > |a|n^{1/q}/4\}$$
$$+ \sum_{n=1}^{\infty} n^{r-2} h(n) P\{|S_n| > |a|n^{1/q}/4\}.$$

The series on the right-hand side of the inequality converge. The first series converges by Theorem 1, while the second one by the assumption. We obtain a contradiction, and hence $a = 0$.

For $r > 1$ we now prove the equivalence of (8)–(11). It is obvious that (10) follows from (11). Assume that (10) is true. Therefore (12) is also true. Note that

$$\sum_{n=3}^{\infty} n^{r-2} h(n) P\{\sup_{m\geqslant n} |m^{-1/q} S_m^{(s)}| > \varepsilon\}$$
$$\leqslant \sum_{j=1}^{\infty} \sum_{n=2^j+1}^{2^{j+1}-1} n^{r-2} h(n) P\{\sup_{m\geqslant n} |m^{-1/q} S_m^{(s)}| > \varepsilon\}$$
$$\leqslant \sum_{j=1}^{\infty} 2^{(j+1)(r-1)} h(2^{j+1}) P\{\sup_{m\geqslant 2^j} |m^{-1/q} S_m^{(s)}| > \varepsilon\}$$
$$\leqslant \sum_{j=1}^{\infty} 2^{(j+1)(r-1)} h(2^{j+1}) \sum_{i=j}^{\infty} P\{\max_{2^i\leqslant m<2^{i+1}} |m^{-1/q} S_m^{(s)}| > \varepsilon\}.$$

By Lévy's maximal inequality for sums of independent symmetrically distributed random variables we have

$$P\{\max_{2^i\leqslant m<2^{i+1}} |m^{-1/q} S_m^{(s)}| > \varepsilon\} \leqslant 2P\{|S_{2^{i+1}}^{(s)}| > \varepsilon 2^{i/q}\}.$$

Hence

$$\text{(14)} \quad \sum_{n=3}^{\infty} n^{r-2} h(n) P\{\sup_{m\geqslant n} |m^{-1/q} S_m^{(s)}| > \varepsilon\}$$
$$\leqslant 2 \sum_{j=1}^{\infty} 2^{(j+1)(r-1)} h(2^{j+1}) \sum_{i=j}^{\infty} P\{|S_{2^{i+1}}^{(s)}| > \varepsilon 2^{i/q}\}.$$

The iterated series on the right-hand side may be estimated as follows:

$$\sum_{j=1}^{\infty} 2^{(j+1)(r-1)} h(2^{j+1}) \sum_{i=j}^{\infty} P\{|S_{2^{i+1}}^{(s)}| > \varepsilon 2^{i/q}\}$$
$$= \sum_{i=1}^{\infty} \sum_{j=1}^{i} 2^{(j+1)(r-1)} h(2^{j+1}) P\{|S_{2^{i+1}}^{(s)}| > \varepsilon 2^{i/q}\}$$
$$\leqslant (2^{r-1}-1)^{-1} \sum_{i=1}^{\infty} 2^{(i+2)(r-1)} h(2^{i+1}) P\{|S_{2^{i+1}}^{(s)}| > \varepsilon 2^{i/q}\}.$$

On the other hand,

$$\infty > \sum_{n=1}^{\infty} n^{r-2} h(n) P\{|S_n^{(s)}| > 2^{-2/q} \varepsilon n^{1/q}\} \geqslant$$

$$\geqslant \sum_{i=1}^{\infty} \sum_{n=2^i+1}^{2^{i+1}-1} n^{r-2} h(n) P\{|S_n^{(s)}| > 2^{-2/q} \varepsilon n^{1/q}\} \geqslant \sum_{i=1}^{\infty} 2^{i(r-1)} h(2^i) P\{|S_{2^i}^{(s)}| > \varepsilon 2^{(i-1)/q}\}$$

$$\geqslant \sum_{i=1}^{\infty} 2^{(i+1)(r-1)} h(2^{i+1}) P\{|S_{2^{i+1}}^{(s)}| > \varepsilon 2^{i/q}\}.$$

Here we applied (12) and Lévy's maximal inequality for sums of independent symmetrically distributed random variables:

$$P\{|S_n^{(s)}| > 2^{-2/q} \varepsilon n^{1/q}\} \geqslant P\{|S_{2^i}^{(s)}| > 2^{-2/q} \varepsilon n^{1/q}\} \quad \text{for } 2^i \leqslant n < 2^{i+1}.$$

The estimations above and (14) imply that

$$\sum_{n=1}^{\infty} n^{r-2} h(n) P\{\sup_{m \geqslant n} |m^{-1/q} S_n^{(s)}| > \varepsilon\} < \infty.$$

By the symmetrization inequality (Loève (1977), Part III, Chapter V, Section 18.1 A) we have

$$P\{\sup_{m \geqslant n} |m^{-1/q} S_m - \operatorname{med}(m^{-1/q} S_m)| > \varepsilon\} \leqslant 2P\{\sup_{m \geqslant n} |m^{-1/q} S_m^{(s)}| > \varepsilon\}.$$

Hence

$$\sum_{n=1}^{\infty} n^{r-2} h(n) P\{\sup_{m \geqslant n} |m^{-1/q} S_m - \operatorname{med}(m^{-1/q} S_m)| > \varepsilon\} < \infty. \tag{15}$$

Since (10) implies (8), we have $E|X_1|^q < \infty$ and $EX_1 = 0$ in the case $1 \leqslant q < 2$. By the strong law of large numbers (Loève (1977), Part III, Chapter V, Section 17.1 A 4^0), the sequence $\{S_n/n^{1/q}, n \geqslant 1\}$ converges to zero almost surely, and hence

$$\lim_{n \to \infty} \sup_{m \geqslant n} |\operatorname{med}(m^{-1/q} S_m)| = 0.$$

This and (15) implies (11). ∎

3. AUXILIARY RESULTS

Here we prove some lemmas which were used previously.

LEMMA 1. *Let ξ and η be non-negative random variables. If $P\{\xi > x\} \leqslant DP\{\eta > x\}$ for some $D > 0$ and for all $x > 0$, then for $p > 0$, $b > a \geqslant 0$*

$$E(\xi^p I[a < \xi \leqslant b]) \leqslant Da^p P\{\eta > a\} + Db^p P\{\eta > b\} + DE(\eta^p I[a < \eta \leqslant b]).$$

If $E\xi^p < \infty$, then $E(\xi^p I[\xi > a]) \leqslant Da^p P\{\eta > a\} + DE(\eta^p I[\eta > a])$.

Proof. The inequalities can be proved with the help of integration by parts. ∎

LEMMA 2. *Let $h(x)$, $x\in[0,\infty)$, be a non-decreasing non-negative function, ξ be a non-negative random variable, $r\geqslant 1$, $p>0$, $\{b_n, n\geqslant 1\}$ be a non-decreasing sequence of positive numbers, $b_0=0$. Then*

$$\sum_{n=1}^{\infty} b_n^{r-1} h(b_n)(b_n-b_{n-1}) P\{\xi > b_n^{1/p}\} \leqslant E\big(\xi^{rp} h(\xi^p)\big).$$

Proof. The inequality is implied by the following relations:

$$\begin{aligned}\sum_{n=1}^{\infty} b_n^{r-1} h(b_n)(b_n-b_{n-1}) P\{\xi > b_n^{1/p}\} &= \sum_{n=1}^{\infty} b_n^{r-1} h(b_n)(b_n-b_{n-1}) \sum_{k=n}^{\infty} P\{b_k < \xi^p \leqslant b_{k+1}\}\\ &= \sum_{k=1}^{\infty}\sum_{n=1}^{k} b_n^{r-1} h(b_n)(b_n-b_{n-1}) P\{b_k < \xi^p \leqslant b_{k+1}\}\\ &\leqslant \sum_{k=1}^{\infty} b_k^{r} h(b_k) P\{b_k < \xi^p \leqslant b_{k+1}\} \leqslant E\big(\xi^{rp} h(\xi^p)\big). \blacksquare\end{aligned}$$

LEMMA 3. *Let $h(x)$, $x\in[0,\infty)$, be a non-decreasing non-negative function such that*

$$\limsup_{x\to\infty} h(2x)/h(x) < \infty,$$

ξ be a non-negative random variable, $r\geqslant 1$, $p>0$, and $\{b_n, n\geqslant 1\}$ be an unbounded non-decreasing sequence of positive numbers with the property (3), *$b_0=0$. Then the exist an integer $k_0\geqslant 1$ and $d>0$ such that*

$$db^{-r} E\big(\xi^{rp} h(\xi^p/b)\big) - d(b_{k_0}/b)^{rp} h(b_{k_0}^p/b) \leqslant \sum_{n=1}^{\infty} b_n^{r-1} h(b_n)(b_n-b_{n-1}) P\{\xi > b_n^{1/p}\}.$$

Proof. Note that

$$\sum_{n=1}^{k} b_n^{r-1} h(b_n)(b_n-b_{n-1}) \geqslant \sum_{n=2}^{k}\int_{b_{n-1}}^{b_n} x^{r-1} h(x)\,dx = \int_{b_1}^{b_k} x^{r-1} h(x)\,dx.$$

The properties of the function h imply the existence of numbers $y_0\geqslant b_1$ and $c>0$ such that $0<h(y)\leqslant ch(y/2)$ for $y\geqslant y_0$. If $y>2y_0$, then

$$y^{-r} h(y)^{-1}\int_{b_1}^{y} x^{r-1} h(x)\,dx \geqslant y^{-r} h(y)^{-1}\int_{y/2}^{y} x^{r-1} h(x)\,dx \geqslant (1-2^{-r})\frac{1}{rc}.$$

Consequently, there exist an integer $k_0\geqslant 1$ and $d>0$ such that

$$\sum_{n=1}^{k} b_n^{r-1} h(b_n)(b_n-b_{n-1}) \geqslant db_k^{r} h(b_k) \qquad \text{for all } k\geqslant k_0.$$

By (3) the inequality $b_{k+1} \leqslant bb_k$ is true for all $k = 1, 2, \ldots$ and for some $b \geqslant 1$. With these remarks we obtain

$$\sum_{n=1}^{\infty} b_n^{r-1} h(b_n)(b_n - b_{n-1}) P\{\xi > b_n^{1/p}\}$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{k} b_n^{r-1} h(b_n)(b_n - b_{n-1}) P\{b_k < \xi^p \leqslant b_{k+1}\}$$

$$\geqslant d \sum_{k=k_0}^{\infty} b_k^r h(b_k) P\{b_k < \xi^p \leqslant b_{k+1}\}$$

$$\geqslant d \sum_{k=k_0}^{\infty} (b_{k+1}/b)^r h(b_{k+1}/b) P\{b_k < \xi^p \leqslant b_{k+1}\}$$

$$\geqslant db^{-r} E\big(\xi^{rp} h(\xi^p/b)\big) - d(b_{k_0}/b)^{rp} h(b_{k_0}/b). \blacksquare$$

LEMMA 4. *Let $\{b_n, n \geqslant 1\}$ be an unbounded non-decreasing sequence of positive numbers with the property* (3), $b_0 = 0$, $p \in (0, 2)$, $h \in \mathscr{H}_p$, ξ *be a non-negative random variable. Then there exist an integer $k_0 \geqslant 1$ and $K > 0$ such that*

$$\sum_{n=1}^{\infty} h(b_n) \frac{b_n - b_{n-1}}{b_n^{2/p}} E(\xi^2 I[\xi \leqslant b_n^{1/p}]) \leqslant k_0 b_{k_0} h(b_{k_0}) + K(k_0+1) E\big(\xi^p h(b^2 \xi^p)\big).$$

Proof. By (3) the inequality $b_{n+1} \leqslant bb_n$ is true with some $b \geqslant 1$. Hence

$$\sum_{n=k}^{\infty} h(b_n) \frac{b_n - b_{n-1}}{b_n^{2/p}} \leqslant \frac{h(bb_k)}{b_k^{2/p-1}} + \sum_{n=k+1}^{\infty} h(bb_{n-1}) \frac{b_n - b_{n-1}}{b_n^{2/p}}.$$

The last series can be estimated as follows:

$$\sum_{n=k+1}^{\infty} h(bb_{n-1}) \frac{b_n - b_{n-1}}{b_n^{2/p}} \leqslant \sum_{n=k+1}^{\infty} \int_{b_{n-1}}^{b_n} h(bx) x^{-2/p} dx = \int_{b_k}^{\infty} h(bx) x^{-2/p} dx.$$

Since $h \in \mathscr{H}_p$, we have

$$\limsup_{y \to \infty} \frac{1}{h(by) y^{1-2/p}} \int_{y}^{\infty} h(bx) x^{-2/p} dx < \infty.$$

Hence there exists a constant $C > 0$ such that

$$\int_{b_k}^{\infty} h(bx) x^{-2/p} dx \leqslant Ch(bb_k) b_k^{1-2/p}$$

for all k greater than some integer $k_0 \geqslant 1$. It follows from the above estimates that

$$\sum_{n=k}^{\infty} h(b_n) \frac{b_n - b_{n-1}}{b_n^{2/p}} \leqslant K \frac{h(bb_k)}{b_k^{2/p-1}} \quad \text{for all } k \geqslant k_0,$$

where $K = 1+C$. With the help of this estimate we obtain

$$\sum_{n=1}^{\infty} h(b_n)\frac{b_n-b_{n-1}}{b_n^{2/p}} E(\xi^2 I[\xi \leqslant b_n^{1/p}])$$

$$= \Big(\sum_{k=1}^{k_0-1}\Big(\sum_{n=k}^{k_0-1}+\sum_{n=k_0}^{\infty}\Big)+\sum_{k=k_0}^{\infty}\sum_{n=k}^{\infty}\Big) h(b_n)\frac{b_n-b_{n-1}}{b_n^{2/p}} E(\xi^2 I[b_{k-1} < \xi^p \leqslant b_k])$$

$$\leqslant k_0 b_{k_0} h(b_{k_0}) + K(k_0+1)\sum_{k=1}^{\infty}\frac{h(bb_k)}{b_k^{2/p-1}} E(\xi^2 I[b_{k-1} < \xi^p \leqslant b_k])$$

$$\leqslant k_0 b_{k_0} h(b_{k_0}) + K(k_0+1) E\big(\xi^p h(b^2\xi^p)\big). \quad \blacksquare$$

LEMMA 5. *Let* $\{(X_{nk},\ 1 \leqslant k \leqslant m_n),\ n \geqslant 1\}$ *be an array of rowwise independent random variables, with means zero when they exist, which is stochastically dominated in mean of order* α *by a random variable* X *with respect to an unbounded non-decreasing sequence* $\{b_n,\ n \geqslant 1\}$ *of positive numbers,* $m_n \leqslant cb_n$, $n \geqslant 1$, $E|X|^{rq/(1-\alpha q)} < \infty$ *for some* $0 < q < 2$, $r \geqslant 1$, $\alpha < 1/q-1/2$, $c > 0$. *Then*

$$\lim_{n\to\infty}\frac{1}{b_n^q}\max_{1\leqslant m\leqslant m_n}\Big|\sum_{k=1}^{m} E(X_{nk} I[|X_{nk}| \leqslant b_n^{1/q}])\Big| = 0. \tag{16}$$

Proof. The exact value of constants c and D in (1) plays no role in the proof and we assume them equal to one. Let $rq/(1-\alpha q) < 1$. For any $\varepsilon > 0$ there exists $a > 0$ such that

$$E(|X|^{rq/(1-\alpha q)} I[|X| > a]) < \varepsilon.$$

Since $\alpha < 1/q$ and $\lim_{n\to\infty} b_n = \infty$, we have $b_n^{1/q} > ab_n^\alpha$ for all n greater than some n_0. If $n \geqslant n_0$, then

$$\sum_{k=1}^{m_n} E(|X_{nk}| I[|X_{nk}| \leqslant b_n^{1/q}])$$

$$= \sum_{k=1}^{m_n} E(|X_{nk}| I[|X_{nk}| \leqslant ab_n^\alpha]) + \sum_{k=1}^{m_n} E(|X_{nk}| I[ab_n^\alpha < |X_{nk}| \leqslant n^{1/q}])$$

$$\leqslant m_n ab_n^\alpha + \sum_{k=1}^{m_n} E(|X_{nk}| I[ab_n^\alpha < |X_{nk}| \leqslant b_n^{1/q}]).$$

By (5), Lemma 1, and the inequality $m_n \leqslant b_n$ we obtain

$$\sum_{k=1}^{m_n} E(|X_{nk}| I[ab_n^\alpha < |X_{nk}| \leqslant b_n^{1/q}]) \leqslant ab_n^{\alpha+1} P\{|X| > a\} + b_n^{1/q+1} P\{|X| > b_n^{1/q-\alpha}\}$$

$$+ b_n^{\alpha+1} E(|X| I[a < |X| \leqslant b_n^{1/q-\alpha}]).$$

Since $0 < rq/(1-\alpha q) < 1$, we have

$$E(|X| I[a < |X| \leqslant b_n^{1/q-\alpha}]) \leqslant b_n^{1/q-\alpha-r} E(|X|^{rq/(1-\alpha q)} I[|X| > a]).$$

Therefore

$$\frac{1}{b_n^{1/q}} \sum_{k=1}^{m_n} E(|X_{nk}| I[|X_{nk}| \leqslant b_n^{1/q}]) \leqslant \frac{2a}{b_n^{1/q-\alpha-1}} + b_n P\{|X| > b_n^{1/q-\alpha}\} + \frac{\varepsilon}{b_n^{r-1}}.$$

Then (16) follows from the above, since $\lim_{n\to\infty} b_n P\{|X| > b_n^{1/q-\alpha}\} = 0$, and $1/q-\alpha-1 > 0$. The last inequality follows from the assumption that $rq/(1-\alpha q) < 1$ and $r \geqslant 1$. Let $rq/(1-\alpha q) \geqslant 1$. In this case, by assumption, $EX_{nk} = 0$ for all $k = 1, \ldots, m_n$, $n \geqslant 1$. By (5) and Lemma 1 we have

$$\Big|\sum_{k=1}^{m} EX_{nk} I[|X_{nk}| \leqslant b_n^{1/q}]\Big| = \Big|\sum_{k=1}^{m} EX_{nk} I[|X_{nk}| > b_n^{1/q}]\Big|$$
$$\leqslant b_n^{1/q+1} P\{|X| > b_n^{1/q-\alpha}\} + b_n^{\alpha+1} E(|X| I[|X| > b_n^{1/q-\alpha}]).$$

Since $rq/(1-\alpha q) \geqslant 1$, we obtain

$$E(|X| I[|X| > b_n^{1/q-\alpha}]) \leqslant \frac{1}{b_n^{r-1/q+\alpha}} E(|X|^{rq/(1-\alpha)} I[|X| > b_n^{1/q-\alpha}]).$$

Therefore

$$\frac{1}{b_n^{1/q}} \Big|\sum_{k=1}^{m} EX_{nk} I[|X_{nk}| \leqslant b_n^{1/q}]\Big|$$
$$\leqslant b_n P\{|X| > b_n^{1/q-\alpha}\} + \frac{1}{b_n^{r-1}} E(|X|^{rq/(1-\alpha q)} I[|X| > b_n^{1/q-\alpha}]).$$

Both terms on the right-hand side tend to zero as $n \to \infty$, which implies (16). ■

Acknowledgements. The research of Victor Kruglov has been supported by two grants from the Russian Fond of Fundamental Research 05-01-00535, 05-01-00583. The work of Andrei Volodin has been partially supported by a grant from the National Science and Engineering Research Council of Canada. The authors are grateful to Professor Vladimir Zolotarev (Russian Academy of Science) and anonymous referee for helpful remarks that improved presentation of the results.

REFERENCES

[1] Z. Bai and Ch. Su, *The complete convergence for partial sums of iid random variables*, Scientia Sinica. Series A, 28 (1985), pp. 1261–1277.

[2] K. B. Baum and M. Katz, *Convergence rates in the law of large numbers*, Trans. Amer. Math. Soc. 120 (1965), pp. 108–123.

[3] P. Erdös, *On a theorem of Hsu and Robbins*, Ann. Math. Statist. 20 (1949), pp. 286–291.
[4] P. Erdös, *Remark on my paper "On a theorem of Hsu and Robbins"*, Ann. Math. Statist. 21 (1950), p. 138.
[5] A. Gut, *On complete convergence in the law of large numbers for subsequences*, Ann. Probab. 13 (1985), pp. 1286–1291.
[6] A. Gut, *Complete convergence for arrays*, Period. Math. Hungar. 25 (1992), pp. 51–75.
[7] P. L. Hsu and H. Robbins, *Complete convergence and the law of large numbers*, Proc. Natl. Acad. Sci. USA 33 (1947), pp. 25–31.
[8] T. C. Hu, F. Moricz and R. L. Taylor, *Strong law of large numbers for arrays of rowwise independent random variables*, Acta Math. Acad. Sci. Hungar. 54 (1989), pp. 153–162.
[9] V. M. Kruglov, A. Volodin and T.-C. Hu, *On complete convergence for arrays*, Statist. Probab. Lett. 76 (2006), pp. 1631–1640.
[10] M. Loève, *Probability Theory I*, Springer, 1977.
[11] M. Maejima, *A theorem on convergence for weighted sums of i.i.d. random variables*, Rep. Statist. Appl. Res. Un. Japan. Sci. Engrs. 24 (1977), pp. 1–4.
[12] F. L. Spitzer, *A combinatorial lemma and its applications*, Trans. Amer. Math. Soc. 82 (1956), pp. 323–339.

Department of Mathematical Statistics
Faculty of Applied Mathematics and Cybernetics
Moscow State University
Vorob'evy Gory
119899, Moscow, Russia
E-mail: vkrug@dataforce.net

Department of Mathematics and Statistics
University of Regina
Regina, Saskatchewan S4S0A2
Canada
E-mail: volodin@math.uregina.ca

Received on 18.5.2006