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EXPONENTIAL FORMULA RELATED TO THE GAUSS SEMIGROUP

BY

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To the memory of Professor Kazimierz Urbanik

Abstract. For the second derivative, an analogue of the classical Taylor's formula is considered on a suitable function space. The sum of the "Taylor series" represents the Gauss semigroup. This may be useful in describing the trajectories of some functions under the action of the Gauss semigroup.

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It is well known that the second derivative is the infinitesimal generator of a semigroup of operators in $L_p(-\infty, \infty)$, $1 \leqslant p < \infty$, given by the formula

$$(T_t f)(x) = \int_{\boldsymbol{R}} f(x+w)\gamma_t(dw), \qquad t > 0, \tag{1}$$

where

$$\gamma_t(dw) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{w^2}{4t}\right) dw. \tag{2}$$

More exactly, the generator A of (1) is of the form

$$(Af)(x) = f''(x), \qquad x \in \boldsymbol{R},$$

with $D(A) = \{f \in L_p\colon f' \text{ is absolutely continuous and } f'' \in L_p(-\infty, \infty)\}$.

For $A = d^2/dx^2$, let us write formally

$$(e^{tA} f)(x) = \sum_{k=0}^{\infty} \frac{f^{(2k)}(x)}{k!} t^k, \qquad x \in \boldsymbol{R},\ t > 0. \tag{3}$$

We construct a function space $\mathscr{X}$ on which a C_0-semigroup $(e^{tA}, t \geqslant 0)$ closely related to (1) can be defined via formula (3).

Let us fix a sequence $0 < t_n \nearrow \infty$. For $f \in C^\infty$, we set

$$\|f\|_{n,m} = \sum_{k=0}^{\infty} \frac{m^k}{k!} \max_{|x| \leqslant n} |f^{(2k)}(x)|, \qquad n, m = 1, 2, \ldots,$$

(4)

$$\|f\|_m^* = \int_{\boldsymbol{R}} |f(w)|\, \gamma_{t_m}(dw) = \|f\|_{L^1(\boldsymbol{R}, \gamma_{t_m})}, \qquad m = 1, 2, \ldots$$

We put

$$\mathscr{X}_0 = \{f \in C^\infty;\ \|f\|_{n,m} < \infty,\ \|f\|_m^* < \infty,\ n, m = 1, 2, \ldots\}.$$

$\mathscr{X}_0$ with seminorms (4) is a Fréchet space, i.e. metric complete and locally convex. We omit a standard proof of completeness. Evidently, $\mathscr{X}_0$ embraces all polynomials. We define the space $\mathscr{X}$ by putting

$$\mathscr{X} = [\text{Polynomials}]^{-}_{\mathscr{X}_0},$$

the closure of polynomials with respect to seminorms (4).

THEOREM. *Let $A = d^2/dx^2$ and let $f \in \mathscr{X}$. For $t \geqslant 0$ we set*

$$(e^{tA} f)(x) = \sum_{k=0}^{\infty} \frac{f^{(2k)}(x)}{k!} t^k. \tag{5}$$

Then $(e^{tA}, t \geqslant 0)$ is a C_0-semigroup of continuous operators acting in $\mathscr{X}$. Moreover,

$$(e^{tA} f)(x) = \int_{\boldsymbol{R}} f(x+w)\, \gamma_t(dw), \qquad t > 0,\ f \in \mathscr{X}, \tag{6}$$

γ_t being the Gaussian measure given by (2).

Before starting the proof of the Theorem we continue with a few remarks.

Clearly, our theorem states that the Gauss semigroup acts on $\mathscr{X}$ as a C_0-semigroup of ordinary exponential operators. The exponential formulae (5) and (6) can be treated as an analogue for the second derivative of the classical Taylor's formula.

For an analytic function f, the condition $|f^{(k)}(0)|^{1/k} = o(k^{1/2})$ implies $f \in \mathscr{X}$. Indeed, it is enough to estimate the series

$$\sum_{k=0}^{\infty} \frac{|f^{(k)}(0)|}{k!} \int_{-\infty}^{\infty} |x|^k \gamma_t(dx),$$

having in mind that the $(2k)$-th moment of γ_t equals $((2k)!/k!)\, t^k$, and using the Stirling formula.

The equality

$$(T_t f)(x) = \sum_{k=0}^{\infty} \frac{f^{(2k)}(x)}{k!} t^k$$

is the *raison d'être* of the space $\mathscr{X}$. Calculating or estimating the Gaussian integrals (1) for $f \in \mathscr{X}$, we can take advantage of the simplicity of "Taylor's series" (5). Let us consider some examples.

I. Let $[y]$ denote the integral part of y. For any complex number α, the function

$$f_\alpha(x) = \sum_{n=0}^{\infty} \alpha^{[n/2]} \frac{x^n}{n!}$$

is the "eigenfunction" for (T_t) in the sense that $(T_t f_\alpha)(x) = e^{t\alpha} f_\alpha(x)$ for all $t \geqslant 0$. This follows immediately from the equalities

$$\left[\frac{n+2k}{2}\right] = \left[\frac{n}{2}\right] + k \quad \text{and} \quad f^{(2k)}(x) = \sum_{\nu=0}^{\infty} \frac{f^{(2k+\nu)}(0)}{\nu!} x^\nu.$$

II. For the Bessel function

$$J_p = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p},$$

we set $I_p(x) = J_p(\sqrt{x})$. In particular,

$$I_0(x) = \sum \frac{(-1)^n}{(n!)^2} \frac{x^n}{4^n}.$$

Clearly, $I_p \in \mathscr{X}$ for $p \geqslant 0$. The integral $(T_t I_0)(x) = \int_{-\infty}^{\infty} I_0(x+w)\gamma_t(dw)$ can be expressed in terms of I_{2k}, $k = 0, 1, \ldots$ Namely,

$$(*) \qquad (T_t I_0)(x) = \sum_{k=0}^{\infty} \left(\frac{t}{4x}\right)^k \frac{1}{k!} I_{2k}(x)$$

(there is no singularity at $x = 0$). Indeed,

$$I_0^{(n)}(0) = \frac{(-1)^n}{n!} \frac{1}{4^n}.$$

Thus we have

$$I_0^{(2k)}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+2k)!} \frac{1}{4^{n+2k}},$$

and consequently we get $(*)$.

III. Obviously, for $f \in \mathscr{X}$, the function $v(t, x) = (T_t f)(x)$ is a solution of the initial value problem for the heat equation and even more. Namely, we have

$$\frac{\partial^k v}{\partial t^k} = \frac{\partial^{2k} v}{\partial x^{2k}}, \qquad k = 1, 2, \ldots,$$

and

$$\frac{\partial^k}{\partial t^k} v(0, x) = f^{(2k)}(x), \quad k = 0, 1, \ldots$$

Proof of the Theorem. For a monomial $f(w) = w^k$, let us put

$$\psi_t((w^k)) = \psi_t(k) = \begin{cases} 1 & \text{for } k = 0, \\ 0 & \text{for } k \text{ odd}, \\ \dfrac{(2\nu)!}{\nu!} t^\nu & \text{for } k \text{ even}. \end{cases}$$

For a polynomial

$$(x+w)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} w^k,$$

we set

$$\phi_t[(x+w)^n] = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \psi_t(k).$$

An elementary though laborious argument leads to the formula

$$e^{tA}(x^n) = \sum_{k=0}^{\infty} \frac{(x^n)^{(2k)}}{k!} t^k = \phi_t[(x+w)^n].$$

Since $\psi_t(k)$ is the k-th moment of the Gaussian measure γ_t, we get immediately

$$(e^{tA} f)(x) = \int_{\boldsymbol{R}} f(x+w) \gamma_t(dw) \tag{7}$$

for any polynomial f.

Let $f \in \mathscr{X}$ and let (P_n) be a sequence of polynomials tending to f in $\mathscr{X}$ (i.e. in seminorms (4)). Since the shift τ_x: $f(w) \to f(w+x)$ is continuous in $L^1(\boldsymbol{R}, \gamma_t)$, $t > 0$, we get in particular

$$(e^{tA} P_n)(x) = \int_{\boldsymbol{R}} P_n(x+w) \gamma_t(dw) \to \int_{\boldsymbol{R}} f(x+w) \gamma_t(dw), \tag{8}$$

where the convergence is almost uniform on $\boldsymbol{R}$.

Let $t > 0$, $|x| \leqslant n$, $m > t$. For $\varepsilon > 0$ we find s_0 such that $\|P_s - f\|_{n,m} < \varepsilon$ for $s \geqslant s_0$. Then we have

$$\max_{|x| \leqslant n} \left| \sum_{k=0}^{\infty} \frac{f^{(2k)}(x)}{k!} t^k - \sum_{k=0}^{\infty} \frac{P_s^{(2k)}(x)}{k!} t^k \right|$$

$$\leqslant \sum_{k=0}^{\infty} \max_{|x| \leqslant n} |f^{(2k)}(x) - P_s^{(2k)}(x)| \cdot \frac{m^k}{k!} < \varepsilon \quad \text{for } s \geqslant s_0.$$

Let us remark that, for $f \in \mathscr{X}$, $s > 0$, $n = 1, 2, \ldots$, $A = d^2/dx^2$,

$$A^n(e^{sA} f) = e^{sA}(A^n f). \tag{9}$$

Indeed, let

$$\sigma_N(x) = \sum_{k=0}^{N} \frac{s^k}{k!} (A^k f)(x).$$

Since $\|f\|_{n,m} < \infty$, for $n, m = 1, 2, \ldots$

$$\sigma_N(x) \to (e^{sA} f)(x) \text{ almost uniformly on } \boldsymbol{R}.$$

Moreover, $\|A^k f\|_{n,m} < \infty$, so we also have

$$A^n \sigma_N(x) = \sigma_N(A^n f)(x) \to e^{sA}(A^n f)$$

and, consequently, we get

$$A^n(e^{sA} f) = e^{sA}(A^n f), \qquad f \in \mathscr{X},\ n = 1, 2, \ldots \tag{10}$$

We are going to prove that $f \in \mathscr{X}$ implies $e^{sA} f \in \mathscr{X}$, $s > 0$. Let $f \in \mathscr{X}$. It means that $\|f\|_{n,m} < \infty$, $\|f\|_m^* < \infty$ and there is a sequence of polynomials, say (P_s), such that

$$\|P_s - f\|_{n,m} \to 0 \quad \text{and} \quad \|P_s - f\|_m^* \to 0 \quad \text{as } s \to \infty.$$

We have to show the same for $e^{sA} f$.

Since $A^k(e^{sA} f) = e^{sA} A^k f$, we get

$$\begin{aligned} \|e^{sA} f\|_{n,m} &= \sum_{k=0}^{\infty} \frac{m^k}{k!} \max_{|x| \leqslant n} |A^k(e^{sA} f)(x)| = \sum_{k=0}^{\infty} \frac{m^k}{k!} \max_{|x| \leqslant n} (e^{sA} A^k f)(x) \\ &= \sum_{k=0}^{\infty} \frac{m^k}{k!} \max_{|x| \leqslant n} \Big| \sum_{\nu=0}^{\infty} \frac{s^\nu}{\nu!} (A^{k+\nu} f)(x) \Big| \leqslant \sum_{k=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{m^k s^\nu}{k!\, \nu!} \max_{|x| \leqslant n} |(A^{k+\nu} f)(x)| \\ &= \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{k=0}^{N} \binom{N}{k} m^k s^{N-k} \max_{|x| \leqslant n} |(A^N f)(x)| \\ &= \sum_{N=0}^{\infty} \frac{(s+m)^N}{N!} \max_{|x| \leqslant n} |(A^N f)(x)| \leqslant \|f\|_{n, m+m_0} < \infty \quad (\text{here } m_0 \geqslant s). \end{aligned} \tag{11}$$

Moreover, since the formula (6) follows easily from the classical facts and the definition of the seminorms (4), we can write

$$\begin{aligned} \|e^{sA} f\|_m^* &= \int_{\boldsymbol{R}} \Big| \int_{\boldsymbol{R}} f(x+w)\, \gamma_s(dw) \Big|\, \gamma_{t_m}(dx) \\ &\leqslant \int_{\boldsymbol{R}} \int_{\boldsymbol{R}} |f(x+w)|\, \gamma_s(dw)\, \gamma_{t_m}(dx) = \int_{\boldsymbol{R}} |f(u)|\, \gamma_{s+t_m}(du) \\ &\leqslant \|f\|_{m+m_0}^* < \infty \quad (\text{here } s \leqslant m_0). \end{aligned} \tag{12}$$

Now, take $Q_n = e^{sA} P_n$ $(n = 1, 2, \ldots)$. We shall show that $Q_n \to e^{sA} f$ in $\mathscr{X}$. We have

$$\|e^{sA}(f-P_n)\|_m^* = \int_R \Big|\int_R \big(f(x+w)-P_n(x+w)\big)\gamma_s(dw)\Big|\,\gamma_{s+t_m}(dx)$$

$$\leqslant \int_R\int_R |f(x+w)-P_n(x+w)|\,\gamma_s(dw)\gamma_{t_m}(dx) = \int_R |f(u)-P_n(u)|\,\gamma_{s+t_m}(du) \to 0,$$

since $\|f-P_n\|_m^* \to 0$ as $n\to\infty$ for each $m = 1, 2, \ldots$

Moreover,

$$\|e^{sA}(P_r-f)\|_{n,m} = \sum_{k=0}^{\infty} \frac{m^k}{k!}\max_{|x|\leqslant n}\left|\left(e^{sA}(P_r-f)(x)\right)^{(2k)}\right|$$

$$\leqslant \sum_{k=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{m^k}{k!}\frac{s^\nu}{\nu!}\max_{|x|\leqslant n}|(P_r-f)(x)^{(2(k+\nu))}|$$

$$= \sum_{N=0}^{\infty}\frac{(s+m)^N}{N!}\max_{|x|\leqslant n}|(P_r-f)^{(2N)}(x)| = \|P_r-f\|_{n,m+m_0} \to 0 \qquad (\text{here } m_0 \geqslant s).$$

Thus the operators e^{sA} act from $\mathscr{X}$ to $\mathscr{X}$. Moreover, for $s, t > 0$, $f\in\mathscr{X}$, we have

$$e^{sA}(e^{tA}f) = \sum_{k=0}^{\infty}\frac{s^k}{k!}A^k(e^{tA}f) = \sum_{k=0}^{\infty}\frac{s^k}{k!}\sum_{\nu=0}^{\infty}\frac{t^\nu}{\nu!}(A^{k+\nu}f)$$

$$= \sum_{N=0}^{\infty}\frac{1}{N!}\sum_{k=0}^{N}\binom{N}{k}s^k t^{N-k}A^N f = e^{(s+t)A}f$$

(since the convergence of the above series is absolute with respect to all seminorms $\|\cdot\|_{n,m}$ and $\|\cdot\|_m^*$).

Consequently, we have

$$e^{sA}(e^{tA}f) = e^{(s+t)A}f$$

for any $s, t\geqslant 0$ and $f\in\mathscr{X}$.

By (11) and (12), the operators $e^{sA}\colon \mathscr{X}\to\mathscr{X}$ are continuous in $\mathscr{X}$. It remains to show the continuity of the semigroup $(e^{sA},\ s\geqslant 0)$.

Let us note that, for $n, m = 1, 2, \ldots,$

$$\|e^{tA}f-f\|_{n,m}\to 0 \quad \text{as } t\to 0^+.$$

Indeed,

$$\|e^{tA}f-f\|_{n,m} = \Big\|\sum_{\nu=1}^{\infty}\frac{t^\nu}{\nu!}A^\nu f\Big\|_{n,m} \leqslant \sum_{k=0}^{\infty}\sum_{\nu=1}^{\infty}\frac{t^\nu}{\nu!}\frac{m^k}{k!}\max_{|x|\leqslant n}|(A^{\nu+k}f)(x)|$$

$$= \sum_{N=0}^{\infty}\frac{(t+m)^N}{N!}\max_{|x|\leqslant n}|(A^N f)(x)| - \sum_{k=0}^{\infty}\frac{m^k}{k!}\max_{|x|\leqslant n}|(A^k f)(x)| \to 0 \quad \text{as } t\to 0.$$

To show that $\|e^{tA}f-f\|_m^*\to 0$ let us first remark that, for every m and $f_s\to 0$ in $\mathscr{X}$, $\|e^{tA}f_s\|_m^*\to 0$ as $s\to\infty$, uniformly with respect to $0\leqslant t\leqslant 1$. In fact,

$$\|e^{tA}f_s\|_m^* = \int_{\boldsymbol{R}}\Big|\int_{\boldsymbol{R}} f_s(x+w)\,\gamma_t(dw)\Big|\,\gamma_{t_m}(dx)$$

$$\leqslant \int_{\boldsymbol{R}} |f_s(u)|\,\gamma_{t+t_m}(du)\leqslant C\int_{\boldsymbol{R}} |f_s(u)|\,\gamma_{t_m+1}\,du\leqslant C\,\|f_s\|_{m_0}^*$$

with $t_{m_0}>t_m+1$ and some $C>0$.

For any polynomial P,

$$\|e^{sA}P-P\|_m^*\to 0 \qquad \text{as } s\to 0\ (m=1,2,\ldots).$$

Indeed, since

$$\big((e^{sA}P)(x)-P(x)\big)^2=\sum_{k=1}^{L} c_k(x)\,s^k$$

for some polynomials $c_k(x)$, we get

$$\int_{\boldsymbol{R}}\big((e^{sA}P)(x)-P(x)\big)^2\,\gamma_{t_m}(dx)\to 0 \qquad \text{as } s\to 0,$$

so

$$\|e^{sA}P-P\|_m^*\leqslant \|e^{sA}P-P\|_{L^2(\boldsymbol{R},\gamma_{t_m})}\to 0 \qquad \text{as } s\to 0.$$

Let $f\in\mathscr{X}$ and let P_k be polynomials such that $\|P_k-f\|_m^*\to 0$ as $k\to\infty$ $(m=1,2,\ldots)$. Assuming $0\leqslant s\leqslant 1$, we have

$$\|e^{sA}f-f\|_m^*\leqslant \|e^{sA}(f-P_k)\|_m^*+\|e^{sA}P_k-P_k\|_m^*+\|P_k-f\|_m^*.$$

Let $\varepsilon>0$. We find a k_0 such that

$$\|f-P_{k_0}\|_m^*<\varepsilon/3, \qquad \sup_{0\leqslant s\leqslant 1}\|e^{sA}(f-P_{k_0})\|_m^*<\varepsilon/3$$

and, finally,

$$\|e^{sA}P_{k_0}-P_{k_0}\|<\varepsilon/3 \qquad \text{for } s<s_0.$$

Thus $\|e^{sA}f-f\|_m^*<\varepsilon$ for $s<s_0$, so $\|e^{sA}f-f\|_m^*\to 0$ $(s\to 0)$.

In a standard way we show the continuity of $(e^{sA}, s>0)$ at any point $s_0>0$, which completes the proof of the Theorem. ∎

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