PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 26, Fasc. 2 (2006), pp. 245–260

EXTENSIONS OF PONTRYAGIN HYPERGROUPS

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To the memory of Professor Kazimierz Urbanik

Abstract. The purpose of this paper is to investigate the extension problem for the category of commutative hypergroups. In fact, by applying the new notion of a field of compact subhypergroups, sufficiently many extensions can be established, and among them splitting extensions can be characterized. Moreover, the duality of extensions will be studied via duality of fields of hypergroups. The method of extension via fields of hypergroups yields the construction of Pontryagin hypergroups which do not arise from group-theoretic objects.

2000 Mathematics Subject Classification: 43A62.

Key words and phrases: Hypergroups, extensions, duality.

1. INTRODUCTION

The concept of convolution of measures on a locally compact group has been generalized in various directions. One direction is based on the notion of generalized convolutions of probability measures on $\boldsymbol{R}_+$ introduced by K. Urbanik in his pioneering work [10] of 1964. Another direction of extending convolutions of measures beyond the group case culminated in the axiomatic setting of a hypergroup due to C. F. Dunkl, R. I. Jewett, and R. Spector around 1975.

Roughly speaking, the hypergroup convolution is a probabilistic extension of the group convolution. In fact, it is possible to consider random walks on groups in terms of Pontryagin hypergroups. For example, there exists a random walk on $\boldsymbol{Z}$ with transition probabilities

$$p(n-1\,|\,n) = p(n+1\,|\,n) = \tfrac{1}{2} \quad (n \in \boldsymbol{Z}),$$

which can be associated with the discrete Pontryagin hypergroup $D := \{l_0, l_1, l_2, \ldots, l_n, \ldots\}$ equipped with the convolution

$$\varepsilon_{l_m} * \varepsilon_{l_n} = \tfrac{1}{2}\varepsilon_{l_{|m-n|}} + \tfrac{1}{2}\varepsilon_{l_{m+n}} \quad (m, n = 0, 1, 2, \ldots).$$

Since the hypergroup dual of D is isomorphic to the compact hypergroup $C:=[-1, 1]$ equipped with the convolution

$$\varepsilon_{\cos\theta_1} * \varepsilon_{\cos\theta_2} = \tfrac{1}{2}\varepsilon_{\cos(\theta_1+\theta_2)} + \tfrac{1}{2}\varepsilon_{\cos(\theta_1-\theta_2)} \qquad (\theta_1, \theta_2 \in [0, \pi]),$$

it corresponds to a random walk on the one-dimensional torus $\boldsymbol{T}$. But then the product hypergroup $C\times D$ corresponds to a random walk on $\boldsymbol{T}\times\boldsymbol{Z}$. An application of the present discussion is the construction of sufficiently many models of random walks on $\boldsymbol{T}\times\boldsymbol{Z}$ in terms of Pontryagin hypergroups which are obtained via hypergroup extensions of D by C as shown in Example 7.3.

Let H and L be hypergroups. Then a hypergroup K is called an *extension* of L by H if the sequence

$$1 \to H \to K \to L \to 1$$

is exact. If the quotient hypergroup K/H is defined, this is equivalent to the fact that K/H is isomorphic to L. Here the notions of subhypergroup, quotient hypergroup and isomorphism between hypergroups are taken from [1], a source from which all the basic knowledge on hypergroups needed in the sequel will be taken.

There exist several methods of constructing extensions of hypergroups from given ones. These methods lead to an insight into the structure of hypergroups. One of the methods is based on the notion of hypergroup join as introduced by Jewett [7] and further developed by Dunkl and Ramirez [2], Fournier and Ross [3], Voit [12], Vrem [14], and Zeuner [16]. The join $H\vee L$ of a compact commutative hypergroup H and a discrete commutative hypergroup L can be interpreted as the minimal extension of L by H. On the other hand, the maximal extension of L by H is the product hypergroup $H\times L$. The purpose of the present discussion is to construct, by generalizing the method of join, sufficiently many extensions which in some sense are larger than the join and smaller than the product. The method of substitution introduced by Voit [12] is another generalization of the join which provides extensions of hypergroups. The relation between the construction presented in this work and the substitution will be clarified in Section 6.

In the course of the paper, for two commutative hypergroups H and L such that each connected component of L is an open set, we shall give the definition of a field $\varphi: L\ni l\mapsto H(l)\subset H$ of compact subhypergroups $H(l)$ of H based on L, and show that every field φ gives rise to an extension $K(H, \varphi, L)$ of L by H as described in Theorem 3.1. Moreover, for strong hypergroups H and L such that each connected component of both L and the dual $\hat{H}$ of H is an open set, we shall introduce the dual $\hat{\varphi}: \hat{H}\ni\chi\mapsto Z(\chi)\subset\hat{L}$ of the field φ and show in Theorem 4.4 that the extension $K(\hat{L}, \hat{\varphi}, \hat{H})$ of $\hat{H}$ by $\hat{L}$ is isomorphic to the dual of $K(H, \varphi, L)$. The latter property implies that if both H and L are Pontryagin hypergroups, then $K(H, \varphi, L)$ is also a Pontryagin hypergroup. By applying the method of fields one can also obtain Pontryagin hypergroups not arising from group-theoretic objects as for example orbital actions and Gelfand pairs. This new aspect is illustrated in Examples 7.2 and 7.3.

In order to investigate the structure of hypergroups it will be essential to determine all extensions K of L by H for given commutative hypergroups H and L. In the corresponding discussion we give a characterization of extensions obtained by a field of compact subhypergroups. Those extensions will be called *splitting extensions*. If L is a discrete commutative hypergroup, it will be shown in Theorem 5.1 that all splitting extensions of L by H are determined by the construction via fields of compact subhypergroups. It is known that in general there are extensions which do not split. It remains still an open problem to determine all extensions of commutative hypergroups, a problem that waits for a solution.

2. PRELIMINARIES

In this section we recapitulate the principal notions from the basic theory of hypergroups by stressing those definitions and properties which are essential in the course of the discussion. We start with the definition of a hypergroup along the axiomatics established by Dunkl, Jewett, and Spector. Further elements of the theory can be taken from the monograph [1].

Let K be a locally compact (Hausdorff) space. We write $C(K)$ for the space of continuous complex-valued functions on K. The space $C(K)$ has various distinguished subspaces, $C_b(K)$, $C_0(K)$, and $C_c(K)$, the spaces of bounded continuous functions, those that vanish at infinity, and those with compact support, respectively. Both $C_b(K)$ and $C_0(K)$ are topologized by the uniform norm $\|\cdot\|_\infty$. We denote by $M_b(K)$, $M_b^+(K)$ and $M^1(K)$ the spaces of bounded measures, non-negative bounded measures and probability measures on K, respectively. For each $\mu \in M_b(K)$ the support of μ is denoted by $\operatorname{supp}(\mu)$ and the norm of μ is given by

$$\|\mu\| := \sup\{|\mu(f)| : f \in C_c(K), \|f\|_\infty \leqslant 1\}.$$

The symbol ε_x stands for the Dirac measure at $x \in K$. By $\mathscr{C}(K)$ we denote the space of non-empty compact subsets of K, furnished with the Michael–Hausdorff topology.

DEFINITION. A *hypergroup* $K := (K, *)$ consists of a locally compact space together with an associative product (called *convolution*) $*$ on $M_b(K)$ satisfying the following conditions:

(1) The space $M_b(K)$ admits a convolution $*$ such that $(M_b(K), *)$ is a Banach algebra with respect to the norm $\|\cdot\|$.

(2) The mapping $(\mu, \nu) \mapsto \mu * \nu$ from $M_b^+(K) \times M_b^+(K)$ into $M_b^+(K)$ is continuous with respect to the weak topology in $M_b(K)$.

(3) For $x, y \in K$ the convolution product $\varepsilon_x * \varepsilon_y$ belongs to $M^1(K)$ and $\operatorname{supp}(\varepsilon_x * \varepsilon_y)$ is compact.

(4) The mapping $K \times K \ni (x, y) \mapsto \operatorname{supp}(\varepsilon_x * \varepsilon_y) \in \mathscr{C}(K)$ is continuous.

(5) There exisits a unit element e of K such that $\varepsilon_e * \varepsilon_x = \varepsilon_x * \varepsilon_e = \varepsilon_x$ for all $x \in K$.

(6) There exists an involutive homeomorphism $x \mapsto x^-$ in K such that $(\varepsilon_x * \varepsilon_y)^- = \varepsilon_y^- * \varepsilon_x^-$ and $e \in \operatorname{supp}(\varepsilon_x * \varepsilon_y)$ if and only if $x = y^-$ for all $x, y \in K$.

A hypergroup K is said to be *commutative* if the convolution $*$ in $M_b(K)$ is commutative, and *hermitian* if the involution $^-$ is the identity mapping. There are prominent classes of commutative hypergroups arising from orbital actions and Gelfand pairs, and also large classes of examples constructed on $\boldsymbol{Z}_+$ and $\boldsymbol{R}_+$ by polynomial and Sturm–Liouville methods, respectively. The reader is encouraged to check the details in [1].

For subsets A and B of K one defines

$$A * B = \bigcup_{x \in A, y \in B} \operatorname{supp}(\varepsilon_x * \varepsilon_y).$$

If $x \in K$, we write $x * A$ or $A * x$ instead of $\{x\} * A$ or $A * \{x\}$, respectively.

A non-empty closed subset H of K is called a *subhypergroup* if $H * H = H = H^-$, where $H^- = \{x \in K: x^- \in H\}$. A subhypergroup H is said to be *normal* if $x * H = H * x$, and *supernormal* if $x^- * H * x \subset H$ for all $x \in K$.

Let $(K, *)$ and $(L, \circ)$ be two hypergroups with units e_K and e_L, respectively. A continuous mapping $\varphi: K \to L$ is said to be a *hypergroup homomorphism* if $\varphi(e_K) = e_L$ and

$$\varepsilon_{\varphi(x)} \circ \varepsilon_{\varphi(y)} = \varphi(\varepsilon_x * \varepsilon_y)$$

whenever $x, y \in K$. A hypergroup homomorphism $\varphi: K \to L$ is said to be an *isomorphism* if φ is a homeomorphism. If $\iota: H \to K$ is an injective hypergroup homomorphism and $p: K \to L$ is a surjective hypergroup homomorphism such that $\iota(H) = p^{-1}(L)$, one says that the sequence

$$1 \to H \to K \to L \to 1$$

is *exact* and that K is an *extension* of L by H. We note that the quotient K/H does not necessarily have a hypergroup structure in this situation.

Here we shall recall some facts on quotient hypergroups. Let $p: K \to L$ be an open and surjective hypergroup homomorphism. Then $H := p^{-1}(L)$ is a normal subhypergroup of K, $K/H := \{x * H: x \in K\}$ is a locally compact space with respect to the quotient topology, and the formula

$$(*) \qquad \varepsilon_{x*H} * \varepsilon_{y*H} := \int_K \varepsilon_{z*H}(\varepsilon_x * \varepsilon_y)(dz)$$

for all $x, y \in K$ defines a hypergroup structure on K/H such that K/H is isomorphic to L, where $(*)$ is understood as an equality of linear functionals on

$C_c(K/H)$. Conversely, if H is a normal subhypergroup of K such that $(*)$ defines a hypergroup structure, then the mapping $x \mapsto x * H$ from K onto K/H is an open hypergroup homomorphism. This statement is especially available if H is a compact normal subhypergroup. Moreover, if H is supernormal in K or a closed subgroup in K or if H is contained in a compact subgroup in K, then K/H is always a hypergroup. For details see [9] and [13].

Next we shall review the notion of substitution introduced by Voit in [12]. Let H and M be hypergroups and $\pi: H \to M$ be a proper and open hypergroup homomorphism. We put $Q := \pi(H) \subset M$ and $L := M/Q$. Then Voit in [12] established a hypergroup

$$S(M, Q \to H) := (H \cup (M \backslash Q), *)$$

by substituting the open subhypergroup Q in M to H via π which is an extension of L by H. It is clear that the hypergroup join $H \vee L$ of a compact hypergroup H and a discrete hypergroup L coincides with the substitution $S(L, \{e_L\} \to H)$ when the unit e_L of L is replaced by H and $\pi: H \to \{e_L\} \subset L$ is the trivial hypergroup homomorphism. Both the substitution and the join will serve as motivating examples for the extensions to be discussed in this work.

Now we shall describe some facts from the duality theory of commutative hypergroups. Let K be a commutative hypergroup. For a Borel measurable function f on K and $x, y \in K$ we write

$$f(x * y) := \int_K f(z)\, d(\varepsilon_x * \varepsilon_y)(z)$$

if this integral exists. For each $x \in K$ the translation T^x on such functions f and on measures μ is defined by

$$(T^x f)(y) = f(x * y) \ (y \in K) \quad \text{and} \quad (T^x \mu)(f) = \mu(T^x f).$$

A nonnegative measure $\omega \neq 0$ is called a *Haar measure* of K if it satisfies the equality $T^x \omega = \omega$ for all $x \in K$. It is known that every commutative hypergroup K has a Haar measure ω_K which is unique up to a positive multiplicative constant. If K is compact, ω_K is finite, and hence can be normalized to become a probability measure.

A complex-valued function χ on K is called a *character* of K if χ is a bounded continuous function on K satisfying

$$\chi(e) = 1, \quad \chi(x * y) = \chi(x)\chi(y), \quad \text{and} \quad \chi(x^-) = \overline{\chi(x)}$$

for all $x, y \in K$. The set $\hat{K}$ of all characters of K becomes a locally compact space with respect to the topology of uniform convergence on compact sets. One calls $\hat{K}$ the *dual* of K. In general, the dual $\hat{K}$ is not necessarily a hypergroup. If $(\hat{K}, \hat{*})$ becomes a hypergroup with respect to a convolution $\hat{*}$ which is defined by the product of characters on K, then K is said to be a *strong* hypergroup. In this case $\hat{\hat{K}} := (\widehat{\hat{K}})$ is also defined as a locally compact space. If $\hat{\hat{K}}$ is a hypergroup and is isomorphic to K, then K is called a *Pontryagin* hypergroup.

Let $\mathscr{S}$ and $\mathscr{P}$ be the symbols for the classes of strong and Pontryagin hypergroups, respectively. The classes $\mathscr{P} \subset \mathscr{S}$ are rather small as one can see from Zeuner's characterizations of Pontryagin hypergroups within the classes of polynomial hypergroups and Sturm–Liouville hypergroups (see [16]). Nevertheless, there are the following hereditary properties of hypergroups in $\mathscr{S}$ and $\mathscr{P}$.

1. Let $K \in \mathscr{S}$ and let $G(K)$ be the *maximal subgroup* of K defined by

$$G(K) := \{x \in K\colon\ \varepsilon_x * \varepsilon_{x^-} = \varepsilon_e = \varepsilon_{x^-} * \varepsilon_x\}.$$

If $G(K)$ is open, then $K \in \mathscr{P}$.

Consequently:

2. If $K \in \mathscr{S}$ is discrete, then $K \in \mathscr{P}$.
3. If H is a compact subhypergroup of $K \in \mathscr{P}$, then $H \in \mathscr{P}$.
4. If H is a subhypergroup of $K \in \mathscr{S}$ and $K/H \in \mathscr{P}$, then $K \in \mathscr{P}$.

But:

5. For a subhypergroup H of K such that $H \in \mathscr{P}$ and $K/H \in \mathscr{P}$, $K \notin \mathscr{P}$ even if in addition $K \in \mathscr{S}$.

Moreover, we quote the following facts.

6. The class $\mathscr{P}$ is closed under the formation of inductive limits and projective limits.
7. If $H, M \in \mathscr{P}$, then $S(M, Q \to H) \in \mathscr{P}$.

Proofs of the above properties 6 and 7 can be found in [11] and [4].

3. FIELDS OF COMPACT SUBHYPERGROUPS

Let $H = (H, *)$ and $L = (L, \circ)$ be commutative hypergroups with units e_H and e_L, respectively. We assume that each connected component of L is an open set.

DEFINITION. A family $\{H(l)\colon l \in L\}$ of subsets of H will be called a *field of compact subhypergroups of* H *based on* L and denoted by $\varphi\colon L \ni l \mapsto H(l) \subset H$ if it satisfies the following conditions:

(1) Each $H(l)$ is a compact subhypergroup of H with $H(e_L) = \{e_H\}$ and $H(l^-) = H(l)$ $(l \in L)$.

(2) For l_1, l_2, and $l \in L$ such that $l \in \operatorname{supp}(\varepsilon_{l_1} \circ \varepsilon_{l_2})$ we have

$$[H(l_1) * H(l_2)] \supset H(l),$$

where $[H(l_1) * H(l_2)]$ is the closed hypergroup generated by $H(l_1)$ and $H(l_2)$.

(3) For l_1 and l_2 contained in a connected component of L, $H(l_1)=H(l_2)$ holds.

Let $\omega(l)$ denote the normalized Haar measure of $H(l)$. Then condition (2) is equivalent to

(4) $\omega(l_1)*\omega(l_2)=\omega(l_1)*\omega(l_2)*\omega(l)$ whenever $l\in\operatorname{supp}(\varepsilon_{l_1}\circ\varepsilon_{l_2})$.

Now let $Q(l)$ denote the quotient hypergroup $H/H(l)$, and let K denote the disjoint union of the hypergroups $Q(l)$ $(l\in L)$, i.e.

$$K:=\bigcup_{l\in L}Q(l)=\{(h*H(l),\,l):\ h\in H,\ l\in L\}.$$

The topology of K is induced by the canonical mapping

$$\pi:\ H\times L\ni(h,\,l)\mapsto(h*H(l),\,l)\in K.$$

It is easy to deduce from conditions (1)–(3) that K is a locally compact space. The Dirac measure of an element $(h*H(l),\,l)\in K$ is given as the measure

$$(\varepsilon_h*\omega(l))\otimes\varepsilon_l\in M_b(H)\otimes M_b(L),$$

and the convolution $*_\varphi$ in $M_b(H)\otimes M_b(L)$ is well defined by

$$\big((\varepsilon_{h_1}*\omega(l_1))\otimes\varepsilon_{l_1}\big)*_\varphi\big((\varepsilon_{h_2}*\omega(l_2))\otimes\varepsilon_{l_2}\big)=(\varepsilon_{h_1}*\varepsilon_{h_2}*\omega(l_1)*\omega(l_2))\otimes\varepsilon_{l_1}\circ\varepsilon_{l_2}.$$

The set K together with the convolution $*_\varphi$ associated with the field $\varphi:\ L\ni l\mapsto H(l)\subset H$ will be denoted by $K(H,\,\varphi,\,L)$. We get the following

THEOREM 3.1. *Let H and L be commutative hypergroups such that every connected component of L is an open set, and let $\varphi:\ L\ni l\mapsto H(l)\subset H$ be a field of compact subhypergroups of H based on L. Then $K(H,\,\varphi,\,L)$ is a commutative hypergroup and an extension of L by H.*

Proof. Let X be the set of all connected components of L. We denote the connected component containing $l\in L$ by $C(l)$. Then the induced topology of X defined by the mapping

$$p:\ L\ni l\to C(l)\in X$$

is discrete due to the assumption that all connected components of L are open.

For $C\in X$ put

$$Q(C):=\{(h*H(l),\,l):\ h\in H,\ l\in C\}.$$

By condition (2) of the field we observe that

$$Q(C)=Q(l)\times C\quad\text{ for } l\in C.$$

Hence we see that K is the disjoint union $\bigcup_{C\in X}Q(C)$ of the locally compact spaces $Q(C)$ indexed by the discrete space X. This implies that K is a locally compact space.

Condition (4) assures the well-definedness of the convolution $*_\varphi$ in $M_b(H) \otimes M_b(L)$.

We note that the unit e_K of K is (e_H, e_L) and that $(h * H(l), l)^- = (h^- * H(l^-), l^-)$, since $H(l^-) = H(l)$.

Now it is an easy task to check that K satisfies all axioms (1)–(6) of a hypergroup. We omit the details. ∎

4. DUALITY OF FIELDS AND HYPERGROUPS

Let H and L be strong hypergroups such that every connected component of both L and the dual $\hat{H}$ of H is an open set, and let $\varphi: L \ni l \mapsto H(l) \subset H$ be a field of compact subhypergroups of H based on L. Then for each $l \in L$ we choose $X(l)$ to be the annihilator

$$A(\hat{H}, H(l)) := \{\chi \in \hat{H} : \chi(x) = 1 \text{ for all } x \in H(l)\}$$

of $H(l)$ in the dual $\hat{H}$ of H.

LEMMA 4.1. *The family $\{X(l) \subset \hat{H} : l \in L\}$ satisfies the following conditions:*

(1) *Each $X(l)$ is an open subhypergroup of $\hat{H}$ with the properties that $X(e_L) = H$ and $X(l^-) = X(l)$.*

(2) *For l_1, l_2, and $l \in L$ such that $l \in \operatorname{supp}(\varepsilon_{l_1} \circ \varepsilon_{l_2})$, $X(l_1) \cap X(l_2) \subset X(l)$ holds.*

(3) *For l_1 and l_2 contained in a connected component of L, $X(l_1) = X(l_2)$ holds.*

Proof. Statements (1), (2), and (3) on $X(l)$ follow naturally from conditions (1), (2), and (3) on $H(l)$, respectively. We omit the details. ∎

Next, for each $\chi \in \hat{H}$ set $Y(\chi) = \{l \in L : \chi \in X(l)\}$.

LEMMA 4.2. *The family $\{Y(\chi) \subset L : \chi \in \hat{H}\}$ satisfies the following conditions:*

(1) *Each $Y(\chi)$ is an open subhypergroup of L with $Y(e_{\hat{H}}) = L$ and $Y(\chi^-) = Y(\chi)$.*

(2) *For χ_1, χ_2, and $\chi \in \hat{H}$ such that $\chi \in \operatorname{supp}(\varepsilon_{\chi_1} * \varepsilon_{\chi_2})$, $Y(\chi_1) \cap Y(\chi_2) \subset Y(\chi)$ holds.*

(3) *For χ_1 and χ_2 contained in a connected component of $\hat{H}$, $Y(\chi_1) = Y(\chi_2)$ holds.*

Proof. (1) We denote by $C(l)$ the connected component of L containing $l \in L$. Then for each $l \in Y(\chi)$ we see that $C(l) \subset Y(\chi)$ by (3) of Lemma 4.1. Hence the set $Y(\chi)$ can be written as

$$Y(\chi) = \bigcup_{l \in Y(\chi)} C(l),$$

which means that $Y(\chi)$ is an open set because all $C(l)$ are open by the assumption on L. For $l \in Y(\chi) * Y(\chi)$ we can take $l_1, l_2 \in Y(\chi)$ such that $l \in \operatorname{supp}(\varepsilon_{l_1} * \varepsilon_{l_2})$. The conditions $l_1 \in Y(\chi)$ and $l_2 \in Y(\chi)$ imply that $\chi \in X(l_1)$ and $\chi \in X(l_2)$. By statement (2) of Lemma 4.1 we see that $\chi \in X(l)$, i.e. $l \in Y(\chi)$. Hence we obtain $Y(\chi) * Y(\chi) = Y(\chi)$. The property $Y(\chi)^- = Y(\chi)$ follows directly from $X(l^-) = X(l)$. Therefore $Y(\chi)$ is seen to be an open subhypergroup of L. The conditions $Y(e_{\hat{H}}) = L$ and $Y(\chi^-) = Y(\chi)$ are just translations of the facts that $e_{\hat{H}} \in X(l)$ and $X(l)^- = X(l)$ for all $l \in L$, respectively.

(2) Take $l \in Y(\chi_1) \cap Y(\chi_2)$ for $\chi_1, \chi_2 \in \hat{H}$. Then we see that $\chi_1, \chi_2 \in X(l)$ by the definition of $Y(\chi)$. Since $X(l)$ is a subhypergroup of $\hat{H}$ by (1) of Lemma 4.1, for $\chi \in \hat{H}$ such that $\chi \in \operatorname{supp}(\varepsilon_{\chi_1} * \varepsilon_{\chi_2})$ we see that $\chi \in X(l)$, i.e. $l \in Y(\chi)$. Hence we proved that $Y(\chi_1) \cap Y(\chi_2) \subset Y(\chi)$.

(3) Let χ_1 and χ_2 be in a connected component C of $\hat{H}$. Take $l \in Y(\chi_1)$, then $\chi_1 \in X(l)$. Since $X(l)$ is open and closed, we see that $C \subset X(l)$. Hence $\chi_2 \in X(l)$, i.e. $l \in Y(\chi_2)$. This fact implies that $Y(\chi_1) = Y(\chi_2)$. ∎

Finally, for each $\chi \in \hat{H}$ we introduce $Z(\chi) = A(\hat{L}, Y(\chi))$ and obtain the following

PROPOSITION 4.3. *The family $\{Z(\chi) \subset \hat{L} : \chi \in \hat{H}\}$ gives rise to a field $\hat{\varphi}$: $\hat{H} \ni \chi \mapsto Z(\chi) \subset \hat{L}$ of compact subhypergroups of $\hat{L}$ based on $\hat{H}$.*

Proof. Since $Y(\chi)$ is an open subhypergroup of the strong hypergroup L, the annihilator $Z(\chi)$ of $Y(\chi)$ in $\hat{L}$ is known to be a compact subhypergroup of $\hat{L}$ which is isomorphic to the dual of the discrete commutative hypergroup $L/Y(\chi)$. Conditions (1), (2), and (3) of a field on $Z(\chi)$ are easily obtained from the respective facts (1), (2), and (3) on $Y(\chi)$ described in Lemma 4.2. We omit the details. ∎

We call the field

$$\hat{\varphi} : \hat{H} \ni \chi \mapsto Z(\chi) \subset \hat{L}$$

the *dual field* of $\varphi : L \ni l \mapsto H(l) \subset H$. Associated with the dual field $\hat{\varphi}$ one can construct an extension $K(\hat{L}, \hat{\varphi}, \hat{H})$ of $\hat{H}$ by $\hat{L}$. We arrive at the following duality theorem.

THEOREM 4.4. *Let $\varphi : L \ni l \mapsto H(l) \subset H$ be a field of compact subhypergroups of a strong hypergroup H based on a strong hypergroup L such that all connected components of L and $\hat{H}$ are open sets. Then*

(1) $K(\hat{L}, \hat{\varphi}, \hat{H}) \cong \hat{K}(H, \varphi, L)$.

Moreover, if both H and L are Pontryagin hypergroups, then $K(H, \varphi, L)$ is also a Pontryagin hypergroup and

(2) $\hat{K}(\hat{L}, \hat{\varphi}, \hat{H}) \cong K(H, \varphi, L)$.

Proof. (1) Let $\omega(l)$ and $\omega(\chi)$ denote the normalized Haar measures of $H(l)$ and $Z(\chi)$, respectively. We note that $\chi(\omega(l)) = \omega(\chi)(\varepsilon_l) = 1$ if and only if

$\chi \in X(l)$, which is equivalent to $l \in Y(\chi)$. Otherwise we see that $\chi(\omega(l)) = \omega(\chi)(\varepsilon_l) = 0$. Applying this formula it is easy to check that each element of $K(\hat{L}, \hat{\varphi}, \hat{H})$ defines a character of $K(H, \varphi, L)$. We show the converse. Take a character τ of $K(H, \varphi, L)$. Then there exist $\chi \in \hat{H}$ and $\rho \in \hat{L}$ such that

$$\chi(\varepsilon_h) = \tau(\varepsilon_h \otimes e_L) \quad \text{and} \quad \tau(\omega(l) \otimes \varepsilon_l) = \chi(\omega(l))\rho(\varepsilon_l).$$

Hence we obtain

$$\tau(\varepsilon_h * \omega(l) \otimes \varepsilon_l) = \rho(\varepsilon_l)\,\omega(\chi)(\varepsilon_l)\,\chi(\varepsilon_h).$$

This implies that

$$\tau = (\rho * Z(\chi), \chi) \in K(\hat{L}, \hat{\varphi}, \hat{H}).$$

(2) For Pontryagin hypergroups H and L we see that the dual of the dual field $\hat{\varphi}$ coincides with the original field φ so that the formula (2) follows from (1). ■

5. SPLITTING EXTENSIONS

Let $H = (H, *)$ and $L = (L, \circ)$ be commutative hypergroups, and let K be an extension of L by H, i.e., the sequence

$$1 \to H \to K \to L \to 1$$

is exact. We say that the extension K of L by H *splits* or that K is a *splitting extension* if K satisfies the following conditions:

There exists a proper and continuous injective mapping ϕ from L into K such that:

(1) $\phi(e_L) = e_K$ and $\phi(l^-) = \phi(l)^-$.

(2) The sets $H(l) = \{h \in H\colon \varepsilon_h * \varepsilon_{\phi(l)} = \varepsilon_{\phi(l)}\}$ are compact subhypergroups of H with $H(l^-) = H(l)$.

(3) $\varepsilon_{\phi(l_1)} * \varepsilon_{\phi(l_2)} = \phi(\varepsilon_{l_1} \circ \varepsilon_{l_2}) * \omega(l_1) * \omega(l_2)$ for l_1 and $l_2 \in L$, where $\omega(l)$ denotes the normalized Haar measure of $H(l)$.

(4) $\omega(l_1) * \omega(l_2) * \omega(l) = \omega(l_1) * \omega(l_2)$ for l_1, l_2, and $l \in L$ such that $l \in \operatorname{supp}(\varepsilon_{l_1} \circ \varepsilon_{l_2})$.

(5) $K = \{h * \phi(l)\colon h \in H \text{ and } l \in L\}$ and $H \cap \phi(L) = \{e_K\}$.

The subsequent result provides a characterization of extensions associated with a field of hypergroups as splitting extensions.

THEOREM 5.1. *Let H and L be commutative hypergroups such that every connected component of L is an open set. Then the extension $K(H, \varphi, L)$ associated with a field $\varphi\colon L \ni l \mapsto H(l) \subset H$ splits. Conversely, if L is a discrete commutative hypergroup, then all splitting extensions of L by H are obtained in this way.*

Proof. Let ϕ be a mapping from L into $K(H, \varphi, L)$ defined by

$$\phi(l) = (H(l), l) \quad \text{for all } l \in L.$$

Then the former part of the statement is clear. We only show the latter part. Let K be a splitting extension of L by H, and let $H(l)$ be the compact subhypergroup of H for $l \in L$ given by

$$H(l) = \{h \in H: \ \varepsilon_h * \varepsilon_{\phi(l)} = \varepsilon_{\phi(l)}\}.$$

For each $h \in H$ put

$$H(h, l) = \{k \in H: \ \varepsilon_k * \varepsilon_h * \varepsilon_{\phi(l)} = \varepsilon_h * \varepsilon_{\phi(l)}\}.$$

We show that $H(h, l) = H(l)$ for all $h \in H$. Since $H(h, l) \supset H(l)$ is clear, it remains to show the inverse inclusion. Take $k \in H(h, l)$. Then we have

$$\varepsilon_k * \varepsilon_h * \varepsilon_{\phi(l)} * \varepsilon_{\overline{\phi(l)}} = \varepsilon_h * \varepsilon_{\phi(l)} * \varepsilon_{\overline{\phi(l)}}.$$

By the condition

$$\varepsilon_{\phi(l)} * \varepsilon_{\overline{\phi(l)}} = \phi(\varepsilon_l * \varepsilon_{l^-})\, \omega(l)$$

we obtain

$$H \cap \operatorname{supp}(\varepsilon_h * \varepsilon_{\phi(l)} * \varepsilon_{\overline{\phi(l)}}) = h * H(l).$$

From this equality we see that k satisfies

$$k * h * H(l) = h * H(l),$$

which implies that $k \in H(l)$. Hence we have checked that $H(h, l) = H(l)$ for all $h \in H$.

If L is a discrete commutative hypergroup, a family $\{H(l) \subset H: l \in L\}$ gives rise to a field $\varphi: L \ni l \mapsto H(l) \subset H$, and one can construct

$$K(H, \varphi, L) = \{(h * H(l), l): \ h \in H, \ l \in L\}.$$

The mapping

$$\psi: K(H, \varphi, L) \ni (h * H(l), l) \mapsto h * \phi(l) \in K$$

is well defined by the above argument, and it is easy to see that ψ is a hypergroup isomorphism from $K(H, \varphi, L)$ onto K. ∎

6. RELATIONSHIP BETWEEN SUBSTITUTION AND EXTENSIONS

Let H be a compact commutative hypergroup, and let L be a discrete commutative hypergroup. Then the hypergroup join $H \vee L$ is canonically defined and appears as a typical extension of H by L. In [12], Voit developed the notion of substitution as a generalization of the hypergroup join. From the point of view of extension of hypergroups one can reformulate the notion of substitution in the following way.

For two exact sequences

$$1 \to W \to H \to Q \to 1 \quad \text{and} \quad 1 \to Q \to M \to L \to 1$$

the substitution $K = S(M, Q \to H) = \big(H \cup (M \backslash Q), \circ\big)$ is defined. K is called the *hypergroup obtained by substitution Q in M by H via* $\pi\colon H \to Q \subset M$, and it satisfies the exact sequences

$$1 \to H \to K \to L \to 1 \quad \text{and} \quad 1 \to W \to K \to M \to 1.$$

This extension K of L by H strongly depends on M. Our method of constructing the extensions associated with a field is different from the notion of substitution. However, there is some relationship between substitution and extension as shown below.

Case 1. If M is given as $K(Q, \psi, L)$ for some field $\psi\colon L \ni l \mapsto Q(l) \subset Q$, the associated field $\varphi\colon L \ni l \mapsto H(l) \subset H$ is canonically defined by $H(l) = \pi^{-1}(Q(l))$, and we see that

$$S\big(K(Q, \psi, L), Q \to H\big) = K(H, \varphi, L).$$

Case 2. For a field $\varphi\colon L \ni l \mapsto H(l) \subset H$ of compact subhypergroups of H based on L, take the common compact subhypergroup W of $H(l)$ for all $l \in L$ except $l = e_L$, for example,

$$W = \bigcap_{l \in L \backslash \{e_L\}} H(l).$$

Setting $Q = H/W$ and $Q(l) = H(l)/W \subset Q$ we obtain a field $\psi\colon L \ni l \mapsto Q(l) \subset Q$. In this case we can take M as $K(Q, \psi, L)$, and we see that

$$K(H, \varphi, L) = S\big(K(Q, \psi, L), Q \to H\big).$$

If for each $l \in L$ except for $l = e_L$, $H(l)$ is equal to the fixed compact subhypergroup W of H, then

$$K(H, \varphi, L) = S(Q \times L, Q \to H).$$

Remark. Here we note the triviality of substitution. If $W = \{e_H\}$, we see that $Q = H$ and $S(M, Q \to H) = M$. This is the trivial substitution. For $k \in S(M, Q \to H)$ such that $k \notin H$,

$$H \cap \operatorname{supp}(\varepsilon_k * \varepsilon_{k^-}) \supset W$$

always holds. Therefore, if the condition

$$H \cap \operatorname{supp}(\varepsilon_k * \varepsilon_{k^-}) = \{e_H\}$$

holds for some $k \in S(M, Q \to H)$ with $k \notin H$, the substitution must be trivial.

If for an extension K of L by H the condition

$$H \cap \operatorname{supp}(\varepsilon_k * \varepsilon_{k^-}) = \{e_K\}$$

holds for some $k \in K$ with $k \notin H$, K does not arise from non-trivial substitution. Consequently, $K(H, \varphi, L)$ does not arise from non-trivial substitution if $H(l) = \{e_H\}$ for some $l \in L$ $(l \neq e_L)$. We note that this situation often occurs as will be shown in the next section.

7. APPLICATIONS AND EXAMPLES

In the category of commutative hypergroups there are only few Pontryagin hypergroups which are not of group-theoretic origin in the sense that they do not arise from orbital actions and Gelfand pairs. Applying the method of fields of hypergroups one can provide many new examples of Pontryagin hypergroups. These examples show the strength of the method of fields of hypergroups and indicate the possibility of further investigations on the structure of commutative hypergroups.

Before describing our examples we prepare some well-known simple facts.

Let A be the smallest non-trivial hypergroup with

$$A = \{l_0, l_1\}, \quad l_1^2 = pl_0 + (1-p)l_1,$$

where l_0 is the unit, $0 < p \leqslant 1$, and we write $l_i l_j$ instead of $\varepsilon_{l_i} * \varepsilon_{l_j}$.

Let B be $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$, namely,

$$B = \{l_0, l_1, l_2, l_3\},$$

$$l_1^2 = l_2^2 = l_3^2 = l_0, \quad l_1 l_2 = l_3, \quad l_1 l_3 = l_2, \quad l_2 l_3 = l_1.$$

Let C denote the simplest compact hypergroup which is given as an orbital hypergroup of the one-dimensional torus $\boldsymbol{T}$ by the action of $\boldsymbol{Z}_2$, i.e.

$$C = ([-1, 1], *),$$

$$\varepsilon_{\cos\theta_1} * \varepsilon_{\cos\theta_2} = \tfrac{1}{2}\varepsilon_{\cos(\theta_1+\theta_2)} + \tfrac{1}{2}\varepsilon_{\cos(\theta_1-\theta_2)}.$$

Finally, let D denote the simplest discrete hypergroup which arises from a random walk on $\boldsymbol{Z}$, i.e.

$$D = \{l_0, l_1, l_2, \ldots, l_n, \ldots\},$$

$$l_m l_n = \tfrac{1}{2} l_{|m-n|} + \tfrac{1}{2} l_{m+n} \quad (m, n = 0, 1, 2, \ldots).$$

Here we note that A and B are self-dual and $\hat{D} \cong C$, $\hat{C} \cong D$. These facts imply that A, B, C, and D are all Pontryagin hypergroups.

For a natural number a, $D(a)$ and $F(a)$ denote the subhypergroups of D and C which are defined by

$$D(a) = \{l_{an}\colon n = 0, 1, 2, \ldots\}$$

and

$$F(a) = \{\cos(2k\pi/a)\colon k = 0, 1, 2, \ldots, a-1\},$$

respectively.

Observe that

$$F(a) = A(C, D(a)), \quad D(a) = A(D, F(a)).$$

We denote the quotient hypergroup $C/F(a)$ by $C(a)$ and write

$$C(a) = ([\cos(\pi/a), 1], *).$$

Example 7.1. Let H be a compact Pontryagin hypergroup and let $L = A = \{l_0, l_1\}$. Take any closed subhypergroup W of H and denote H/W by Q. Then we obtain a field $\varphi: L \ni l \mapsto H(l) \subset H$, where $H(l_0) = \{e_H\}$ and $H(l_1) = W$. This field φ gives rise to an extension of L by H of the form

$$K(H, \varphi, L) = S(Q \times L, Q \to H).$$

If we choose $H = C$ and $W = F(a)$, we get the concrete model

$$K(a) = \{[-1, 1] \cup [\cos(\pi/a), 1], *\}$$

with a parameter a from the set of natural numbers.

Example 7.2. Let W_1 and W_2 be two compact subhypergroups of a compact Pontryagin hypergroup H and let $L = B = \{l_0, l_1, l_2, l_3\}$. When we put

$$H(l_0) = \{e_H\}, \quad H(l_1) = W_1, \quad H(l_2) = W_2, \quad H(l_3) = [W_1 * W_2],$$

we obtain a field $\varphi: L \ni l \mapsto H(l) \subset H$ and an extension $K(H, \varphi, L)$ of L by H. With the choice $H = C$ and $W_1 = F(a)$, $W_2 = F(b)$ we see that $[W_1 * W_2] = F(c)$ for a natural number c which is the least common multiple of a and b. Hence, we arrive at an extension $K = K(a, b)$ which is concretely represented as

$$K(a, b) = ([-1, 1] \cup [\cos(\pi/a), 1] \cup [\cos(\pi/b), 1] \cup [\cos(\pi/c), 1], *).$$

In a similar way one can get the extensions $K_n = K(H, \varphi_n, L_n)$ for $L_n = B \times B \times \ldots \times B$ and $K_\infty = K(H, \varphi_\infty, L_\infty)$ with $L_\infty = B \times B \times \ldots \times B \times \ldots$ We note that L_∞ is the inductive limit of the sequence $\{L_n: n = 1, 2, \ldots\}$ and K_∞ is the inductive limit of the sequence $\{K_n: n = 1, 2, \ldots\}$.

Example 7.3. Let W_1 and W_2 be two compact subhypergroups of a compact Pontryagin hypergroup H, and let $L = D = \{l_0, l_1, l_2, \ldots, l_n, \ldots\}$. Putting

$$H(l_0) = \{e_H\}, \quad H(l_1) = [W_1 * W_2], \quad H(l_2) = W_1,$$

$$H(l_3) = W_2, \quad H(l_4) = W_1, \quad H(l_5) = [W_1 * W_2], \quad H(l_6) = W_1 \cap W_2$$

and

$$H(l_n) = H(l_k) \quad (n \equiv k \ (\mathrm{mod}\ 6),\ n \neq 0 \text{ and } k = 1, 2, 3, 4, 5, 6)$$

we obtain a field $\varphi: L \ni l \mapsto H(l) \subset H$ and an extension $K(H, \varphi, L)$ of L by H. If $H = C$, $W_1 = F(a)$, and $W_2 = F(b)$, we see that, as above, $[W_1 * W_2] = F(c)$ for a natural number c which is the least common multiple of a and b, and $W_1 \cap W_2 = F(d)$ for a natural number d which is the greatest common divisor of a and b. Thus we have an extension $K = K(a, b)$, where a and b are natural numbers.

It is easy to see that the dual hypergroup of $K(a, b)$ can be concretely described by the dual field $\hat{\varphi}: \hat{H} \ni \chi \mapsto Z(\chi) \subset \hat{L}$. We give the description in the case when $1 < d < a < b < c$:

$$\hat{H} = \{\chi_0, \chi_1, \chi_2, \ldots, \chi_n, \ldots\} \cong D \quad \text{and} \quad \hat{L} \cong C = ([-1, 1], *),$$

$Z(\chi_n) = F(1)$ for $n \equiv 0 \pmod{c}$,

$Z(\chi_n) = F(2)$ for $n \equiv 0 \pmod{a}$ except $n \equiv 0 \pmod{b}$,

$Z(\chi_n) = F(3)$ for $n \equiv 0 \pmod{b}$ except $n \equiv 0 \pmod{a}$,

$Z(\chi_n) = F(6)$ for $n \equiv 0 \pmod{d}$ except $n \equiv 0 \pmod{a}$ and $n \equiv 0 \pmod{b}$,

$Z(\chi_n) = \hat{L}$ for any other n.

We list further properties of the Pontryagin hypergroup $K(a, b)$:

(1) $K(a_1, b_1) \cong K(a_2, b_2)$ if and only if $a_1 = a_2$ and $b_1 = b_2$.

(2) $K(1, 1) \cong C \times D$.

(3) $K(a, a) = S(C(a) \times D, C(a) \to C)$.

(4) $K(a, b)$ is self-dual if and only if $a = 2$ and $b = 3$.

(5) For the greatest common divisor d of a and b,

$$K(a, b) = S(M(d), C(d) \to C) \quad \text{for } M(d) = K(C(d), \psi, D).$$

(6) If a and b are coprime, $K(a, b)$ does not arise from non-trivial substitution.

This follows from the facts that $H(l_6) = F(1) = \{e_H)$ and

$$H \cap \operatorname{supp}((\varepsilon_{e_H} \otimes \varepsilon_{l_6})^- *_\varphi (\varepsilon_{e_H} \otimes \varepsilon_{l_6})) = \{e_K\}.$$

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Received on 2.7.2006