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ON AN EXTENSION OF MIN-SEMISTABLE DISTRIBUTIONS

BY

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Abstract. This work focuses on a functional equation which extends the notion of min-semistable distributions. Our main results are an existence theorem and a characterization theorem for its solutions. The first establishes the existence of a class of solutions of this equation under a condition on the first zero on the positive axis of the associated structure function. The second shows that solutions belonging to a subclass of complementary distribution functions can be identified by their behavior at the origin. Our constructed solutions are in this subclass. The uniqueness question is also discussed.

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1. INTRODUCTION

In this paper we shall consider the functional equation defined on the space of complementary cumulative probability distribution functions (for short, ccdf) $\bar{F}$ with support $[0, \infty]$:

$$(1.1) \qquad (E):\ \bar{F}(x) = E\Big[\prod_{i=1}^{M} \bar{F}(C_i x)^{\Gamma_i}\Big].$$

Here $M \in N^*$ is an integer-valued random variable and $(C_i, i \geqslant 1)$ and $(\Gamma_i, i \geqslant 1)$ are sequences of random variables such that $C_i > 0$, $\Gamma_i \geqslant 1$. In the statistical literature, the function $\bar{F}$ is also called the *survival* or *survivor function.* The solution $\bar{F}$ of (E) can be regarded as a fixed point of the transformation T defined on the set of complementary cumulative distribution functions by

$$T\bar{F}(x) = E\Big[\prod_{i=1}^{M} \bar{F}(C_i x)^{\Gamma_i}\Big].$$

Let X be the random variable with ccdf $\bar{F}$ satisfying (E). When Γ_i are integer-valued random variables, the equation (E) reads in terms of random variables

$$X \stackrel{d}{=} \min_{1 \leqslant i \leqslant M} \min_{1 \leqslant j \leqslant \Gamma_i} C_{i,j} X_{i,j}.$$

Here the $X_{i,j}$ are i.i.d. copies of X; for each i, $C_{i,j}$ are i.i.d. copies of C_i, and $X_{i,j}$ are independent of $C_{i,j}$, Γ_i and M. After a suitable identification of variables, this distributional equality can be put into the simpler form

$$X \stackrel{d}{=} \min_{1 \leqslant i \leqslant N} A_i X_i \tag{1.2}$$

in terms of new random variables $N \in N^*$ and $\{A_i,\ i \geqslant 1\}$ positive. Here, X_i are i.i.d. copies of $X \geqslant 0$ and independent of the random variables $\{N,\ A_i,\ i \geqslant 1\}$. This identity in law expresses the invariance property under weighted minima considered by Alsmeyer and Rösler [1].

Let again Γ_i be integer-valued random variables. Equation (1.1), on the space of Laplace–Stieltjes transforms instead of the space of ccdf yields an equation similar to (1.2), namely

$$X \stackrel{d}{=} \sum_{i=1}^{N} A_i X_i. \tag{1.3}$$

Under this form, it has been intensively studied by several authors.

Initially, the functional equation associated with (1.3) was introduced in Mandelbrot [19] and [20] in the context of a model for turbulence. Later, Kahane and Peyrière [16] obtained necessary and sufficient conditions for the existence of solutions of (1.3), when the A_i are independent and identically distributed and N is a constant. Holley and Liggett [14] obtained the same kind of results when A_i are a fixed multiple of a given random variable.

On physical grounds, such distributions provided examples of invariant measures for infinite interacting particle systems. Motivated by questions raised by these works on the nature of such invariant measures, their ergodic behavior, notably the possible display of phase transitions, Durrett and Liggett [11] studied (1.3) in a quite general setting. More precisely, taking N constant and A_i non-negative with arbitrary law, they gave necessary and sufficient conditions for the existence of solutions under a sole condition on the moments of the A_i. Moreover, they characterized all these solutions and proved some convergence results.

Random variables satisfying (1.3) can also be viewed as a generalization of semistable laws, in that they are stable under random weighted means. In this view, Guivarc'h [13] discussed equation (1.3) when the A_i are independent identically distributed variables and N is constant. He gave theorems on existence and uniqueness of solutions and analyzed particularly their behavior at infinity.

More recently, Liu [17], [18] extended the results of [11] on equation (1.3) allowing N to be an almost surely finite random variable, finding the optimal conditions for the existence of its solutions. As reviewed in [17], equation (1.3) or some its variants, arises in several other application fields: for instance, it defines distributions appearing as limiting distributions of some branching processes (either of the Bellman–Harris or of the Crump–Mode types) or Hausdorff measures of some random fractal sets [17]. See also Caliebe [7], [8] for recent results and references.

Coming back to equation (1.1), the idea of taking non-integral powers $\Gamma_i > 1$ in a similar equation is initially due to Barral [2]. Considering the following functional equation

$$f(x) = \big(E\big(f(Cx)\big)\big)^{\gamma},$$

where C is a positive random variable and $\gamma \geqslant 1$ is non-random, he was able to obtain analogous results as in [11] and [18] by studying it in a space containing the space of Laplace–Stieltjes transforms and included in the space of complementary distribution functions.

On the other hand, in [3], the problem of characterizing the cumulative distribution functions (for short, cdf) with support $[0, \infty]$, say G, satisfying the functional equation

$$G(x) = \prod_{i=1}^{m} G(x/c_i)^{\gamma_i} \tag{1.4}$$

for some integer $m > 1$, and real numbers $c_i > 0$, $\gamma_i > 0$, $i = 1, \ldots, m$, was considered. These have been called *multiscaling max-semistable distributions.* The functional equation (1.4) may be viewed as a version of the integrated Cauchy functional equation whose solution can be defined by appealing to Corollary 2.3.2 of [21]. This constitutes a by-product of Deny's theorem (see [21]).

Setting $\bar{F}(x) = G(1/x)$ when $x > 0$ and $\bar{F}(x) = 1$ for $x \leqslant 0$, the complementary cumulative distribution function $\bar{F}$, with support $[0, \infty]$, is a solution to

$$\bar{F}(x) = \prod_{i=1}^{m} \bar{F}(c_i x)^{\gamma_i} \tag{1.5}$$

and we can deduce similarly the class of the so-called *multiscaling min-semistable distributions.*

In [3], the physical meaning of the functional equation (1.4) has been discussed to some extent. Essentially, it was emphasized that any strictly positive random variable, interpreted as some observable, can be viewed as the maximum of a Poisson number of "micro-events". The model (1.4) expresses that the observable under concern might as well result from the aggregation of $m > 1$ independent observations of statistically similar events, each with its specific intensity γ_i and scale c_i (in other words, it might as well result from more frequent micro-events but with smaller reduced amplitudes); it translates an amplitude and scale invariance principle for the observable. Such a fixed-point equation also appears in discrete scale invariance in Renormalization Group Theory in Physics. This model exhibits log-periodic features, whose empirical evidence was underlined in diverse application fields such as finance, turbulence, rupture theory, DLA growth, geophysics and frustrated systems' statistics (see [15] and references therein). In a concrete physical situation, it seems natural to imagine that the intensity and scale parameters are unknown or, more realistically, modelled by some random variables. This motivates the randomization of this model.

The functional equation (E) given by (1.1) can indeed be viewed as a randomization of the equation (1.4). By putting $G(x) = \bar{F}(1/x)$ when $x > 0$ and $G(x) = 0$ when $x \leqslant 0$, conclusions drawn from (E) can readily be translated to the randomization of the equation (1.4), namely

$$G(x) = \boldsymbol{E}\Big[\prod_{i=1}^{M} G(x/C_i)^{\Gamma_i}\Big]. \tag{1.6}$$

Central to the solution of the functional equation (1.4) was the Kahane–Peyrière–Mandelbrot (KPM) real-valued structure function defined by

$$q \to \sum_{i=1}^{m} \gamma_i c_i^q, \qquad q \in \boldsymbol{R}.$$

In its randomized version, the KPM structure function now reads

$$\tau(q) = \boldsymbol{E}\Big[\sum_{i=1}^{M} \Gamma_i C_i^q\Big], \qquad q \in \boldsymbol{R}. \tag{1.7}$$

We shall assume that $\tau(q) < \infty$ whenever $q \geqslant 0$. Essentially, this function is convex. We note that $\tau(0) \geqslant 1$ and $\tau(0) = 1$ correspond to the case $M = \Gamma_1 = 1$ and the equation (E) admits a non-degenerate solution if and only if $C_1 = 1$. This trivial situation will be avoided in the sequel by assuming $\tau(0) > 1$.

The first main result is an existence theorem, which establishes the existence of solutions under a condition on the first zero on the positive axis of the structure function (1.7). Following [11], [13], [17] and [18], we first prove the existence of solutions of (E) in the special case, where $\tau(1) = 1$ and $\tau'(1) < 0$. Then the general case is investigated by introducing a transport operator. Our techniques follow the lines of Durrett and Liggett [11], and Liu [17], [18].

Next, we exhibit a large space of complementary distribution functions containing the given solutions, namely, with $F := 1 - \bar{F}$,

$$\mathscr{F} = \{\bar{F} \in C^0(\boldsymbol{R}_+, [0, 1]) :$$
$$\exists \lambda > 0,\, c > 0, \text{ satisfying } F(ax)/F(x) \leqslant c a^{\lambda},\ \forall a > 1,\, x > 0\}.$$

Then we show a characterization theorem, which states that the solutions of (E) belonging to $\mathscr{F}$ can be identified by their behavior at the origin.

The paper is organized as follows. In Section 2, the existence of solutions of the equation (E) in the special and general cases is studied. In Section 3, the main characterization theorem is first stated. The core of Section 3 is devoted to the proof of some technical results, which will contribute to elucidate the behavior at the origin of the solutions belonging to space $\mathscr{F}$. In Section 4, we discuss the uniqueness of solution.

2. EXISTENCE OF SOLUTIONS

2.1. The special case: existence of a solution. In this section we suppose that, with $\log_+ x := 0 \vee \log x$, $x > 0$,

$$\text{(i)}\ \boldsymbol{E}\Big[\sum_{i=1}^{M} \Gamma_i C_i \log_+\Big(\sum_{i=1}^{M} \Gamma_i C_i\Big)\Big] < \infty,$$

$$\text{(ii)}\ \tau(1) = \boldsymbol{E}\Big[\sum_{i=1}^{M} \Gamma_i C_i\Big] = 1 \quad \text{and} \quad \text{(iii)}\ \tau'(1) < 0.$$

We note that $\tau(0) > 1$. If conditions (ii) and (iii) are fulfilled, we shall refer to the special case. Define

$$\mathscr{E} = \{\bar{F} \text{ ccdf: } \bar{F} \text{ convex with } -\infty < \bar{F}'(0) < 0\},$$

and let $\mathscr{E}_1 := \{\bar{F} \text{ ccdf: } \bar{F} \text{ convex with } \bar{F}'(0) = -1\}$. Note that if $\bar{F} \in \mathscr{E}$, then F is absolutely continuous with respect to Lebesgue measure. In the following theorem we give sufficient conditions which guarantee the existence of a non-degenerate solution to the functional equation (E). This result is obtained by adapting the proof of Theorem 3.1 of Liu [17]. Liu himself used techniques developed in Durrett and Liggett [11] and some ideas of Doney and Biggins (see [9], [10], [4]). For the reader's convenience the proof of some technical arguments used in Theorem 2.1 below will be postponed to Section 3.

THEOREM 2.1. *Under the above conditions* (i), (ii) *and* (iii), *there exists a solution of* (E) *in* $\mathscr{E}_1$, *implying, in particular,* $F(x)/x \to 1$ *as* $x \downarrow 0$.

Proof. For a complementary cumulative distribution function (ccdf) $\bar{F}$, we define non-negative functions D and G on $\boldsymbol{R}$ by

$$D(z) = \frac{1-\bar{F}(e^{-z})}{e^{-z}} \tag{2.1}$$

and

$$G(z) = e^{z} \boldsymbol{E}\Big[\Big(\prod_{i=1}^{M} \bar{F}(e^{-z} C_i)^{\Gamma_i}\Big) - 1 + \sum_{i=1}^{M} \Gamma_i\big(1-\bar{F}(e^{-z} C_i)\big)\Big]. \tag{2.2}$$

Let Z be a random variable with distribution determined by

(2.3) $$E(\Psi(Z)) = E\Big(\sum_{i=1}^{M} \Gamma_i C_i \Psi(-\log(C_i))\Big)$$

for all bounded measurable functions Ψ. Since $\tau(1)$ is finite, $\Psi(Z)$ is integrable.

Let $\bar{F}_0(x) = e^{-x}\mathbf{1}_{(x \geqslant 0)} + \mathbf{1}_{(x<0)}$ and $\bar{F}_{n+1} = T\bar{F}_n$, $n \geqslant 1$. Replacing $\bar{F}$ by $\bar{F}_n$ in equations (2.1) and (2.2) we obtain the associated functions denoted by D_n and G_n in place of D and G for all $n \in N$. Noticing that, for $x \geqslant 0$,

(2.4) $$\begin{aligned}\bar{F}_1(x) &= E\Big[\exp\Big(-x \sum_{i=1}^{M} \Gamma_i C_i\Big)\Big] \\ &\geqslant \exp\Big[-xE\Big(\sum_{i=1}^{M} \Gamma_i C_i\Big)\Big] = \exp(-x) = \bar{F}_0(x)\end{aligned}$$

and $\bar{F}_1(x) = \bar{F}_0(x) = 1$ for $x < 0$, we deduce, by the monotonicity of T, that $\bar{F}_{n+1} \geqslant \bar{F}_n$. By Lemma 3.2 (iii) below, $G_{n+1} \leqslant G_n$, and by Lemma 3.1 we have

(2.5) $$D_{n+1}(z) = E\big(D_n(z+Z)\big) - G_n(z) \geqslant E\big(D_n(z+Z)\big) - G_0(z).$$

Thus,

(2.6) $$D_n(z) \geqslant E\big(D_0(z+S_n)\big) - \sum_{k=0}^{n-1} E\big(G_0(z+S_k)\big)$$

for all $n \geqslant 1$. Here, $S_n := \sum_{k=0}^{n} Z_k$, where $(Z_k)_{k \geqslant 1}$ is a sequence of independent random variables with the same distribution as Z, and $S_0 = 0$. As

(2.7) $$E(Z) = -E\Big(\sum_{i=1}^{M} \Gamma_i C_i \log(C_i)\Big) > 0,$$

S_n goes almost surely to $+\infty$ when n tends to infinity. Since $D_0(z)$ is bounded and $\lim_{z \to +\infty} D_0(z) = 1$, we get

(2.8) $$\lim_{n \to +\infty} E\big(D_0(z+S_n)\big) = 1.$$

The function $f(z) := \sum_{k=0}^{\infty} E\big(G_0(z+S_k)\big)$ satisfies the renewal equation $f = G_0 + F_{-Z} * f$, where F_{-Z} is the cdf of the random variable $-Z$, with $-\infty < E(-Z) < 0$. When G_0 is direct Riemann integrable, as we will show below, the renewal theorem yields $\lim_{z \uparrow \infty} f(z) = 0$ ([12], p. 381). This result, together with (2.6) and (2.8) implies

$$\lim_{z \uparrow \infty} \lim_{n \uparrow \infty} D_n(z) \geqslant 1.$$

But using $D_{n+1} \leqslant D_n \leqslant \ldots \leqslant D_0$ we obtain

$$\lim_{z \uparrow \infty} \lim_{n \uparrow \infty} D_n(z) \leqslant \lim_{z \uparrow \infty} D_0(z) = 1.$$

This shows that $\lim_{z\uparrow\infty}\lim_{n\uparrow\infty} D_n(z) = 1$. Calling $\bar{F}_\infty(x)$ the limiting ccdf of $\bar{F}_n(x)$, we infer that $\bar{F}_\infty(x)$ is derivable at point 0 with $\bar{F}'_\infty(0) = -1$.

Next, we show that the sequence $\bar{F}_n$ remains in $\mathscr{E}_1 \cap C^1(\boldsymbol{R}_+, [0, 1])$. In other words, suppose $\bar{F}_n \in C^1(\boldsymbol{R}_+, [0, 1])$ with $\bar{F}_n$ convex and $\bar{F}'_n(0) = -1$; let us show that this also holds for $\bar{F}_{n+1} = T\bar{F}_n$. By the dominated convergence theorem,

$$\bar{F}'_{n+1}(x) = -\boldsymbol{E}\Big[\sum_{i=1}^{M} \Gamma_i C_i \prod_{j\neq i} \bar{F}_n(C_j x)^{\Gamma_j} \bar{F}_n(C_i x)^{\Gamma_i - 1}\big(-\bar{F}'_n(C_i x)\big)\Big]$$

because the term in the brackets is bounded from above by $\sum_{i=1}^{M} \Gamma_i C_i$, which is integrable. Hence $\bar{F}_n \in \mathscr{E}_1 \cap C^1(\boldsymbol{R}_+, [0, 1])$. By passing to limit, the convexity property is preserved.

Now, it remains to prove direct Riemann integrability of G_0. By Lemma 3.2 (ii), $e^{-z} G_0(z)$ is a decreasing function of z and, following [11], p. 287, it suffices to show that G_0 is Lebesgue integrable. Using $u \leqslant e^{-(1-u)}$, when $u \in [0, 1]$, we get

$$G_0(z) \leqslant e^z \Big[\boldsymbol{E}\phi\Big(\sum_{i=1}^{M} \Gamma_i\big(1-\bar{F}_0(C_i e^{-z})\big)\Big)\Big], \tag{2.9}$$

where $\phi(x) := e^{-x} - 1 + x$, $x \geqslant 0$. We shall split $\int_{\boldsymbol{R}} G_0(z)\,dz$ into two parts.

- For $z < 0$, we note that ϕ is decreasing and $\phi(x) < x$. Therefore,

$$\int_{-\infty}^{0} G_0(z)\,dz < \boldsymbol{E}\Big(\sum_{i=1}^{M} \Gamma_i\Big) \int_{-\infty}^{0} e^z\,dz < \infty.$$

- For $z > 0$, using the inequality $1 - e^{-x} \leqslant x$, $x \geqslant 0$, and recalling that $\bar{F}_0(x) = e^{-x}$, $x > 0$, we obtain

$$G_0(z) \leqslant e^z \boldsymbol{E}\phi\Big(\sum_{i=1}^{M} \Gamma_i C_i e^{-z}\Big).$$

As a result, we have

$$\int_{0}^{\infty} G_0(z)\,dz \leqslant \int_{0}^{\infty} e^z \boldsymbol{E}\phi\Big(\sum_{i=1}^{M} \Gamma_i C_i e^{-z}\Big)\,dz.$$

Introducing the random variable $S = \sum_{i=1}^{M} \Gamma_i C_i$ and letting $u = e^{-z}$, we get

$$\int_{0}^{\infty} G_0(z)\,dz \leqslant \int_{0}^{1} \frac{1}{u^2} \boldsymbol{E}\phi(Su)\,du$$

which by Theorem B of Bingham and Doney ([5], p. 718) is finite if and only if $\boldsymbol{E}S\log_+ S < \infty$. This condition has been imposed. ∎

2.2. Behavior of solutions in the special case. Let us distinguish the *lattice* and *non-lattice* cases.

DEFINITION 2.1. We will speak of the *lattice* case when a common span of $-\log C_i$, $i \geqslant 1$, exists and is $-\log c$, $c > 0$.

We consider the random walk previously defined by $S_n = \sum_{k=0}^{n} Z_k$, where $(Z_k)_{k \geqslant 1}$ are i.i.d. random variables with the same distribution as Z given in equation (2.3), and $S_0 = 0$. It is easy to check that

PROPOSITION 2.1. *The random variables* $-\log C_i$ *have a common span* $-\log c$ *if and only if the random walk* S_n *is arithmetic in the sense that the support of the distribution of* S_n *is* $\{-k \log c\}_{k \in \boldsymbol{Z}}$.

Let us give the following definition:

DEFINITION 2.2. We denote by $\mathscr{S}_c$ the set of functions $s(\cdot): \boldsymbol{R} \to \boldsymbol{R}_+$ satisfying:

In the lattice case with common span $-\log c$, $c > 0$, $s(z) := e^{-\nu(z)}$ for some right-continuous bounded periodic function $\nu(\cdot)$ on $\boldsymbol{R}$ with period $-\log c$, such that $z - \nu(z)$ is a non-decreasing function.

In the non-lattice case, $s(z) := s > 0$, the constant function for all $z \in \boldsymbol{R}$.

The following corollary is easily obtained from Theorem 2.1.

COROLLARY 2.1. *In the special case, if* $\bar{F} \in \mathscr{E}_1$ *is a solution to the functional equation* (*E*), *then* $\bar{F}_s(x) := \bar{F}(xs(-\log x))$, *where* $s \in \mathscr{S}_c$, *is also a solution to the same equation. The solution* $\bar{F}_s(x)$ *now satisfies the property*

$$\frac{F_s(x)}{xs(-\log x)} \xrightarrow{x \downarrow 0} 1.$$

This means that in the special case the solutions to (*E*) are determined modulo a scaling factor s which can be a log-periodic function in the lattice case.

2.3. Existence of a solution in the general case. Consider the functional equation (*E*). We recall that $\tau(0) > 1$ and τ is convex. Under the condition on τ, we obtain the following existence theorem:

THEOREM 2.2. *Suppose that there exists* $0 < \alpha < \infty$ *such that* $\tau(\alpha) = 1$ *and* $\tau'(\alpha) \leqslant 0$. *Two cases arise*:

(i) *If* $\tau'(\alpha) < 0$ *and* $E\left[\sum_{i=1}^{M} \Gamma_i C_i^{\alpha} \log_+ \left(\sum_{i=1}^{M} \Gamma_i C_i^{\alpha}\right)\right] < \infty$, *then there exists a non-trivial ccdf* $\bar{F}$ *solution to* (*E*).

(ii) *If* $\tau'(\alpha) = 0$ *and* $E\left[\sum_{i=1}^{M} \Gamma_i C_i^{\beta} \log_+ \left(\sum_{i=1}^{M} \Gamma_i C_i^{\beta}\right)\right] < \infty$ *for all* $\beta < \alpha$, *then there exists a non-trivial ccdf* $\bar{F}$ *solution to* (*E*).

Proof. (i) Suppose $\tau'(\alpha) < 0$. Consider the ccdf $\bar{F}_\alpha$ as a solution to the functional equation

$$(E_\alpha):\ \bar{F}_\alpha(x) = E\left[\sum_{i=1}^{M} \bar{F}_\alpha(C_i^{\alpha} x)^{\Gamma_i}\right]. \tag{2.10}$$

The associated structure function is $\tau_\alpha(q) = \tau(\alpha q)$ with $\tau_\alpha(1) = 1$, $\tau'_\alpha(1) = \alpha\tau'(\alpha) < 0$. The existence of $\bar{F}_\alpha$ in $\mathscr{E}_1$ is given by Theorem 2.1, in the special case, when substituting C_i^α by C_i. Finally, the ccdf $\bar{F}(x) = \bar{F}_\alpha(x^\alpha)$ solves the functional equation (E).

(ii) Suppose $\tau'(\alpha) = 0$. Let $0 < \beta < \alpha$. Consider the random variables $C_i(\beta) = C_i^\beta \tau(\beta)^{-1}$ and introduce the functional equation

$$(2.11) \qquad (E_\beta)\colon\ \bar{F}_\beta(x) = E\Big[\prod_{i=1}^{M} \bar{F}_\beta\big(C_i(\beta)x\big)^{\Gamma_i}\Big].$$

Its associated structure function is $\tau_\beta(q) = \tau(\beta q)/\tau(\beta)^q$. We have $\tau_\beta(1) = 1$. As $\tau(0) > 1$ and τ is convex, $\tau(\beta) > 1$ and $\tau'(\beta) < 0$ for each $\beta < \alpha$. We have

$$\tau'_\beta(1) = \frac{\beta\tau'(\beta) - \tau(\beta)\log\tau(\beta)}{\tau(\beta)} < 0.$$

Consider now a sequence β_n with $0 < \beta_n < \alpha$, and $\beta_n \to \alpha$ as $n \to \infty$. By Theorem 2.1 and Corollary 2.1, (E_{β_n}) has a solution, say $\bar{F}_{\beta_n}$, in $\mathscr{E}$ satisfying $\bar{F}_{\beta_n}(1) = 1/2$. The sequence $\bar{F}_{\beta_n} \in \mathscr{E}$ is an equi-continuous sequence of functions $[0, \infty) \to [0, 1]$, because, for all $x > 0$, $F_{\beta_n}(x)/x$ is non-increasing. By an extended version of Arzelà's theorem [6], one can extract a convergent sub-sequence. By the same transformation as in (i), the ccdf $\bar{F}(x) = \bar{F}_\alpha(x^\alpha)$ also solves the functional equation (E) in this case. ∎

Remark 2.1. From the proof above we infer that when $\alpha \leqslant 1$, the constructed solution is convex.

3. CHARACTERIZATION OF SOLUTIONS

The space of solutions. We will look for a solution of equation (E) in the space $\mathscr{F}$. We recall that $F = 1 - \bar{F}$ and

$$\mathscr{F} = \{\bar{F} \in C^0(\boldsymbol{R}_+, [0, 1])\colon \ \exists\lambda > 0,\ c > 0, \text{ satisfying } F(ax)/F(x) \leqslant ca^\lambda,\ \forall a > 1,\ x > 0\}.$$

We note that this space contains the space of all absolutely continuous distributions with density f such that xf/F is bounded, which itself contains $\mathscr{E}$. For the first inclusion, there exists $\lambda > 0$ such that $xf(x)/F(x) < \lambda$. Then, for $a > 1$ and $x > 0$, integrating on the interval $[x, ax]$, we get $F(ax)/F(x) \leqslant a^\lambda$. For the second inclusion, as $\bar{F}$ is convex, $\big(1 - \bar{F}(x)\big)/x$ is decreasing. Differentiating, we obtain $xf(x)/F(x) < 1$. Moreover, we have $\mathscr{E} \subset \mathscr{F}$.

As recalled in the Introduction, Barral (in his paper [2]) studied a similar equation and found out a space of continuous functions having some key properties. We go further along this way, defining a space $\mathscr{F}$ containing the constructed solutions given by Theorems 2.1 and 2.2.

3.1. Behavior of the solutions in $\mathscr{F}$. We now come to the behavior at the origin of the solutions to (E) belonging to $\mathscr{F}$. In [11], Durrett and Liggett characterize the behavior at the origin of the solutions of the functional equation for Laplace transforms corresponding to the identity in law given in (1.3). This is found in Theorem 2.18 of [11], pp. 288–291, and is based on several technical results, namely, Lemma 2.3, Corollary 2.17 and Theorem 2.12. Replacing them by our Lemma 3.1, Corollary 3.1 and Theorem 3.2, respectively, we can adapt their proofs and obtain the following theorem. For the reader's convenience the statement and proofs of the quoted technical results are postponed to a subsequent subsection.

THEOREM 3.1. *Suppose the following condition* (H_δ) *holds*:

$$(H_\delta)\colon \ \exists\, \delta > 0,\ \forall q \in \boldsymbol{R}_+,\ \sum_{i=1}^{M} \Gamma_i C_i^q \in L_{1+\delta}.$$

Suppose also that there is an $\alpha > 0$ *such that* $\tau(\alpha) = 1$, $\tau'(\alpha) \leqslant 0$. *Then, if* $\bar{F}$ *is a solution to* (E) *and if* $\bar{F} \in \mathscr{F}$, *there exists* $s(\cdot)\colon \boldsymbol{R} \to \boldsymbol{R}_+$, *continuous and periodic with period* $-\log c$, $c > 0$, *in the lattice case and constant in the non-lattice case, such that* $x \to x^\alpha s(-\log x)$ *is increasing, with*

$$\text{(i)} \qquad \frac{F(x)}{x^\alpha s(-\log x)} \xrightarrow{x \downarrow 0} 1 \quad \textit{if } \tau'(\alpha) < 0,$$

$$\text{(ii)} \qquad \frac{F(x)}{x^\alpha |\log x|\, s(-\log x)} \xrightarrow{x \downarrow 0} 1 \quad \textit{if } \tau'(\alpha) = 0.$$

3.2. Technical results. In order to adapt the techniques developed in Durrett and Liggett [11] and Liu [17] we start by giving several technical lemmas which are essential to obtain Theorem 3.2 and Corollary 3.1. Finally, we derive our main Theorem 3.1. We recall that $\tau(q) < \infty$ whenever $q \geqslant 0$. Let us define a random variable Z_α, $\alpha > 0$, by the equality

$$E\Psi(Z_\alpha) = \tau(\alpha)^{-1} E\Big(\sum_{i=1}^{M} \Gamma_i C_i^\alpha \Psi(-\log C_i)\Big) \tag{3.1}$$

for all bounded measurable functions Ψ.

For an arbitrary ccdf $\bar{F}$, we define the functions D_α and G_α by

$$D_\alpha(z) = \frac{1 - \bar{F}(e^{-z})}{e^{-\alpha z}}, \tag{3.2}$$

$$G_\alpha(z) = e^{\alpha z} E\Big[\Big(\prod_{i=1}^{M} \bar{F}(e^{-z} C_i)^{\Gamma_i}\Big) - 1 + \sum_{i=1}^{M} \Gamma_i\big(1 - \bar{F}(e^{-z} C_i)\big)\Big]. \tag{3.3}$$

Let $\bar{F}_1$ be an arbitrary ccdf and $\bar{F}_2 = T\bar{F}_1$. We denote by $D_{\alpha,i}$ and $G_{\alpha,i}$ the corresponding functions associated with $\bar{F}_i$, $i = 1, 2$. We first give a series of lemmas.

LEMMA 3.1. *We have*

$$D_{\alpha,2}(z) = \tau(\alpha) E D_{\alpha,1}(z+Z_\alpha) - G_{\alpha,1}(z).$$

Proof. We have

$$D_{\alpha,2}(z) = e^{\alpha z}(1-\bar{F}_2(e^{-z})) = e^{\alpha z} E\Big[\sum_{i=1}^{M} \Gamma_i(1-\bar{F}_1(C_i e^{-z}))\Big] - G_{\alpha,1}(z)$$

$$= E\Big[\sum_{i=1}^{M} \Gamma_i C_i^{\alpha} D_{\alpha,1}(z-\log C_i)\Big] - G_{\alpha,1}(z) = \tau(\alpha) E D_{\alpha,1}(z+Z_\alpha) - G_{\alpha,1}(z). \blacksquare$$

LEMMA 3.2. *We have*:

(i) $G_\alpha(z) \geqslant 0$.

(ii) $e^{-\alpha z} G_\alpha(z)$ *is a decreasing function of* z.

(iii) *If* $\bar{F}_2 \geqslant \bar{F}_1$, *then, for all* z, $G_{\alpha,2}(z) \leqslant G_{\alpha,1}(z)$.

Proof. From the inequality

$$(3.4)\qquad \Big(\prod_{i=1}^{M} u_i^{\Gamma_i}\Big) - 1 + \sum_{i=1}^{M} \Gamma_i(1-u_i) \geqslant \Big(\prod_{i=1}^{M} v_i^{\Gamma_i}\Big) - 1 + \sum_{i=1}^{M} \Gamma_i(1-v_i),$$

$0 \leqslant u_i \leqslant v_i \leqslant 1$, we deduce the monotone decreasing feature of the function $e^{-\alpha z} G_\alpha(z)$. Let $\bar{F}_1$ and $\bar{F}_2$ be two ccdf's with $\bar{F}_1 \leqslant \bar{F}_2$. Replacing $\bar{F}$ by $\bar{F}_1$ or by $\bar{F}_2$, respectively, in equation (3.3), we obtain their associated functions $G_{\alpha,1}$ and $G_{\alpha,2}$. From the above inequality we have $G_{\alpha,2} \leqslant G_{\alpha,1}$. Finally, inequality (3.4) can be checked by observing

$$\partial_{u_j}\Big[\Big(\prod_{i=1}^{M} u_i^{\Gamma_i}\Big) - 1 + \sum_{i=1}^{M} \Gamma_i(1-u_i)\Big] = \Gamma_j\Big(\Big(\prod_{i\neq j} u_i^{\Gamma_i} u_j^{\Gamma_j - 1}\Big) - 1\Big) \leqslant 0. \blacksquare$$

LEMMA 3.3. *With* $\phi(u) := e^{-u} - 1 + u$ *and a ccdf* $\bar{F} \in \mathscr{F}$ *with corresponding* λ, *we have*

$$\text{(i)}\qquad G_\alpha(z) \leqslant e^{\alpha z} E\big[\phi(W D_\alpha(z) e^{-\alpha z})\big],$$

where $W := \sum_{i=1}^{M} \Gamma_i \max(cC_i^{\lambda}, 1)$;

$$\text{(ii)}\qquad \lim_{z\uparrow\infty} \frac{G_\alpha(z)}{D_\alpha(z)} = 0.$$

Proof. (i) Using $u \leqslant e^{-(1-u)}$ if $0 < u < 1$, we get

$$G_\alpha(z) \leqslant e^{\alpha z} E\Big[\Big(\prod_{i=1}^{M} \big[\exp\big(-(1-\bar{F}(C_i e^{-z}))\big)\big]^{\Gamma_i}\Big) - 1 + \sum_{i=1}^{M} \Gamma_i(1-\bar{F}(C_i e^{-z}))\Big]$$

$$\leqslant e^{\alpha z} E\Big[\exp\Big(-\sum_{i=1}^{M} \Gamma_i(1-\bar{F}(C_i e^{-z}))\Big) - 1 + \sum_{i=1}^{M} \Gamma_i(1-\bar{F}(C_i e^{-z}))\Big]$$

$$\leqslant e^{\alpha z} E\Big[\phi\Big(\sum_{i=1}^{M} \Gamma_i(1-\bar{F}(C_i e^{-z}))\Big)\Big].$$

Now, two cases arise:

- if $C_i \leqslant 1$, then $C_i e^{-z} \leqslant e^{-z}$ and $1-\bar{F}(C_i e^{-z}) \leqslant 1-\bar{F}(e^{-z})$;
- if $C_i \geqslant 1$, then $F(C_i e^{-z}) \leqslant cC_i^{\lambda} F(e^{-z})$, and so

$$1-\bar{F}(C_i e^{-z}) \leqslant cC_i^{\lambda}\big(1-\bar{F}(e^{-z})\big).$$

Consequently, $1-\bar{F}(C_i e^{-z}) \leqslant (cC_i^{\lambda} \vee 1)\big(1-\bar{F}(e^{-z})\big)$ and function $u \to \phi(u)$ being monotone increasing it follows

$$E\Big[\phi\Big(\sum_{i=1}^{M} \Gamma_i\big(1-\bar{F}(C_i e^{-z})\big)\Big)\Big] \leqslant E\Big[\phi\Big(\sum_{i=1}^{M} \Gamma_i (cC_i^{\lambda} \vee 1)\big(1-\bar{F}(e^{-z})\big)\Big)\Big].$$

Finally, $G_\alpha(z) \leqslant e^{\alpha z} E\big[\phi\big(WD_\alpha(z)e^{-\alpha z}\big)\big]$.

(ii) We first note that $e^{-\alpha z} D_\alpha(z) \xrightarrow{z\uparrow\infty} 0$. To prove (ii), we need to check

$$\lim_{t\downarrow 0} E\{\phi(Wt)/t\} = 0.$$

Now, $\phi(u)/u$ is bounded, and so $|\phi(Wt)/t| < K\cdot W$ for a suitable constant $K > 0$. Further, W is integrable since

$$W \leqslant \sum_{i=1}^{M} \Gamma_i + \sum_{i=1}^{M} c\Gamma_i C_i^{\lambda} \quad \text{and} \quad EW \leqslant \tau(0)+c\tau(\lambda) < \infty. \ \blacksquare$$

LEMMA 3.4. *Let* $\alpha \in \boldsymbol{R}$ *and* Z_α *be defined by* (3.1). *Let* g *be a non-negative function on* $\boldsymbol{R}$. *If* $g(y) = \tau(\alpha) Eg(y+Z_\alpha)$, *then*

$$g(y) = \sum_{\beta\in\mathscr{S}} \zeta_\beta(y) \exp\big(-(\beta-\alpha)y\big),$$

where $\zeta_\beta(y) \geqslant 0$ *with* $\zeta_\beta(x+y) = \zeta_\beta(y)$ *for all* $x \in \operatorname{Supp}(Z_\alpha)$, *and* $\mathscr{S} := \{\beta\colon \tau(\beta) = 1\}$ *with* $|\mathscr{S}| \in \{0, 1, 2\}$. *If* $|\mathscr{S}| = 0$, *we use the convention* $g = 0$.

Proof. Following the Lau–Rao–Shanbhag theorem [21]: if $|\mathscr{S}| = 0$, then $g = 0$; if $|\mathscr{S}| \in \{1, 2\}$, we get

$$g(y) = \sum_{\beta\in\mathscr{S}} \zeta_\beta(y) \exp(-\eta_\beta y),$$

where η_β satisfies

$$\tau(\alpha) E\big(\exp(-\eta_\beta Z_\alpha)\big) = 1.$$

Clearly,

$$E\Big(\sum_{i=1}^{M} \Gamma_i C_i^{\alpha} \exp(\eta_\beta \log C_i)\Big) = E\Big(\sum_{i=1}^{M} \Gamma_i C_i^{\alpha+\eta_\beta}\Big) = 1,$$

which implies $\beta = \alpha+\eta_\beta$, where $\beta \in \mathscr{S}$. $\blacksquare$

Remark 3.1. In the lattice case, ζ_β are periodic functions, and in the non-lattice case, ζ_β are constants. Under the additional hypothesis, $g(0) = 1$ when $|\mathscr{S}| = 2$, necessarily $\zeta_{\beta_2}(0) = 1-\zeta_{\beta_1}(0)$, where (β_1, β_2) are two solutions to $\tau(\beta) = 1$.

THEOREM 3.2. *Assume that $\bar{F} \in \mathscr{F}$ and that $\bar{F} \in \mathscr{F}$ is a solution of (E). Then:*
(i) *There exists $\alpha > 0$ such that $\tau(\alpha) = 1$.*
(ii) *Let $\alpha > 0$ satisfy $\tau(\alpha) = 1$ and $\tau'(\alpha) \leqslant 0$; then*

$$\limsup_{x \uparrow \infty} \frac{D_\alpha(x+y)}{D_\alpha(x)} \leqslant 1 \quad \text{if } \tau'(\alpha) < 0$$

and

$$\lim_{x \uparrow \infty} \frac{D_\alpha(x+y)}{D_\alpha(x)} = 1 \quad \text{if } \tau'(\alpha) = 0,$$

where y is any non-negative multiple of $-\log c$ in the lattice case, and $y \in \boldsymbol{R}_+$ in the non-lattice case.

Proof. Following [11], let $\alpha > 0$ and

$$h_x(y) = \frac{D_\alpha(x+y)}{D_\alpha(x)}.$$

We have

$$h_x(y) = \tau(\alpha) E h_x(y+Z_\alpha) - \frac{G_\alpha(x+y)}{D_\alpha(x+y)} h_x(y).$$

Note that $\bar{F}$ is not necessarily convex as in the Laplace transform context. Nevertheless we can adapt the proof of Theorem 2.12 in [11] to ccdf's $\bar{F} \in \mathscr{F}$. When $\bar{F} \in \mathscr{F}$, there exist $\lambda > 0$ and $c > 0$ such that

$$h_x(y) \leqslant e^{\alpha y} \mathbf{1}_{(y \geqslant 0)} + c e^{(\alpha - \lambda) y} \mathbf{1}_{(y < 0)}.$$

Consequently, the set $\{h_x(\cdot),\ x \in \boldsymbol{R}\}$ is uniformly bounded and equi-continuous on the bounded subsets of $\boldsymbol{R}$. We can therefore extract a subsequence h_{x_n} converging uniformly on the bounded subsets of $\boldsymbol{R}$ to some function h. The sequence $(x_n)_{n \in \boldsymbol{N}}$ converges to infinity when n tends to ∞. From the inequality above we infer that $h_{x_n}(y+Z_\alpha)$ is dominated by

$$\exp\big(\alpha(y+Z_\alpha)\big) \mathbf{1}_{((y+Z_\alpha) \geqslant 0)} + c \exp\big((\alpha-\lambda)(y+Z_\alpha)\big) \mathbf{1}_{((y+Z_\alpha) < 0)}$$

and

$$E\big[\exp\big(\alpha(y+Z_\alpha)\big) \mathbf{1}_{((y+Z_\alpha) \geqslant 0)} + c \exp\big((\alpha-\lambda)(y+Z_\alpha)\big) \mathbf{1}_{((y+Z_\alpha) < 0)}\big]$$
$$\leqslant e^{\alpha y} \frac{\tau(0)}{\tau(\alpha)} + c e^{(\alpha-\lambda) y} \frac{\tau(\lambda)}{\tau(\alpha)} < \infty.$$

By the dominated convergence theorem and Lemma 3.3 (ii), we obtain

$$h(y) = \tau(\alpha) E h(y+Z_\alpha).$$

Consider an $\alpha > 0$ satisfying $\tau(\alpha) > 1$. From Lemma 3.4 we infer that there exists $\beta \in \boldsymbol{R}$ such that $\tau(\beta) = 1$; equivalently, β satisfies $E \exp\big(-(\beta-\alpha) Z_\alpha\big) = 1/\tau(\alpha)$, and hence $\beta > \alpha > 0$. This proves (i). In the non-lattice case, assuming

$|\mathcal{S}| = 2$ with $\mathcal{S} := \{\beta : \tau(\beta) = 1\}$, for $\zeta_\beta > 0$ we obtain

$$h(y) = \sum_{\beta \in \mathcal{S}} \zeta_\beta \exp\big(-(\beta - \alpha) y\big).$$

Assuming $\beta_1 < \beta_2$, taking $\alpha = \beta_1$ and recalling that $h(0) = 1$, we have for $y > 0$

$$h(y) = \zeta_{\beta_1} + (1 - \zeta_{\beta_1}) \exp\big(-(\beta_2 - \alpha) y\big) \leqslant 1.$$

In the case $\beta_1 = \beta_2$ with $\tau'(\beta_1) = 0$, we have $h(y) = \zeta_{\beta_1} = 1$. In the lattice case, for y a multiple of $-\log c$, we get

$$h(-k \log c) = \sum_{\beta \in \mathcal{S}} \zeta_\beta(0) \exp\big(-(\beta - \alpha) y\big),$$

and using similar arguments we obtain $h(y) \leqslant 1$ if $\tau'(\beta_1) < 0$, $h(y) = 1$ if $\tau'(\beta_1) = 0$.

Fix $y > 0$. Let $(x_n)_{n \in \boldsymbol{N}}$ be a sequence converging to infinity when n tends to ∞ such that

$$\limsup_{x \uparrow \infty} \frac{D_\alpha(x+y)}{D_\alpha(x)} = \lim_{n \uparrow \infty} h_{x_n}(y).$$

Extracting a convergent subsequence from $\{h_{x_n}\}_{n \geqslant 1}$, converging uniformly to some h on bounded subsets of $\boldsymbol{R}$, we see that this function h fulfills the above conditions and similar arguments apply, completing the proof. ∎

Corollary 3.1. *Let $\bar{F} \in \mathcal{F}$ and assume that there is an $\alpha > 0$ satisfying $\tau(\alpha) = 1$ and $\tau'(\alpha) \leqslant 0$. Then, under the condition (H_δ) which states that*

$$\exists \delta > 0,\ \forall q \in \boldsymbol{R}_+,\ \sum_{i=1}^{M} \Gamma_i C_i^q \in \boldsymbol{L}_{1+\delta},$$

$G_\alpha(x)$ is direct Riemann integrable on $\boldsymbol{R}$.

Proof. We first note, following [11], that if $G_\alpha(x)$ is integrable, and if, as follows from Lemma 3.2, $e^{-\alpha x} G_\alpha(x)$ is a decreasing function of x, then $G_\alpha(x)$ is direct Riemann integrable. Hence it suffices to show that G_α is integrable. By Lemma 3.3, for $\bar{F} \in \mathcal{F}$ we have

$$0 \leqslant G_\alpha(z) \leqslant e^{\alpha z} \boldsymbol{E}\big[\phi\big(W D_\alpha(z) e^{-\alpha z}\big)\big],$$

where $\phi(u) = e^{-u} - 1 + u$ and $W = \sum_{i=1}^{M} \Gamma_i \max(c C_i^\lambda, 1)$.

At $z = -\infty$, from monotonicity of ϕ and recalling that $\phi(x) \leqslant x$ we obtain

$$\phi\big(W D_\alpha(z) e^{-\alpha z}\big) \leqslant \phi(W) \leqslant W \leqslant \sum_{i=1}^{M} \Gamma_i + \sum_{i=1}^{M} c \Gamma_i C_i^\lambda.$$

Hence $G_\alpha(z) \leqslant e^{\alpha z}\big(\tau(0) + c\tau(\lambda)\big)$, which is integrable at $z = -\infty$.

At $z=+\infty$, as $e^{-\alpha z}D_\alpha(z)\xrightarrow{z\uparrow\infty}0$, there exists z_0 such that, for all $z\geqslant z_0$, $D_\alpha(z)\leqslant e^{\beta z}$ for some $0<\beta<\alpha$. By Lemma 3.3,

$$\int_{z_0}^{+\infty} G_\alpha(x)\,dx \leqslant \int_{z_0}^{+\infty} e^{\alpha x}E\phi(WD_\alpha(x)e^{-\alpha x})\,dx \leqslant \int_{z_0}^{+\infty} e^{\alpha x}E\phi(We^{(\beta-\alpha)x})\,dx.$$

Passing to the variable $u=e^{(\beta-\alpha)x}$, letting $u_0=e^{(\beta-\alpha)z_0}$, we obtain

$$\int_{z_0}^{+\infty} G_\alpha(x)\,dx \leqslant \frac{1}{\alpha-\beta}\int_0^{u_0}\frac{E\phi(Wu)}{u^{2+\beta/(\alpha-\beta)}}\,du = \frac{1}{\alpha-\beta}\int_0^{u_0}\frac{\varphi_W(u)-1+uEW}{u^{2+\beta/(\alpha-\beta)}}\,du,$$

where $\varphi_W(u)$ is the Laplace–Stieltjes transform of W. By Theorem B of Bingham and Doney ([5], p. 718), this integral is finite if and only if $E\phi(W^{1+\beta/(\alpha-\beta)})<\infty$. Assuming $0<\beta/(\alpha-\beta)<1$, this condition holds as soon as we have

$$0<\frac{\beta}{\alpha-\beta}<1\wedge\delta \quad\text{or}\quad 0<\beta<\frac{\alpha\delta}{\delta+1}\wedge\frac{\alpha}{2}. \ ■$$

3.3. Characterization of the constructed solutions. We now give a more precise statement on the constructed solutions of equation (E) as given by Theorems 2.1 and 2.2. Using Theorem 3.1 we are able to describe more precisely their behavior at the origin. Definition 2.2 of the space $\mathscr{S}_c$ is adapted to the special case. We introduce a more general space $\mathscr{S}_{\alpha,c}$ which will be used in the general case and for which $\mathscr{S}_{1,c}:=\mathscr{S}_c$.

DEFINITION 3.1. Define the space $\mathscr{S}_{\alpha,c}$ as the set of functions $s(\cdot)\colon \boldsymbol{R}\to\boldsymbol{R}_+$ satisfying:

In the lattice case with common span $-\log c$, $c>0$, $s(z):=e^{-\alpha v(z)}$ for some right-continuous bounded periodic function $v(\cdot)$ on $\boldsymbol{R}$ with period $-\log c$, such that $z-v(z)$ is a non-decreasing function.

In the non-lattice case, $s(z):=s>0$, the constant function for all $z\in\boldsymbol{R}$.

THEOREM 3.3. *Suppose that there exists $0<\alpha<\infty$ such that $\tau(\alpha)=1$ and $\tau'(\alpha)\leqslant 0$. Two cases arise:*

(i) *If $\tau'(\alpha)<0$ and $\boldsymbol{E}\left[\sum_{i=1}^M\Gamma_iC_i^\alpha\log_+\left(\sum_{i=1}^M\Gamma_iC_i^\alpha\right)\right]<\infty$, then for each $s\in\mathscr{S}_{\alpha,c}$ there exists a solution $\bar{F}$ of (E) satisfying*

$$\frac{F(x)}{x^\alpha s(-\log x)}\xrightarrow{x\downarrow 0}1.$$

(ii) *If $\tau'(\alpha)=0$ and $\boldsymbol{E}\left[\left(\sum_{i=1}^M\Gamma_iC_i^\beta\right)^{1+\delta}\right]<\infty$ for all $0<\beta<\alpha$, then for each $s\in\mathscr{S}_{\alpha,c}$ there exists a solution $\bar{F}$ of (E) satisfying*

$$\frac{F(x)}{x^\alpha|\log x|\,s(-\log x)}\xrightarrow{x\downarrow 0}1.$$

Proof. Consider the ccdf $\bar{F}_\alpha$ as a solution to the functional equation

$$(E_\alpha):\ \bar{F}_\alpha(x) = E\left[\prod_{i=1}^{M} \bar{F}_\alpha(C_i^\alpha x)^{\Gamma_i}\right]. \tag{3.5}$$

(i) If $\tau'(\alpha) < 0$, reconsidering the proof of Theorem 2.2 concerning this case, we see that there exists $\bar{F}_\alpha$ in $\mathscr{E}_1$, a solution to (E_α). In particular, we have

$$\lim_{x\downarrow 0} \frac{F_\alpha(x)}{x} = 1.$$

Now for each $s \in \mathscr{S}_{\alpha,c}$ the ccdf $\bar{F}(x) = \bar{F}_\alpha(x^\alpha s(-\log x))$ solves the functional equation (E) with the claimed behavior at 0.

(ii) If $\tau'(\alpha) = 0$, reconsidering the proof of Theorem 2.2 concerning this case, we see that there exists $\bar{F}_\alpha$, a solution to (E_α). By construction, $\bar{F}_\alpha$ is convex, and *a fortiori* $\bar{F}_\alpha \in \mathscr{F}$. Hence we can deduce from Theorem 3.1 that

$$\lim_{x\downarrow 0} \frac{F_\alpha(x)}{x\,|\log x|\, s_\alpha(-\log x)} = 1,$$

where $s_\alpha(\cdot)$ is continuous and log-periodic. The structure function $\tau_\alpha(q) = \tau(\alpha q)$ associated with (E_α) now satisfies $\tau_\alpha(1) = 1$ and $\tau'_\alpha(1) = 0$. Following the arguments of [11], p. 290, using the convexity of $\bar{F}_\alpha$ and the fact that the function $s_\alpha(\cdot)$ is monotone and periodic, we can infer that $s_\alpha(\cdot)$ is constant, say $s_\alpha(\cdot) = \kappa > 0$.

Now, for each function $s(\cdot) \in \mathscr{S}_{\alpha,c}$, the ccdf

$$\bar{F}(x) := \bar{F}_\alpha\left(x^\alpha \frac{s(-\log x)}{\kappa\alpha}\right)$$

solves the functional equation (E) and

$$\lim_{x\downarrow 0} \frac{F(x)}{x^\alpha\,|\log x|\, s(-\log x)}$$

$$= \lim_{x\downarrow 0} \left\{ \frac{F_\alpha\left(x^\alpha \frac{s(-\log x)}{\kappa\alpha}\right)}{x^\alpha \frac{s(-\log x)}{\kappa\alpha}\left|\log\left(x^\alpha \frac{s(-\log x)}{\kappa\alpha}\right)\right|} \frac{\left|\log\left(x^\alpha \frac{s(-\log x)}{\kappa\alpha}\right)\right|}{\kappa\alpha\,|\log x|} \right\}.$$

Using the behavior of $F_\alpha(x)$ at 0 and recalling that $s_\alpha(\cdot) = \kappa$, we obtain

$$\lim_{x\downarrow 0} \frac{F(x)}{x^\alpha\,|\log x|\, s(-\log x)} = \kappa \lim_{x\downarrow 0} \frac{\left|\log\left(x^\alpha \frac{s(-\log x)}{\kappa\alpha}\right)\right|}{\kappa\alpha\,|\log x|} = 1.$$

Consequently, $F(x)$ has the claimed behavior at 0. ∎

Remark 3.2. From the proof we observe that, for $\alpha < 1$ and $s(\cdot) = s > 0$ constant, the constructed solution is convex.

4. UNIQUENESS OF SOLUTIONS

In this section we discuss the uniqueness of solutions to equation (E).

As noticed in the Introduction, when Γ_i are integer-valued random variables, *a fortiori* when $\Gamma_i = 1$, equation (E) yields the following equation in distribution:

$$X \stackrel{d}{=} \min_{1 \leqslant i \leqslant N} A_i X_i, \tag{4.1}$$

where $A_i > 0$. We can adapt the proof of the uniqueness theorem given in Liu [18], p. 105, to our ccdf context. We obtain the following result:

THEOREM 4.1. *Let Γ_i be integer-valued random variables. Assume that there is an $\alpha > 0$ satisfying $\tau(\alpha) = 1$ and $\tau'(\alpha) \leqslant 0$. Under the condition (H_δ) of Theorem* 3.1, *the solution to (E) in the space $\mathscr{F}$ is unique. By uniqueness, it is meant that: if $\bar{F}_1$ and $\bar{F}_2$ are solutions whose behaviors in a neighborhood of zero are both given by the same pair $(\alpha, s(\cdot))$ in* (i) *and* (ii) *of Theorem* 3.1, *then $\bar{F}_1 = \bar{F}_2$.*

Sketch of the proof. For all sequences $\sigma \in \bigcup_{i \geqslant 1} N^*$ of positive integers, with $|\sigma|$ the length of σ, let $(A_{\sigma,1}, A_{\sigma,2}, \ldots)$ be i.i.d. copies of $(A_1, A_2, \ldots)$. For a ccdf $\bar{F}$, $T^n \bar{F}$ is the ccdf of $\min_{|\sigma|=n} l_\sigma X_\sigma$, where $l_\sigma := A_{\sigma_1} A_{\sigma_1\sigma_2} \ldots A_{\sigma_1\sigma_2\ldots\sigma_n}$ if $\sigma = \sigma_1\sigma_2\ldots\sigma_n$, $\{X_\sigma : |\sigma| = n\}$ are i.i.d. copies with ccdf $\bar{F}$, independent of $\{A_\sigma : |\sigma| \leqslant n\}$. The results of Lemmas 7.1 and 7.2 of [18], p. 104, still hold because we have the same tree structure. We are in the position to obtain a version of Lemma 7.3 of [18], p. 104, while considering the quantity $T^n \bar{F}(x) = E \prod_{|\sigma|=n} \bar{F}(x l_\sigma)$ replacing Laplace–Stieltjes transforms by ccdf's. Under the condition (H_δ), let $\bar{F}_1$ and $\bar{F}_2$ be two solutions in $\mathscr{F}$ of (E) whose behaviors in a neighborhood of zero are both given by the same pair $(\alpha, s(\cdot))$ in (i) and (ii) of Theorem 3.1. Then $1-\bar{F}_1 \sim 1-\bar{F}_2$ in a neighborhood of zero and, following the steps of Theorem 7.1 in [18], p. 105, we obtain $\lim_{n\uparrow\infty} T^n \bar{F}_1 = \bar{F}_2$. ∎

In the general case, when $\Gamma_i \geqslant 1$ but is not necessarily integer-valued, we obtain the uniqueness in the only case when we suppose that there is an α satisfying $\tau(\alpha) = 1$ and $\tau'(\alpha) < 0$.

THEOREM 4.2. *Assume that there is an $\alpha > 0$ satisfying $\tau(\alpha) = 1$ and $\tau'(\alpha) < 0$. Under the condition (H_δ), the solution to (E) in the space $\mathscr{F}$ is unique: if $\bar{F}_1$ and $\bar{F}_2$ are solutions whose behaviors in a neighborhood of zero are both given by the same pair $(\alpha, s(\cdot))$ in* (i) *of Theorem* 3.1, *then $\bar{F}_1 = \bar{F}_2$.*

Proof. Let us now show that if there are two solutions in $\mathscr{F}$, with similar behavior close to 0, then they coincide. Let $\bar{F}_1$ and $\bar{F}_2$ be two distinct ccdf's in $\mathscr{F}$ which are solutions to (E), with $\bar{F}_1$ and $\bar{F}_2$ both equivalent close to 0 to $x^\alpha s(-\log x)$ with $s(\cdot)$ continuous and periodic by Theorem 3.1. Consider

$$d(\bar{F}_1, \bar{F}_2) := \sup_{x>0}\left|\frac{\bar{F}_1(x)-\bar{F}_2(x)}{x^\alpha s(-\log x)}\right|.$$

As $\bar{F}_1$ and $\bar{F}_2$ are solutions in $\mathscr{F}$, the function

$$x \to \left|\frac{\bar{F}_1(x)-\bar{F}_2(x)}{x^\alpha s(-\log x)}\right|$$

is continuous on $[0, \infty)$, vanishes at ∞, so its supremum is attained at some point x_0 in $(0, \infty)$. Clearly,

$$d(\bar{F}_1, \bar{F}_2) = \left|\frac{T\bar{F}_1(x_0)-T\bar{F}_2(x_0)}{x_0^\alpha s(-\log x_0)}\right|.$$

Now, by Jensen's inequality,

$$|T\bar{F}_1(x_0)-T\bar{F}_2(x_0)| \leqslant \boldsymbol{E}\left|\prod_{i=1}^{M}\bar{F}_1(C_i x_0)^{\Gamma_i}-\prod_{i=1}^{M}\bar{F}_2(C_i x_0)^{\Gamma_i}\right|.$$

Using the inequality $|\prod_{i=1}^M a_i - \prod_{i=1}^M b_i| \leqslant \sum_{i=1}^M |a_i - b_i|$ for $a_i \in [0, 1]$ and $b_i \in [0, 1]$, $i = 1, \ldots, M$, we obtain

$$|T\bar{F}_1(x_0)-T\bar{F}_2(x_0)| \leqslant \boldsymbol{E}\Big[\sum_{i=1}^{M}|\bar{F}_1(C_i x_0)^{\Gamma_i}-\bar{F}_2(C_i x_0)^{\Gamma_i}|\Big]. \tag{4.2}$$

Let $A := \{\bar{F}_1(C_i x_0) \neq \bar{F}_2(C_i x_0),\ \Gamma_i > 1 \text{ for some } i \in \{1, \ldots, M\}\}$.

If $\boldsymbol{P}(A) > 0$, using the Hölderian character of $u \to u^\gamma$, $\gamma > 1$, $u \in [0, 1]$, that is $|x^\gamma - y^\gamma| < \gamma|x-y|$ for $x, y \in [0, 1]$ and $x \neq y$, we get from (4.2) the inequality

$$|T\bar{F}_1(x_0)-T\bar{F}_2(x_0)| < \boldsymbol{E}\Big[\sum_{i=1}^{M}\Gamma_i|\bar{F}_1(C_i x_0)-\bar{F}_2(C_i x_0)|\Big].$$

Now,

$$\begin{aligned} d(\bar{F}_1, \bar{F}_2) &= \left|\frac{T\bar{F}_1(x_0)-T\bar{F}_2(x_0)}{x_0^\alpha s(-\log x_0)}\right| \\ &< \boldsymbol{E}\left[\sum_{i=1}^{M}\Gamma_i C_i^\alpha\left|\frac{\bar{F}_1(C_i x_0)-\bar{F}_2(C_i x_0)}{C_i^\alpha x_0^\alpha s(-\log(C_i x_0))}\right|\right] \leqslant \boldsymbol{E}\Big(\sum_{i=1}^{M}\Gamma_i C_i^\alpha\Big) d(\bar{F}_1, \bar{F}_2). \end{aligned}$$

For the first inequality, we use the fact that $-\log C_i$, $i \geqslant 1$, have a common span $-\log c$, $c > 0$, and $s(\cdot)$ is periodic with period $-\log c$. In this case, $d(\bar{F}_1, \bar{F}_2) < d(\bar{F}_1, \bar{F}_2)$, which is absurd.

If $P(A) = 0$, we have almost surely either $\Gamma_i = 1$ or $\bar{F}_1(C_i x_0) = \bar{F}_2(C_i x_0)$ for all $i \in \{1, \ldots, M\}$. Consequently, we get

$$\sum_{i=1}^{M} |\bar{F}_1(C_i x_0)^{\Gamma_i} - \bar{F}_2(C_i x_0)^{\Gamma_i}| = \sum_{i=1}^{M} |\bar{F}_1(C_i x_0) - \bar{F}_2(C_i x_0)|$$

almost surely. In this case, using the argument above on the support of $-\log C_i$, $i \geqslant 1$, and the periodicity of $s(\cdot)$, we obtain

$$d(\bar{F}_1, \bar{F}_2) = \left|\frac{T\bar{F}_1(x_0) - T\bar{F}_2(x_0)}{x_0^{\alpha} s(-\log x_0)}\right|$$
$$\leqslant E\left[\sum_{i=1}^{M} C_i^{\alpha} \left|\frac{\bar{F}_1(C_i x_0) - \bar{F}_2(C_i x_0)}{C_i^{\alpha} x_0^{\alpha} s(-\log(C_i x_0))}\right|\right] \leqslant E\left(\sum_{i=1}^{M} C_i^{\alpha}\right) d(\bar{F}_1, \bar{F}_2).$$

If $d(\bar{F}_1, \bar{F}_2) \neq 0$, then $E\left(\sum_{i=1}^{M} C_i^{\alpha}\right) \geqslant 1$. Since

$$E\left(\sum_{i=1}^{M} C_i^{\alpha}\right) \leqslant E\left(\sum_{i=1}^{M} \Gamma_i C_i^{\alpha}\right),$$

recalling that $E\left(\sum_{i=1}^{M} \Gamma_i C_i^{\alpha}\right) = 1$, we obtain

$$E\left(\sum_{i=1}^{M} (\Gamma_i - 1) C_i^{\alpha}\right) = 0,$$

which means that $\Gamma_i = 1$ for all $i \in \{1, \ldots, M\}$ almost surely. Hence, we recover the first case, which was dealt with by Theorem 4.1. ∎

5. CONCLUDING REMARK

In this paper, solutions to the functional equation (E), extending min-semistable distributions, are considered. The main extension with respect to previously studied functional equations of the same type is that it involves non-integral random powers. The techniques employed to derive our results are largely inspired from the ones originally designed for Laplace–Stieltjes transforms in the semistable case for sums.

In a special case, we start constructing solutions from scratch in the space $\mathscr{E}_1$ involving convexity. When considering the general case, we need to introduce a larger space, namely the space $\mathscr{F}$. It is the largest space within which solutions can be searched for, with the techniques we use to do so. The behavior at the origin of the solutions within $\mathscr{F}$ is elucidated. The characterization theorem involving the space $\mathscr{S}_{\alpha,c}$ shows that there are solutions to (E) whose behaviors in a neighborhood of the origin are possibly far from regular. This suggests that, due to some restrictions imposed on the solutions (in particular, continuity), we possibly miss some solutions with a wild behavior near zero.

Nevertheless, despite some technical constraints that we feel not intrinsical to the solutions of the posed problem, we hope to have done a further step towards the comprehension of a widely explored functional equation.

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