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ON THE STRONG LAW OF LARGE NUMBERS FOR SEQUENCES OF BLOCKWISE INDEPENDENT AND BLOCKWISE p-ORTHOGONAL RANDOM ELEMENTS IN RADEMACHER TYPE p BANACH SPACES

BY

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Abstract. For a sequence of random elements $\{V_n, n \geqslant 1\}$ taking values in a real separable Rademacher type p ($1 \leqslant p \leqslant 2$) Banach space and positive constants $b_n \uparrow \infty$, conditions are provided for the strong law of large numbers $\sum_{i=1}^{n} V_i/b_n \to 0$ almost surely. We treat the following cases: (i) $\{V_n, n \geqslant 1\}$ is blockwise independent with $EV_n = 0$, $n \geqslant 1$, and (ii) $\{V_n, n \geqslant 1\}$ is blockwise p-orthogonal. The conditions for case (i) are shown to provide an exact characterization of Rademacher type p Banach spaces. The current work extends results of Móricz [12], Móricz et al. [13], and Gaposhkin [8]. Special cases of the main results are presented as corollaries and illustrative examples or counterexamples are provided.

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1. INTRODUCTION

Móricz [12] introduced the concept of *blockwise independence* for a sequence of (real-valued) random variables and extended a classical strong law of large numbers (SLLN) of Kolmogorov (see, e.g., Chow and Teicher [6], p. 124) to the blockwise independent case. (Technical definitions needed in this paper will be discussed in Section 2.) Gaposhkin [7] and [8] also studied the SLLN problem for sequences of blockwise independent random variables; in those papers he also proved SLLNs for sequences of *blockwise orthogonal* random variables.

In the present paper, we consider a sequence of random elements $\{V_n, n \geqslant 1\}$ defined on a probability space $(\Omega, \mathscr{F}, P)$ and taking values in a real separable Banach space $\mathscr{X}$ with norm $\|\cdot\|$. We provide conditions for the SLLN:

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} V_i}{b_n} = 0 \text{ almost surely (a.s.)},$$

where $\{b_n, n \geqslant 1\}$ is a sequence of positive constants with $b_n \uparrow \infty$. The Banach space $\mathscr{X}$ is assumed to be of *Rademacher type p* $(1 \leqslant p \leqslant 2)$. The main findings are Theorems 3.1, 3.2, and 3.3. In Theorem 3.1 the random elements $\{V_n, n \geqslant 1\}$ are assumed to be blockwise independent with $EV_n = 0$, $n \geqslant 1$, whereas in Theorem 3.3 the random elements are assumed to be *blockwise p-orthogonal*. In Theorem 3.2, it is shown that the implication (3.2) $\Rightarrow$ (3.3) in Theorem 3.1 indeed provides an exact characterization of Rademacher type p Banach spaces. Theorems 3.1 and 3.3 are very general results in that they are new when the Banach space $\mathscr{X}$ is the real line $\boldsymbol{R}$. Of course, special cases of Theorems 3.1 and 3.3 are known to hold when $\mathscr{X} = \boldsymbol{R}$.

The present work extends results of Móricz [12], Móricz et al. [13], and Gaposhkin [8]. Our proofs are substantially different from those of the earlier counterparts due to a recent and elementary result of Chobanyan et al. [5] (Lemma 2.5 below) and these differences will be discussed in Remark 3.2.

The plan of the paper is as follows. Technical definitions, notation, lemmas and other results used in the proofs of the main results or their corollaries are given in Section 2. The main results are stated and proved in Section 3. In Section 4, some corollaries and interesting examples or counterexamples are presented.

2. PRELIMINARIES

Some definitions, notation, and preliminary results will be presented prior to establishing the main results. Let $\mathscr{X}$ be a real separable Banach space with norm $\|\cdot\|$. A random element in $\mathscr{X}$ will be denoted by V or V_n, etc.

The *expected value* or *mean* of a random element V, denoted by EV, is defined to be the *Pettis integral* provided it exists. That is, V has an expected value $EV \in \mathscr{X}$ if $f(EV) = E\big(f(V)\big)$ for every $f \in \mathscr{X}^*$, where $\mathscr{X}^*$ denotes the (*dual*) space of all continuous linear functionals on $\mathscr{X}$. If $E\|V\| < \infty$, then (see, e.g., Taylor [18], p. 40) V has an expected value. But the expected value can exist when $E\|V\| = \infty$. For an example, see Taylor [18], p. 41.

Let $\{Y_n, n \geqslant 1\}$ be a symmetric *Bernoulli sequence*; that is, $\{Y_n, n \geqslant 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with $P\{Y_1 = 1\} = P\{Y_1 = -1\} = 1/2$. Let $\mathscr{X}^\infty = \mathscr{X} \times \mathscr{X} \times \mathscr{X} \times \ldots$ and define

$$\mathscr{C}(\mathscr{X}) = \Big\{(v_1, v_2, \ldots) \in \mathscr{X}^\infty : \sum_{n=1}^{\infty} Y_n v_n \text{ converges in probability}\Big\}.$$

Let $1 \leqslant p \leqslant 2$. Then $\mathscr{X}$ is said to be of *Rademacher type p* if there exists a constant $0 < C < \infty$ such that

$$E \left\| \sum_{n=1}^{\infty} Y_n v_n \right\|^p \leqslant C \sum_{n=1}^{\infty} \|v_n\|^p \quad \text{for all } (v_1, v_2, \ldots) \in \mathscr{C}(\mathscr{X}).$$

Hoffmann-Jørgensen and Pisier [9] proved for $1 \leqslant p \leqslant 2$ that a real separable Banach space is of Rademacher type p if and only if there exists a constant $0 < C < \infty$ such that

$$E \left\| \sum_{i=1}^{n} V_i \right\|^p \leqslant C \sum_{i=1}^{n} E \|V_i\|^p \tag{2.1}$$

for every finite collection $\{V_1, \ldots, V_n\}$ of independent mean 0 random elements.

If a real separable Banach space is of Rademacher type p for some $1 < p \leqslant 2$, then it is of Rademacher type q for all $1 \leqslant q < p$. Every real separable Banach space is of Rademacher type (at least) 1 while the $\mathscr{L}_p$-spaces and l_p-spaces are of Rademacher type $2 \wedge p$ for $p \geqslant 1$. Every real separable Hilbert space and real separable finite-dimensional Banach space is of Rademacher type 2. In particular, the real line $\boldsymbol{R}$ is of Rademacher type 2.

A finite collection of random elements $\{V_1, \ldots, V_N\}$ $(N \geqslant 2)$ is said to be *p-orthogonal* $(1 \leqslant p < \infty)$ if $E\|V_n\|^p < \infty$ for all $1 \leqslant n \leqslant N$ and

$$E \left\| \sum_{i=1}^{n} a_{\pi(i)} V_{\pi(i)} \right\|^p \leqslant E \left\| \sum_{i=1}^{m} a_{\pi(i)} V_{\pi(i)} \right\|^p$$

for all choices of $1 \leqslant n < m \leqslant N$, for all constants $\{a_1, \ldots, a_m\}$, and for all permutations π of the integers $\{1, \ldots, m\}$. A sequence of random elements $\{V_n, n \geqslant 1\}$ is said to be *p-orthogonal* $(1 \leqslant p < \infty)$ if $\{V_1, \ldots, V_N\}$ is p-orthogonal for all $N \geqslant 2$. The notion of p-orthogonality was introduced by Howell and Taylor [10]; we refer to Howell and Taylor [10] and Móricz et al. [13] for a detailed discussion of p-orthogonality.

Let $\{\beta_k, k \geqslant 1\}$ be a strictly increasing sequence of positive integers with $\beta_1 = 1$ and set $B_k = [\beta_k, \beta_{k+1})$, $k \geqslant 1$. A sequence of random elements $\{V_n, n \geqslant 1\}$ is said to be *blockwise independent* (resp., *blockwise p-orthogonal* $(1 \leqslant p < \infty)$) with respect to the blocks $\{B_k, k \geqslant 1\}$ if for each $k \geqslant 1$ the random elements $\{V_i, i \in B_k\}$ are independent (resp., p-orthogonal). Thus the random elements with indices in each block are independent (resp., p-orthogonal) but there are no independence (resp., p-orthogonality) requirements between the random elements with indices in different blocks; even repetitions are permitted.

The following notation will be used throughout this paper. For $x \geqslant 0$, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x and let $\lceil x \rceil$ denote the smallest integer greater than or equal to x. We use Log to denote the logarithm to base 2. The symbol C denotes a generic constant $(0 < C < \infty)$ which is not necessarily the same in each appearance.

For $\{\beta_k, k \geqslant 1\}$ and $\{B_k, k \geqslant 1\}$ as above, we introduce the following notation:

$$B_k^{(m)} = B_k \cap [2^m, 2^{m+1}), \qquad k \geqslant 1, m \geqslant 0,$$

$$I_m = \{k \geqslant 1 : B_k^{(m)} \neq \emptyset\}, \qquad m \geqslant 0,$$

$$r_k^{(m)} = \min B_k^{(m)}, \qquad k \in I_m, m \geqslant 0,$$

$$c_m = \operatorname{card} I_m, \qquad m \geqslant 0,$$

$$\varphi(n) = \sum_{m=0}^{\infty} c_m I_{[2^m, 2^{m+1})}(n), \qquad n \geqslant 1,$$

where $I_{[2^m, 2^{m+1})}$ denotes the indicator function of the set $[2^m, 2^{m+1})$, $m \geqslant 0$.

It is easy to verify that the following relations are satisfied:

(i) If $\beta_k = 2^{k-1}$, $k \geqslant 1$, then

$$c_m = 1,\ m \geqslant 0, \quad \text{and} \quad \varphi(n) = 1,\ n \geqslant 1. \tag{2.2}$$

(ii) If $\beta_k = \lfloor q^{k-1} \rfloor$ for all large k, where $q > 1$, then

$$c_m = \mathcal{O}(1) \quad \text{and} \quad \varphi(n) = \mathcal{O}(1). \tag{2.3}$$

(iii) If $\beta_k = \lfloor 2^{k/(\operatorname{Log} k)^\alpha} \rfloor$ for all large k, where $\alpha > 0$, then

$$c_m = \mathcal{O}\big((\operatorname{Log} m)^\alpha\big) \quad \text{and} \quad \varphi(n) = \mathcal{O}\big((\operatorname{Log}\operatorname{Log} n)^\alpha\big). \tag{2.4}$$

(iv) If $\beta_k = \lfloor 2^{k^\alpha} \rfloor$ for all large k, where $0 < \alpha < 1$, then

$$c_m = \mathcal{O}\big(m^{(1-\alpha)/\alpha}\big) \quad \text{and} \quad \varphi(n) = \mathcal{O}\big((\operatorname{Log} n)^{(1-\alpha)/\alpha}\big). \tag{2.5}$$

(v) If $\beta_k = \lfloor k^\alpha \rfloor$, $k \geqslant 1$, where $\alpha > 1$, then

$$c_m = \mathcal{O}(2^{m/\alpha}) \quad \text{and} \quad \varphi(n) = \mathcal{O}(n^{1/\alpha}). \tag{2.6}$$

(vi) If $\beta_k = k$, $k \geqslant 1$, then

$$c_m = 2^m,\ m \geqslant 0, \quad \text{and} \quad \varphi(n) \leqslant n,\ n \geqslant 1. \tag{2.7}$$

The following result is well known when $\mathcal{X} = \boldsymbol{R}$ and $p = 2$.

LEMMA 2.1. *Let $\{V_n, n \geqslant 1\}$ be a sequence of independent mean 0 random elements in a real separable Rademacher type p $(1 \leqslant p \leqslant 2)$ Banach space. Then*

$$E\Big(\big(\max_{1 \leqslant j \leqslant n} \big\|\sum_{i=1}^{j} V_i\big\|\big)^p\Big) \leqslant C \sum_{i=1}^{n} E\|V_i\|^p, \qquad n \geqslant 1, \tag{2.8}$$

where the constant C does not depend on n.

Proof. Let $\mathcal{F}_n = \sigma(X_i, 1 \leqslant i \leqslant n)$, $n \geqslant 1$. Now it is well known but seems to have been first observed by Scalora [17] that $\{\|\sum_{i=1}^{n} V_i\|, \mathcal{F}_n, n \geqslant 1\}$ is a real submartingale. In the case $1 < p \leqslant 2$, by Doob's submartingale maximal inequality (see, e.g., Chow and Teicher [6], p. 255), for $n \geqslant 1$ we have

$$E\Big(\big(\max_{1 \leqslant j \leqslant n} \Big\|\sum_{i=1}^{j} V_i\Big\|\big)^p\Big) \geqslant \Big(\frac{p}{p-1}\Big)^p E\Big\|\sum_{i=1}^{n} V_i\Big\|^p$$

$$\leqslant C \sum_{i=1}^{n} E\|V_i\|^p \quad \text{(by (2.1))},$$

establishing (2.8). In the case $p = 1$, note that for $n \geqslant 1$

$$E\big(\max_{1 \leqslant j \leqslant n} \Big\|\sum_{i=1}^{j} V_i\Big\|\big) \leqslant E\big(\max_{1 \leqslant j \leqslant n} \sum_{i=1}^{j} \|V_i\|\big) = E\big(\sum_{i=1}^{n} \|V_i\|\big) = \sum_{i=1}^{n} E\|V_i\|,$$

again establishing (2.8). ■

PROPOSITION 2.1 (Hoffmann-Jørgensen and Pisier [9]). *Let $1 \leqslant p \leqslant 2$ and let $\mathscr{X}$ be a real separable Banach space. Then the following two statements are equivalent*:

(i) *$\mathscr{X}$ is of Rademacher type p.*

(ii) *For every sequence $\{V_n, n \geqslant 1\}$ of independent mean 0 random elements in $\mathscr{X}$, the condition*

$$\sum_{n=1}^{\infty} \frac{E\|V_n\|^p}{n^p} < \infty$$

implies

$$\frac{\sum_{j=1}^{n} V_j}{n} \to 0 \text{ a.s.}$$

LEMMA 2.2 (Howell and Taylor [10]). *If $\mathscr{X}$ is a real separable Rademacher type p ($1 \leqslant p \leqslant 2$) Banach space, then there exists a constant $C < \infty$ such that*

$$E\Big\|\sum_{i=1}^{n} V_i\Big\|^p \leqslant C \sum_{i=1}^{n} E\|V_i\|^p, \quad n \geqslant 1,$$

for all p-orthogonal sequences of $\mathscr{X}$-valued random elements.

LEMMA 2.3 (Móricz et al. [13]). *Let $\{V_n, n \geqslant 1\}$ be a p-orthogonal ($1 \leqslant p < \infty$) sequence of random elements in a real separable Banach space and suppose that there exists a sequence of nonnegative numbers $\{u_n, n \geqslant 1\}$ such that*

$$E\Big\|\sum_{i=n}^{m} V_i\Big\|^p \leqslant \sum_{i=n}^{m} u_i \quad \text{for all } m \geqslant n \geqslant 1.$$

Then

$$E\Big(\big(\max_{n \leqslant j \leqslant m} \Big\|\sum_{i=n}^{j} V_i\Big\|\big)^p\Big) \leqslant \big(\mathrm{Log}(2(m-n+1))\big)^p \sum_{i=n}^{m} u_i, \quad m \geqslant n \geqslant 1.$$

An immediate consequence of Lemmas 2.2 and 2.3 is that if $\{V_1, \ldots, V_N\}$ is a p-orthogonal ($1 \leqslant p \leqslant 2$) collection of $N \geqslant 2$ random elements in a real separable Rademacher type p Banach space, then

$$(2.9)\qquad E\Big(\max_{n\leqslant j\leqslant m}\Big\|\sum_{i=n}^{j} V_i\Big\|\Big)^p \leqslant C\big(\operatorname{Log}(2(m-n+1))\big)^p\sum_{i=n}^{m} E\|V_i\|^p,\qquad 1\leqslant n\leqslant m\leqslant N,$$

where the constant C does not depend on n, m, or N.

The next lemma is most probably known but we are unable to locate a reference. It follows immediately from Hölder's inequality writing $a_i = a_i\cdot 1$, as was so kindly pointed out to us by the referee.

LEMMA 2.4. *If $a_i \geqslant 0$, $1\leqslant i\leqslant n$, and $p\geqslant 1$, then*

$$\Big(\sum_{i=1}^{n} a_i\Big)^p \leqslant n^{p-1}\sum_{i=1}^{n} a_i^p.$$

The following elementary result was recently obtained by Chobanyan et al. [5] and it will play a key role in the proofs of Theorems 3.1 and 3.3. Lemma 2.5 is closely related to a classical result of Prokhorov [14] and [15] when $\mathscr{X} = \boldsymbol{R}$, the $\{V_n, n\geqslant 1\}$ are independent symmetric random variables, and $b_n = n^\alpha$, $n\geqslant 1$, with $\alpha > 0$.

LEMMA 2.5 (Chobanyan et al. [5]). *Let $\{V_n, n\geqslant 1\}$ be a sequence of random elements in a real separable Banach space $\mathscr{X}$, let $\{b_n, n\geqslant 1\}$ be a nondecreasing sequence of positive constants, and let $\{k_n, n\geqslant 0\}$ be a sequence of positive integers such that*

$$(2.10)\qquad \inf_{n\geqslant 0}\frac{b_{k_{n+1}}}{b_{k_n}} > 1 \quad \textit{and} \quad \sup_{n\geqslant 0}\frac{b_{k_{n+1}}}{b_{k_n}} < \infty.$$

Then

$$(2.11)\qquad \lim_{n\to\infty}\frac{\sum_{i=1}^{n} V_i}{b_n} = 0 \ \textit{a.s.}$$

if and only if

$$(2.12)\qquad \lim_{n\to\infty}\frac{\max_{k_n\leqslant k<k_{n+1}}\|\sum_{i=k_n}^{k} V_i\|}{b_{k_{n+1}} - b_{k_n}} = 0 \ \textit{a.s.}$$

Remark 2.1. (i) Note that the first inequality of (2.10) ensures that $\{k_n, n\geqslant 0\}$ is strictly increasing and $\lim_{n\to\infty} b_n = \infty$.

(ii) It follows that if (2.12) holds for some sequence of positive integers $\{k_n, n\geqslant 0\}$ satisfying (2.10), then (2.12) holds for every sequence of positive integers $\{k_n, n\geqslant 0\}$ satisfying (2.10). Thus, in order to prove the SLLN (2.11), nothing is lost in working with a convenient sequence such as $k_n = 2^n$, $n\geqslant 0$. This remark was made by Chobanyan et al. [5].

(iii) If the random elements $\{V_n, n \geqslant 1\}$ are independent and symmetric, then by the random element version of Lévy's inequality (see, e.g., Araujo and Giné [3], p. 102) and the Borel–Cantelli lemma, (2.12) is equivalent to the structurally simpler and apparently weaker condition

$$\lim_{n\to\infty} \frac{\left\|\sum_{i=k_n}^{k_{n+1}-1} V_i\right\|}{b_{k_{n+1}} - b_{k_n}} = 0 \text{ a.s.}$$

3. THE MAIN RESULTS

With the preliminaries accounted for, the main results may now be established. When $\mathscr{X} = \boldsymbol{R}$ and $b_n \equiv n$, a version of Theorem 3.1 was obtained by Gaposhkin [8] using a substantially more complicated argument. Gaposhkin's [8] condition is slightly different from (3.2) specialized to $\mathscr{X} = \boldsymbol{R}$ and $b_n \equiv n$ and is at least as strong. Gaposhkin's [8] result is an extension of an earlier result of Móricz [12] which was apparently the first SLLN for a sequence of blockwise independent random variables.

THEOREM 3.1. *Let $\{V_n, n \geqslant 1\}$ be a sequence of mean 0 random elements in a real separable Rademacher type p $(1 \leqslant p \leqslant 2)$ Banach space $\mathscr{X}$ and let $\{b_n, n \geqslant 1\}$ be a nondecreasing sequence of positive constants such that*

$$\inf_{n\geqslant 0} \frac{b_{2^{n+1}}}{b_{2^n}} > 1 \quad \textit{and} \quad \sup_{n\geqslant 0} \frac{b_{2^{n+1}}}{b_{2^n}} < \infty. \tag{3.1}$$

If $\{V_n, n \geqslant 1\}$ is blockwise independent with respect to the blocks $\{B_k, k \geqslant 1\}$ and if

$$\sum_{i=1}^{\infty} \frac{E\|V_i\|^p}{b_i^p} (\varphi(i))^{p-1} < \infty, \tag{3.2}$$

then

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} V_i}{b_n} = 0 \textit{ a.s.} \tag{3.3}$$

Proof. Set

$$T_k^{(m)} = \max_{j\in B_k^{(m)}} \Big\| \sum_{i=r_k^{(m)}}^{j} V_i \Big\|, \qquad k \in I_m,\ m \geqslant 0,$$

and

$$T_m = \frac{1}{b_{2^{m+1}}} \sum_{k\in I_m} T_k^{(m)}, \qquad m \geqslant 0.$$

Note that, for $m \geqslant 0$,

$$ET_m^p \leqslant \frac{1}{b_{2^{m+1}}^p} c_m^{p-1} \sum_{k\in I_m} E(T_k^{(m)})^p \qquad \text{(by Lemma 2.4)}$$

$$\leqslant \frac{1}{b_{2^{m+1}}^p} c_m^{p-1} C \sum_{k \in I_m} \sum_{i \in B_k^{(m)}} E\|V_i\|^p \quad \text{(by Lemma 2.1)}$$

$$= \frac{1}{b_{2^{m+1}}^p} c_m^{p-1} C \sum_{i=2^m}^{2^{m+1}-1} E\|V_i\|^p \leqslant C \sum_{i=2^m}^{2^{m+1}-1} \frac{E\|V_i\|^p}{b_i^p} (\varphi(i))^{p-1}.$$

It thus follows from (3.2) that $\sum_{m=0}^{\infty} ET_m^p < \infty$, and so by the Markov inequality and the Borel–Cantelli lemma

$$\lim_{m \to \infty} T_m = 0 \text{ a.s.}$$

Now it follows from the first inequality of (3.1) that

$$\frac{\max_{2^m \leqslant k < 2^{m+1}} \|\sum_{i=2^m}^k V_i\|}{b_{2^{m+1}} - b_{2^m}} \leqslant \frac{C \max_{2^m \leqslant k < 2^{m+1}} \|\sum_{i=2^m}^k V_i\|}{b_{2^{m+1}}} \leqslant CT_m \to 0 \text{ a.s.}$$

The conclusion (3.3) follows immediately from Lemma 2.5. ∎

Remark 3.1. (i) The slower $b_n \uparrow \infty$, the stronger is the assumption (3.2), but so is the conclusion (3.3).

(ii) Theorem 3.1 is an analogue of an SLLN of Adler et al. [2] obtained for a sequence of independent random elements in a Rademacher type p $(1 \leqslant p \leqslant 2)$ Banach space. The Adler et al. [2] result, which extends the implication (i) $\Rightarrow$ (ii) in Proposition 2.1 to more general norming constants $0 < b_n \uparrow \infty$, is a random element analogue of a classical result of Kolmogorov (see, e.g., Loève [11], p. 250). It should be pointed out that this SLLN of Adler et al. [2] does indeed follow immediately from Theorem V.7.5 (or Corollary V.7.5) of Woyczyński [19] and the Kronecker lemma.

(iii) When $p = 1$, Theorem 3.1 is not of interest since the mean 0 assumption, the blockwise independence assumption, and (3.1) are not needed. Indeed, for a sequence of random elements $\{V_n, n \geqslant 1\}$ in a real separable Banach space and constants $0 < b_n \uparrow \infty$, Cantrell and Rosalsky [4] recently proved that if

$$\sum_{i=1}^{\infty} E\left(\frac{\|V_i\|}{\|V_i\| + b_i}\right) < \infty$$

(which is a weaker assumption than $\sum_{i=1}^{\infty} E\|V_i\|/b_i < \infty$), then

$$\lim_{n \to \infty} \sum_{i=1}^{n} V_i/b_n = 0 \text{ a.s.}$$

irrespective of the joint distributions of the $\{V_n, n \geqslant 1\}$.

(iv) If (3.1) holds where $0 < b_n \uparrow$ and

$$c_m = o(a^m) \quad \text{for all } a > 1, \tag{3.4}$$

then a sufficient condition for (3.2) to hold is that

$$\sum_{i=1}^{\infty} \frac{E\|V_i\|^p}{b_i^q} < \infty \quad \text{for some } 0 < q < p. \tag{3.5}$$

To see this, it follows from the first inequality of (3.1) that $b_{2^{n+1}}/b_{2^n} \geqslant 1+\delta$ for some $\delta > 0$ and all $n \geqslant 0$. Thus for all $m \geqslant 1$ we have

$$b_{2^m} = b_1 \prod_{j=0}^{m-1} \frac{b_{2^{j+1}}}{b_{2^j}} \geqslant b_1 (1+\delta)^m. \tag{3.6}$$

Hence

$$\sum_{i=1}^{\infty} \frac{E\|V_i\|^p}{b_i^p} (\varphi(i))^{p-1} = \sum_{m=0}^{\infty} \sum_{i=2^m}^{2^{m+1}-1} \frac{E\|V_i\|^p (\varphi(i))^{p-1}}{b_i^q b_i^{p-q}}$$

$$\leqslant \sum_{m=0}^{\infty} \sum_{i=2^m}^{2^{m+1}-1} \frac{E\|V_i\|^p c_m^{p-1}}{b_i^q b_{2^m}^{p-q}}$$

$$\leqslant \sum_{m=0}^{\infty} \sum_{i=2^m}^{2^{m+1}-1} \frac{E\|V_i\|^p c_m^{p-1}}{b_i^q b_1^{p-q} (1+\delta)^{(p-q)m}} \quad \text{(by (3.6))}$$

$$\leqslant C \sum_{m=0}^{\infty} \sum_{i=2^m}^{2^{m+1}-1} \frac{E\|V_i\|^p}{b_i^q} \quad \text{(by (3.4) with } a = (1+\delta)^{(p-q)/(p-1)})$$

$$= C \sum_{i=1}^{\infty} \frac{E\|V_i\|^p}{b_i^q} < \infty \quad \text{(by (3.5))},$$

thereby establishing (3.2).

In the next theorem, we will show that the implication (3.2) $\Rightarrow$ (3.3) holding indeed completely characterizes Rademacher type p Banach spaces.

THEOREM 3.2. *Let $\mathscr{X}$ be a real separable Banach space and let $1 \leqslant p \leqslant 2$. Then the following two statements are equivalent*:

(i) *$\mathscr{X}$ is of Rademacher type p.*

(ii) *For every sequence of mean* 0 *random elements $\{V_n, n \geqslant 1\}$, which is blockwise independent with respect to some sequence of blocks $\{B_k, k \geqslant 1\}$ and every nondecreasing sequence of positive constants $\{b_n, n \geqslant 1\}$ satisfying* (3.1), *the implication* (3.2) $\Rightarrow$ (3.3) *holds.*

Proof. The implication (i) $\Rightarrow$ (ii) is precisely Theorem 3.1. To verify the implication (ii) $\Rightarrow$ (i), assume that (ii) holds. Let $\{V_n, n \geqslant 1\}$ be a sequence of independent mean 0 random elements in $\mathscr{X}$ such that

$$\sum_{n=1}^{\infty} \frac{E\|V_n\|^p}{n^p} < \infty. \tag{3.7}$$

In view of Proposition 2.1, it suffices to verify that

$$\frac{\sum_{i=1}^{n} V_i}{n} \to 0 \text{ a.s.} \tag{3.8}$$

Note that $\{V_n, n \geqslant 1\}$ is blockwise independent with respect to the blocks $\{[2^{k-1}, 2^k), k \geqslant 1\}$. Set $b_n = n$, $n \geqslant 1$. Then (3.1) holds and, recalling (2.2), we see that (3.7) and (3.2) are the same. Thus, by (ii), we see that (3.8) holds. ∎

The next theorem, when specialized to $\{V_n, n \geqslant 1\}$ being p-orthogonal and $b_n = n^{\alpha}$, $n \geqslant 1$, where $\alpha > 0$, reduces to a result of Móricz et al. [13] by taking $B_k = [2^{k-1}, 2^k)$, $k \geqslant 1$, and recalling (2.2).

THEOREM 3.3. *Let $\{V_n, n \geqslant 1\}$ be a sequence of random elements in a real separable Rademacher type p $(1 \leqslant p \leqslant 2)$ Banach space and let $\{b_n, n \geqslant 1\}$ be a nondecreasing sequence of positive constants satisfying* (3.1). *If $\{V_n, n \geqslant 1\}$ is blockwise p-orthogonal with respect to the blocks $\{B_k, k \geqslant 1\}$ and if*

$$\sum_{i=1}^{\infty} \frac{E\|V_i\|^p}{b_i^p} (\operatorname{Log} i)^p (\varphi(i))^{p-1} < \infty, \tag{3.9}$$

then

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} V_i}{b_n} = 0 \text{ a.s.} \tag{3.10}$$

Proof. Define $T_k^{(m)}$, $k \in I_m$, $m \geqslant 0$, and T_m, $m \geqslant 0$, as in the proof of Theorem 3.1. Note that, for $m \geqslant 0$,

$$\begin{aligned}
ET_m^p &\leqslant \frac{1}{b_{2^{m+1}}^p} c_m^{p-1} \sum_{k\in I_m} E(T_k^{(m)})^p \qquad \text{(by Lemma 2.4)}\\
&\leqslant \frac{1}{b_{2^{m+1}}^p} c_m^{p-1} C \sum_{k\in I_m} \left(\operatorname{Log}(2\operatorname{card} B_k^{(m)})\right)^p \sum_{i\in B_k^{(m)}} E\|V_i\|^p \qquad \text{(by (2.9))}\\
&\leqslant \frac{1}{b_{2^{m+1}}^p} c_m^{p-1} C (\operatorname{Log} 2^{m+1})^p \sum_{i=2^m}^{2^{m+1}-1} E\|V_i\|^p\\
&\leqslant \frac{1}{b_{2^{m+1}}^p} c_m^{p-1} C (\operatorname{Log} 2^{m})^p \sum_{i=2^m}^{2^{m+1}-1} E\|V_i\|^p\\
&\leqslant C \sum_{i=2^m}^{2^{m+1}-1} \frac{E\|V_i\|^p}{b_i^p} (\operatorname{Log} i)^p (\varphi(i))^{p-1}.
\end{aligned}$$

It thus follows from (3.9) that $\sum_{m=0}^{\infty} ET_m^p < \infty$. The rest of the argument is exactly the same as that at the end of the proof of Theorem 3.1. ∎

Remark 3.2. We close this section with discussion of the main difference between the structure of the proofs of Theorems 3.1 and 3.3 and that of the earlier results by Móricz [12], Móricz et al. [13], and Gaposhkin [8]. As in Móricz [12] and Gaposhkin [8], let us now take $b_n = n$, $n \geqslant 1$, and $\mathscr{X} = \boldsymbol{R}$ (although Móricz et al. [13] took $b_n = n^\alpha$, $n \geqslant 1$, where $\alpha > 0$, and they assumed that the Banach space $\mathscr{X}$ is of Rademacher type p $(1 \leqslant p \leqslant 2)$). Móricz [12], Móricz et al. [13], and Gaposhkin [8] bounded $\sum_{i=1}^{n} V_i/n$ (where $2^m \leqslant n < 2^{m+1}$) by

$$\left|\frac{\sum_{i=1}^{n} V_i}{n}\right| \leqslant \left|\frac{\sum_{i=1}^{2m-1} V_i}{2^m}\right| + \frac{1}{2^m} \max_{2^m \leqslant k < 2^{m+1}} \Big|\sum_{i=2^m}^{k} V_i\Big|. \tag{3.11}$$

They then argued that each of the two terms on the right-hand side of (3.11) converges to 0 a.s. as $m \to \infty$. Their arguments that the first term on the right-hand side of (3.11) converges to 0 a.s. as $m \to \infty$ did not use any (type of) independence hypothesis; only (a type of) orthogonality was used. However, in our proof of Theorems 3.1 and 3.3, we only need to argue that the second term on the right-hand side of (3.11) converges to 0 a.s. as $m \to \infty$. This is the case because we then apply Lemma 2.5, and so we are in effect bounding $\sum_{i=1}^{n} V_i/n$ (where $2^m \leqslant n < 2^{m+1}$) by

$$\left|\frac{\sum_{i=1}^{n} V_i}{n}\right| \leqslant \sum_{r=0}^{m} \frac{\max_{2^r \leqslant k < 2^{r+1}} \left|\sum_{i=2^r}^{k} V_i\right|}{2^m}$$
$$= \frac{1}{2^m} \sum_{r=0}^{m} 2^r \frac{\max_{2^r \leqslant k < 2^{r+1}} \left|\sum_{i=2^r}^{k} V_i\right|}{2^r},$$

which via the Toeplitz lemma converges to 0 a.s. as $m \to \infty$ (see Proposition 3.1 and Theorem 9.1 of Chobanyan et al. [5]).

4. COROLLARIES AND EXAMPLES

In this section, some particular cases of Theorems 3.1 and 3.3 are presented as corollaries. Some illustrative examples or counterexamples are also provided.

When the underlying Banach space is the real line, $p = 2$, and $b_n \equiv n$, Corollaries 4.1, 4.3, 4.4 (take $\alpha_2 = 1$), and 4.5 reduce to results of Gaposhkin [8]. This special case of Corollary 4.1 is also the SLLN for a sequence of blockwise independent random variables obtained by Móricz [12]. In view of Remark 3.1 (iii), we formulate Corollaries 4.1–4.5 assuming that $p > 1$.

COROLLARY 4.1. *Let $\{V_n, n \geqslant 1\}$ be a sequence of mean* 0 *random elements in a real separable Rademacher type p $(1 < p \leqslant 2)$ Banach space and let $\{b_n, n \geqslant 1\}$ be a nondecreasing sequence of positive constants such that* (3.1) *holds. If $\{V_n, n \geqslant 1\}$ is blockwise independent with respect to the blocks $\{[2^{k-1}, 2^k), k \geqslant 1\}$ (or, more generally, with respect to the blocks $\{[\beta_k, \beta_{k+1}),$*

$k \geqslant 1\}$, where $\beta_k = \lfloor q^{k-1} \rfloor$ for all large k and $q > 1$) and if

$$\sum_{i=1}^{\infty} \frac{E\|V_i\|^p}{b_i^p} < \infty, \tag{4.1}$$

then (3.3) *holds.*

Proof. Recalling (2.2) and (2.3), we infer that the assumption (4.1) ensures that (3.2) holds. The conclusion follows directly from Theorem 3.1. ∎

COROLLARY 4.2. *Let $\{V_n, n \geqslant 1\}$ be a sequence of mean* 0 *random elements in a real separable Rademacher type p $(1 < p \leqslant 2)$ Banach space and let $\{b_n, n \geqslant 1\}$ be a nondecreasing sequence of positive constants such that* (3.1) *holds. If $\{V_n, n \geqslant 1\}$ is blockwise independent with respect to the blocks $\{[\beta_k, \beta_{k+1}), k \geqslant 1\}$, where $\beta_k = \lfloor 2^{k/(\mathrm{Log}\,k)^\alpha} \rfloor$ for all large k and $\alpha > 0$, and if*

$$\sum_{i=2}^{\infty} \frac{E\|V_i\|^p}{b_i^p} (\mathrm{Log}\,\mathrm{Log}\, i)^{\alpha(p-1)} < \infty, \tag{4.2}$$

then (3.3) *holds.*

Proof. Recalling (2.4), we see that the assumption (4.2) ensures that (3.2) holds. The conclusion follows directly from Theorem 3.1. ∎

COROLLARY 4.3. *Let $\{V_n, n \geqslant 1\}$ be a sequence of mean* 0 *random elements in a real separable Rademacher type p $(1 < p \leqslant 2)$ Banach space and let $\{b_n, n \geqslant 1\}$ be a nondecreasing sequence of positive constants such that* (3.1) *holds. If $\{V_n, n \geqslant 1\}$ is blockwise independent with respect to the blocks $\{[\beta_k, \beta_{k+1}), k \geqslant 1\}$, where $\beta_k = \lfloor 2^{k^\alpha} \rfloor$ for all large k and $0 < \alpha < 1$, and if*

$$\sum_{i=1}^{\infty} \frac{E\|V_i\|^p}{b_i^p} (\mathrm{Log}\, i)^{(\alpha^{-1}-1)(p-1)} < \infty, \tag{4.3}$$

then (3.3) *holds.*

Proof. By (2.5) the assumption (4.3) ensures that (3.2) holds. The conclusion follows directly from Theorem 3.1. ∎

COROLLARY 4.4. *Let $\{V_n, n \geqslant 1\}$ be a sequence of mean* 0 *random elements in a real separable Rademacher type p $(1 < p \leqslant 2)$ Banach space and suppose that $\{V_n, n \geqslant 1\}$ is blockwise independent with respect to the blocks $\{[\lfloor k^{\alpha_1} \rfloor, \lfloor (k+1)^{\alpha_1} \rfloor), k \geqslant 1\}$, where $\alpha_1 > 1$. Let $\alpha_2 \geqslant \alpha_1^{-1}$. If*

$$\sum_{i=1}^{\infty} \frac{E\|V_i\|^p}{i^q} < \infty, \tag{4.4}$$

where $q = \big(p(\alpha_1 \alpha_2 - 1) + 1\big)/\alpha_1$, then

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} V_i}{n^{\alpha_2}} = 0 \ \text{a.s.}$$

Proof. Recalling (2.6), we obtain

$$\sum_{i=1}^{\infty} \frac{E\|V_i\|^p}{i^{\alpha_2 p}} (\varphi(i))^{p-1} \leqslant C \sum_{i=1}^{\infty} \frac{E\|V_i\|^p}{i^{\alpha_2 p}} i^{(p-1)/\alpha_1}$$

$$= C \sum_{i=1}^{\infty} \frac{E\|V_i\|^p}{i^q} < \infty \text{ (by (4.4))}$$

and the conclusion follows from Theorem 3.1. ∎

Remark 4.1. Apropos of the constant q in Corollary 4.4, it may be noted that $\alpha_2 \leqslant q < \alpha_2 p$.

Counterparts to Corollaries 4.1–4.4 can easily be given when $\{V_n, n \geqslant 1\}$ is blockwise p-orthogonal with respect to a sequence of blocks. We content ourselves with only presenting the counterpart to Corollary 4.1. Corollary 4.5 is an extension of the classical SLLN resulting from the celebrated Rademacher–Menšov fundamental convergence theorem for sums of orthogonal random variables (see, e.g., Révész [16], pp. 86–87).

Corollary 4.5. *Let $\{V_n, n \geqslant 1\}$ be a sequence of mean* 0 *random elements in a real separable Rademacher type p* $(1 < p \leqslant 2)$ *Banach space and let $\{b_n, n \geqslant 1\}$ be a nondecreasing sequence of positive constants such that* (3.1) *holds. If $\{V_n, n \geqslant 1\}$ is blockwise p-orthogonal with respect to the blocks $\{[2^{k-1}, 2^k), k \geqslant 1\}$ (or, more generally, with respect to the blocks $\{[\beta_k, \beta_{k+1}), k \geqslant 1\}$, where $\beta_k = \lfloor q^{k-1} \rfloor$ for all large k and $q > 1$) and if*

$$\sum_{i=1}^{\infty} \frac{E\|V_i\|^p}{b_i^p} (\operatorname{Log} i)^p < \infty, \tag{4.5}$$

then (3.10) *holds.*

Proof. Recalling (2.2) and (2.3), we infer that the assumption (4.5) ensures that (3.9) holds. The conclusion follows directly from Theorem 3.3. ∎

We close by presenting four examples. The first example illustrates Theorem 3.1 and Corollary 4.1.

Example 4.1. Let $\{W_n, n \geqslant 1\}$ be a sequence of independent mean 0 random elements in a real separable Rademacher type p $(1 \leqslant p \leqslant 2)$ Banach space and suppose that

$$\sum_{i=1}^{\infty} \frac{E\|W_i\|^p}{i^{\alpha p}} < \infty \quad \text{for some } \alpha > 0. \tag{4.6}$$

Let $V_n = W_{n-2^m+1}$, $2^m \leqslant n < 2^{m+1}$, $m \geqslant 0$. Then $EV_n = 0$, $n \geqslant 1$, and $\{V_n, n \geqslant 1\}$ is blockwise independent with respect to the blocks $\{[2^{k-1}, 2^k),$

$k \geqslant 1\}$. Now, using (4.6), we obtain

$$\sum_{i=1}^{\infty} \frac{E\|V_i\|^p}{i^{\alpha p}} = \sum_{m=0}^{\infty} \sum_{i=2^m}^{2^{m+1}-1} \frac{E\|W_{i-2^m+1}\|^p}{i^{\alpha p}}$$

$$= \sum_{m=0}^{\infty} \sum_{i=1}^{2^m} \frac{E\|W_i\|^p}{(i+2^m-1)^{\alpha p}} = \sum_{i=1}^{\infty} \sum_{m=\lceil \mathrm{Log} i \rceil}^{\infty} \frac{E\|W_i\|^p}{(i+2^m-1)^{\alpha p}}$$

$$\leqslant \sum_{i=1}^{\infty} \sum_{m=\lceil \mathrm{Log} i \rceil}^{\infty} \frac{E\|W_i\|^p}{2^{\alpha pm}} \leqslant C \sum_{i=1}^{\infty} \frac{E\|W_i\|^p}{2^{\alpha p \mathrm{Log} i}} = C \sum_{i=1}^{\infty} \frac{E\|W_i\|^p}{i^{\alpha p}} < \infty.$$

Thus, by (2.2) and Theorem 3.1 (or by Corollary 4.1),

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} V_i}{n^\alpha} = 0 \text{ a.s.}$$

Finally, we note that, by Theorem 1 of Adler et al. [2], we also have

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} W_i}{n^\alpha} = 0 \text{ a.s.}$$

The second example shows that Theorem 3.1 can fail if the series in (3.2) diverges. More specifically, Example 4.2 shows that we cannot replace (3.2) by the weaker condition $E\|V_n\|^p(\varphi(n))^{p-1}/b_n^p = o(1)$.

Example 4.2. Let $\{W_n, n \geqslant 1\}$ be a sequence of independent mean 0 random elements in a real separable Rademacher type p ($1 \leqslant p \leqslant 2$) Banach space where

$$P\{\|W_n\| = 2n-1\} = \frac{1}{1+\mathrm{Log}\, n}, \quad P\{W_n = 0\} = 1 - \frac{1}{1+\mathrm{Log}\, n}, \quad n \geqslant 1.$$

Let $V_n = W_{n-2^m+1}$, $2^m \leqslant n < 2^{m+1}$, $m \geqslant 0$, and let $b_n = n$, $n \geqslant 1$. Then $EV_n = 0$, $n \geqslant 1$, and $\{V_n, n \geqslant 1\}$ is blockwise independent with respect to the blocks $\{[2^{k-1}, 2^k), k \geqslant 1\}$. Now $V_{2^{m+1}-1} = W_{2^m}$, $m \geqslant 0$, and so $\{V_{2^{m+1}-1}, m \geqslant 0\}$ is a sequence of independent random elements with

$$P\{\|V_{2^{m+1}-1}\| = 2^{m+1}-1\} = \frac{1}{m+1}, \quad P\{V_{2^{m+1}-1} = 0\} = \frac{m}{m+1}, \quad m \geqslant 0.$$

Then, recalling (2.2), we obtain

$$\sum_{i=1}^{\infty} \frac{E\|V_i\|^p}{b_i^p}(\varphi(i))^{p-1} \geqslant \sum_{m=0}^{\infty} \frac{E\|V_{2^{m+1}-1}\|^p}{(2^{m+1}-1)^p}$$

$$= \sum_{m=0}^{\infty} \frac{(2^{m+1}-1)^p}{(m+1)(2^{m+1}-1)^p} = \sum_{m=0}^{\infty} \frac{1}{m+1} = \infty,$$

and so (3.2) fails. Moreover, since

$$\sum_{m=0}^{\infty} P\{\|V_{2^{m+1}-1}\| = 2^{m+1}-1\} = \sum_{m=0}^{\infty} \frac{1}{m+1} = \infty,$$

by the Borel–Cantelli lemma we have

$$P\{\|V_{2^{m+1}-1}\| = 2^{m+1}-1 \text{ i.o. } (m)\} = 1.$$

Thus,

$$\begin{aligned} 1 = \limsup_{m\to\infty} \frac{\|V_{2^{m+1}-1}\|}{2^{m+1}-1} &\leqslant \limsup_{n\to\infty} \frac{\|V_n\|}{n} \\ &\leqslant \limsup_{n\to\infty} \frac{\|\sum_{i=1}^{n} V_i\|}{n} + \limsup_{n\to\infty} \frac{\|\sum_{i=1}^{n-1} V_i\|}{n-1} \quad \text{a.s.} \end{aligned}$$

implying

$$\limsup_{n\to\infty} \frac{\|\sum_{i=1}^{n} V_i\|}{b_n} = \limsup_{n\to\infty} \frac{\|\sum_{i=1}^{n} V_i\|}{n} \geqslant \frac{1}{2} \quad \text{a.s.}$$

Thus (3.3) fails. We also note that for all large n, writing $2^m \leqslant n < 2^{m+1}$ and again recalling (2.2), we get

$$\begin{aligned} \frac{E\|V_n\|^p}{b_n^p}(\varphi(n))^{p-1} &= \frac{E\|W_{n-2^m+1}\|^p}{n^p} \\ &= \frac{(2(n-2^m+1)-1)^p}{n^p(1+\operatorname{Log}(n-2^m+1))} \\ &\leqslant \frac{(2(2^{m+1}-1-2^m+1)-1)^p}{2^{mp}(1+\operatorname{Log}(2^{m+1}-1-2^m+1))} \\ &\leqslant \frac{2^p}{m+1} \to 0 \quad \text{as } n\to\infty. \end{aligned}$$

The third example shows that the Rademacher type p hypothesis in Theorem 3.1 cannot in general be dispensed with. Example 4.3 concerns the real separable Banach space l_1 of absolutely summable real sequences $v = \{v_j, j \geqslant 1\}$ with norm $\|v\| = \sum_{j=1}^{\infty} |v_j|$. The element of l_1 having 1 in its nth position and 0 elsewhere will be denoted by $v^{(n)}$, $n \geqslant 1$.

EXAMPLE 4.3. Consider the real separable Banach space l_1. It is well known (see, e.g., Adler et al. [1]) that l_1 is not of Rademacher type p for any $1 < p \leqslant 2$. Define a sequence of random elements $\{W_n, n \geqslant 1\}$ in l_1 by requiring the $\{W_n, n \geqslant 1\}$ to be independent with

$$P\{W_n = v^{(n)}\} = P\{W_n = -v^{(n)}\} = \tfrac{1}{2}, \quad n \geqslant 1.$$

Let $V_n = W_{n-2^m+1}$, $2^m \leqslant n < 2^{m+1}$, $m \geqslant 0$. Let $1 < p \leqslant 2$ and let $b_n = n^\alpha$, $n \geqslant 1$, where $p^{-1} < \alpha \leqslant 1$. Then $EV_n = 0$, $n \geqslant 1$, and $\{V_n, n \geqslant 1\}$ is blockwise independent with respect to the blocks $\{[2^{k-1}, 2^k), k \geqslant 1\}$. Now, by (2.2),

$$\sum_{i=1}^{\infty} \frac{E\|V_i\|^p}{b_i^p} (\varphi(i))^{p-1} = \sum_{i=1}^{\infty} \frac{1}{i^{\alpha p}} < \infty$$

since $\alpha p > 1$, and so (3.2) holds. Next, for $M \in N$,

$$\sum_{i=1}^{2^{M+1}-1} V_i = \sum_{m=0}^{M} \sum_{i=2^m}^{2^{m+1}-1} V_i = \sum_{m=0}^{M} \sum_{i=2^m}^{2^{m+1}-1} W_{i-2^m+1}$$

$$= \sum_{m=0}^{M} \sum_{i=1}^{2^m} W_i = \sum_{i=1}^{2^M} \sum_{m=\lceil \operatorname{Log} i \rceil}^{M} W_i = \sum_{i=1}^{2^M} (M - \lceil \operatorname{Log} i \rceil + 1) W_i,$$

and so

$$\frac{\|\sum_{i=1}^{2^{M+1}-1} V_i\|}{b_{2^{M+1}-1}} = \frac{\|\sum_{i=1}^{2^M} (M - \lceil \operatorname{Log} i \rceil + 1) W_i\|}{(2^{M+1}-1)^\alpha}$$

$$= \frac{\sum_{i=1}^{2^M} (M - \lceil \operatorname{Log} i \rceil + 1)}{(2^{M+1}-1)^\alpha} \geqslant \frac{2^M}{2^{M+1}} = \frac{1}{2} \text{ a.s.}$$

Consequently, (3.3) fails.

The final example, which is a modification of an example of Móricz [12], shows apropos of Corollary 4.1 that under its hypotheses the series $\sum_{i=1}^{n} V_i/b_i$ can diverge a.s. Consequently, the conclusion of Corollary 4.1 (or Theorem 3.1) cannot in general be reached through the well-known Kronecker lemma approach for proving SLLNs.

EXAMPLE 4.4. Let the underlying Banach space be the real line and let $p = 2$. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent mean 0 random variables such that $P\{X_1 \neq 0\} = 1$ and $\sum_{n=1}^{\infty} EX_n^2 < \infty$. Define for $n \geqslant 1$

$$V_n = \begin{cases} nX_1/(1+\operatorname{Log} n), & \operatorname{Log} n \in N, \\ nX_n, & \operatorname{Log} n \notin N, \end{cases}$$

and let $b_n = n$, $n \geqslant 1$. Then $EV_n = 0$, $n \geqslant 1$, and $\{V_n, n \geqslant 1\}$ is blockwise independent with respect to the blocks $\{[2^{k-1}, 2^k), k \geqslant 1\}$. Now

$$\sum_{n=1}^{\infty} \frac{EV_n^2}{b_n^2} = EX_1^2 \sum_{m=0}^{\infty} \frac{1}{(1+m)^2} + \sum_{\substack{n=3 \\ \operatorname{Log} n \notin N}}^{\infty} EX_n^2 < \infty,$$

and so, by Corollary 4.1, (3.3) holds. However,

$$\sum_{\substack{i=1 \\ \operatorname{Log} i \in N}}^{n} \frac{X_1}{1+\operatorname{Log} i} = X_1 \sum_{m=0}^{\lfloor \operatorname{Log} n \rfloor} \frac{1}{1+m} \text{ diverges a.s. as } n \to \infty,$$

and by the Khintchine–Kolmogorov convergence theorem (see, e.g., Chow and Teicher [6], p. 113)

$$\sum_{\substack{i=3 \\ \mathrm{Log}\, i \notin N}}^{n} X_i \text{ converges a.s. as } n \to \infty .$$

Consequently, for $n \geqslant 3$

$$\sum_{i=1}^{n} \frac{V_i}{b_i} = \sum_{\substack{i=1 \\ \mathrm{Log}\, i \in N}}^{n} \frac{X_1}{1+\mathrm{Log}\, i} + \sum_{\substack{i=3 \\ \mathrm{Log}\, i \notin N}}^{n} X_i \text{ diverges a.s. as } n \to \infty .$$

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