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SOME REMARKS ON THE CENTRAL LIMIT THEOREM FOR FUNCTIONALS OF LINEAR PROCESSES UNDER SHORT-RANGE DEPENDENCE

BY

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Abstract. In this paper we consider the central limit theorems for functionals $G: R^m \to R$ of one-sided m-dimensional linear processes $X_t = \sum_{r=0}^{\infty} A_r Z_{t-r}$, where A_r is a nonrandom matrix $m \times m$ and Z_t's are i.i.d. random vectors in R^m.

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1. INTRODUCTION

The one-sided linear process (LP) is defined as follows:

$$X_j = \sum_{r=0}^{\infty} a_r Z_{j-r}, \quad j = 1, 2, \ldots, n,$$

where the innovations Z_n's are i.i.d. random variables and a_i are constant coefficients such that $\sum_{r=0}^{\infty} a_r^2 < \infty$.

Linear processes have a wide range of applications in time series analysis. A large class of time series processes can be modelled in such a way, including a subset of the fractional ARIMA processes (see Brockwell and Davis (1987)). Additionally, we say that (X_n) is a *short-range dependent process* if the following condition is satisfied:

$$\sum_{j=1}^{\infty} |\mathbf{Cov}(X_1, X_{1+j})| < \infty.$$

It is obvious that if $\sum_{j=0}^{\infty} |\sum_{r=0}^{\infty} a_r a_{r+j}| < \infty$ and $E(Z_1^2) < \infty$, then the linear process (X_n) is short-range dependent.

Let G be a measurable function such that $E|G(X_1)| < \infty$. In statistical inference for time series, it is an important problem to investigate the asymptotic behavior of the partial sums $S_n = \sum_{i=1}^{n} (G(X_i) - EG(X_i))$, since many statistics are of this form. This problem has been considered by many authors. For example, Davydov (1970) considers the special case $G(x) = x$, which is also discussed in Giraitis and Surgailis (1986). This problem has been considered recently by Ho and Hsing (1996), (1997), Hsing (2000), Koul and Surgailis (2001), Wu (2002).

The central limit theorem for multiple linear processes with heavy tailed innovations has been proved by Wu (2003). A very simple question is whether one can prove a central limit theorem (CLT) under a short-range dependence without additional assumptions on the (a_r) and (Z_n) and for the most general class of functions G.

The generalization of the one-sided linear process is the multidimensional linear process (MLP), defined as follows:

$$X_j = \sum_{r=0}^{\infty} A_r Z_{j-r},$$

where the innovations $(Z_n) = (Z_n^{(1)}, \ldots, Z_n^{(m)})$ are i.i.d. random vectors in $\boldsymbol{R}^m$, with mean zero and

$$E|Z_1|^2 = 1. \tag{1.1}$$

The (A_r) are the nonrandom matrices, where $\sum_{r=0}^{\infty} \|A_r\|^2 < \infty$, $A_0 = I$, and the matrix norm $\|\cdot\|$ is such that, for every $z \in \boldsymbol{R}^m$,

$$|A_r z| \leqslant \|A_r\| \, |z|, \tag{1.2}$$

where $|\cdot|$ is the usual Euclidean norm. We will also consider the following assumptions:

$$\sum_{r=0}^{\infty} \|A_r\| < \infty, \tag{a_0}$$

$$\sum_{r=j}^{\infty} \|A_r\|^2 = \mathcal{O}(j^{-t}) \quad \text{for some } t > 1, \tag{a_2}$$

(b_1) the density of the vector Z_1 is the Lipschitz function,

$$E|Z_1|^t < \infty \quad \text{for some } t \geqslant 2. \tag{$b_2(t)$}$$

In Section 4 we prove the CLT for the sums of functionals of linear processes. Let $\mathscr{D}[0, 1]$ be the space of all real functions defined on $[0, 1]$ that are right-continuous and have left-side limits with Skorokhod topology (see Billingsley (1968)). We will show that, under some simple conditions,

$$n^{-1/2} S_{[nt]} \Rightarrow \sigma \{B(t), 0 \leqslant t \leqslant 1\} \quad \text{in } \mathscr{D}[0, 1] \text{ for some } \sigma.$$

The basic idea of this proof is the martingale decomposition of the process

$$Y_n = G(X_n) - EG(X_n).$$

This decomposition is of the following form (see also Ho and Hsing (1996) and (1997)):

$$(1.3) \qquad Y_n = \sum_{s=0}^{\infty} U_{n,s}, \quad \text{where } U_{n,s} = E\{Y_n \mid \mathscr{F}_{n-s}\} - E\{Y_n \mid \mathscr{F}_{n-s-1}\},$$

where $\mathscr{F}_n := \sigma(\ldots, Z_{n-1}, Z_n)$ is the σ-field generated by the innovations in the "past" $\leqslant n$, (X_n) is an m-dimensional process (MLP), and $G: \boldsymbol{R}^m \to \boldsymbol{R}$ is a real function such that $E|Y_1|^Q < \infty$ for some $Q \geqslant 2$.

In the proof of the main theorem we use the idea introduced by Koul and Surgailis (1997).

Theorem 2.2 is a simple corollary to the theorem of Maxwell and Woodroofe (see Wu (2002), Corollary 1). In this context see also Ho and Hsing (1997). They proved that if $\sum_{r=0}^{\infty} |a_r| < \infty$, $E(Z_1^4) < \infty$, and (a_0) holds, then under some regularity conditions on G (for example, if the derivative of G is bounded and continuous, or G is any polynomial)

$$S_n/\sqrt{n} \Rightarrow N(0, \sigma^2) \quad \text{for some } \sigma < \infty$$

(see Theorem 4.1 in Ho and Hsing (1997)).

On the other hand, under the assumptions that

$$G_n(x) = EG\big(x + \sum_{i=0}^{n} a_i Z_{n-i}\big)$$

is Lipschitz for all sufficiently large n, $a_n = n^{-\gamma} L(n)$ for some $\gamma > 1$, where $L(\cdot)$ is a slowly varying function, Wu (2002) proved the CLT.

2. MAIN RESULTS

The main results are the central limit theorems for linear process. Let

$$G_s(x) := EG\big(\sum_{r=0}^{s-1} A_r Z_{k-r} + x\big) = \int G(z+x)\, dF_s(z),$$

where F_s is the distribution function of $\sum_{r=0}^{s-1} A_r Z_{k-r}$ and $EG^2(X_1) < \infty$. We consider the following conditions:

(Lip) $$|G_s(x) - G_s(y)| \leqslant \mathrm{Lip}(G_s)|x-y|$$

for each $s = 1, 2, \ldots$ and all $x, y \in \boldsymbol{R}^m$, where

$$\sup_{s \geqslant 0} \mathrm{Lip}(G_s) < C \quad \text{for some constant } C,$$

(Lip*) $$|G_s(x) - G_s(y)| \leqslant \mathrm{Lip}(G_s)|x-y|$$

for sufficiently large s, where $\mathrm{Lip}(G_s) < C$ for some constant C for sufficiently large s.

THEOREM 2.1. *Assume that* (a_0) *holds,* $(b_2(t))$ *is fulfilled with* $t = 2$, *and* **(Lip)** *is satisfied. Then*

$$(2.1)\qquad \left\{\frac{1}{\sqrt{n}}\sum_{j=1}^{[nt]} Y_j,\ 0 \leqslant t \leqslant 1\right\} \Rightarrow \sigma\{\boldsymbol{B}(t),\ 0 \leqslant t \leqslant 1\} \quad \text{in } \mathscr{D}[0, 1],$$

where $\sigma^2 := \boldsymbol{E}(Y_1)^2 + 2\sum_{j=1}^{\infty} \boldsymbol{E}(Y_1 Y_{1+j})$ *and* $\boldsymbol{B}$ *is a standard Brownian motion on* $[0, 1]$.

THEOREM 2.2. *Assume that* (a_2) *holds,* $(b_2(t))$ *is fulfilled with* $t > 2$, *and* **(Lip*)** *is satisfied. Then* (2.1) *holds with* $\sigma^2 := \lim_{n\to\infty} n^{-1}\boldsymbol{E}(S_n^2)$.

Remark 2.1. In comparison with the assumptions of Theorem 4.1 of Ho and Hsing (1997) our assumptions do not require the existence of the fourth moments of innovations and the condition $C(t, \tau, \lambda)$ (*ibidem*, p. 1639). It is easily seen that every Lipschitz function G satisfies the assumption **(Lip)**, but not every Lipschitz function satisfies the assumptions of Theorem 4.1. Similarly, not every integrable function fulfills the assumptions of the Theorem of Ho and Hsing (1997), but every integrable function satisfies the condition **(Lip)** (see Proposition 4.1).

3. AUXILIARY LEMMAS

We need the following lemmas for the proof of Theorem 2.1. The first one follows immediately from general theorems about weak convergence. The second lemma is the CLT for the martingale differences by Billingsley.

Assume that $W_n, V_{u,n}, \ldots$ are stochastic processes for $n = 1, 2, \ldots$

LEMMA 3.1 (Theorem 4.2 in Billingsley (1968)). *Suppose that the following assumptions hold for every* $u \in N$:

(i) $V_{u,n} \Rightarrow \sigma_u \boldsymbol{B}$ *as* $n \to \infty$, *where* $\boldsymbol{B}$ *is a standard Brownian motion on* $[0, 1]$;

(ii) $\sigma^2 = \lim_{u\to\infty} \sigma_u^2 > 0$;

(iii) *we have*

$$\liminf_{u\to\infty} \limsup_{n\to\infty} \boldsymbol{P}(|V_{u,n} - W_n| \geqslant \varepsilon) = 0 \quad \text{for any } \varepsilon > 0.$$

Then

$$W_n \Rightarrow \sigma\boldsymbol{B} \quad \text{as } n \to \infty.$$

LEMMA 3.2 (CLT for the martingale differences, Theorem 23.1 in Billingsley (1968)). *Suppose that the martingale difference sequence* $(M_n, \mathscr{F}_n)_n$ *is stationary, ergodic, and centered, with* $\mathbf{Var}(M_1) < \infty$. *Then*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]} M_i \Rightarrow \sigma\boldsymbol{B} \quad \text{as } n \to \infty \text{ in } \mathscr{D}[0, 1],$$

where $\sigma^2 = \mathbf{Var}(M_1)$ *for each* $t \in [0, 1]$.

Additionally, we will use the following lemmas. Let $\|\cdot\|_2$ denote the norm in $\mathscr{L}^2$, i.e., $\|X\|_2 = [\boldsymbol{E}|X|^2]^{1/2}$ and $U_{j,s}$ be such as in (1.3).

LEMMA 3.3 (Lemma 6.4 in Ho and Hsing (1996)). $\mathrm{Cov}\{U_{j,s}, U_{i,k}\} = 0$ *for any* $i, j \in \boldsymbol{Z}$, $s, k \in \boldsymbol{N}$ *and* $j-s \neq i-k$.

LEMMA 3.4. *The following statements are true*:
If $\|G(X_1)\|_2 < \infty$, *then*

$$\|U_{k,0}\|_2 \leqslant \|G(X_1)\|_2 \tag{3.1}$$

for any $k \in \boldsymbol{N}$.
If (**Lip**) *holds, then*

$$\|U_{k,s}\|_2 \leqslant C\|A_s\| \tag{3.2}$$

for any $k \in \boldsymbol{N}$ *and for* $s = 1, 2, \ldots,$ *where* C *is a constant.*

LEMMA 3.5. *Under the assumptions of Theorem* 2.1

$$\mathbf{Var}\Big(\sum_{s=0}^{u-1} U_{1+s,s}\Big) \to \boldsymbol{E}(Y_1^2) + 2\sum_{j=1}^{\infty} \boldsymbol{E}(Y_1 Y_{1+j}) \quad \text{as } u \to \infty.$$

Additionally, $\sum_{j=1}^{\infty} |\boldsymbol{E}(Y_1 Y_{1+j})| < \infty$.

The proofs of Lemmas 3.4 and 3.5 are contained in the Appendix.

4. PROOF OF THE MAIN RESULTS

In this section we prove Theorems 2.1 and 2.2. Let us start with the proof of Theorem 2.1.

Proof of Theorem 2.1. Let

$$W_n := \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} Y_j, \qquad V_{u,n} := \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} \sum_{s=0}^{u-1} U_{j,s},$$

$$H_{u,n} := \frac{1}{\sqrt{n}} \sum_{s=0}^{u-1} \Big(\sum_{j=1}^{s} U_{j,s} - \sum_{j=[nt]+1}^{[nt]+s} U_{j,s}\Big).$$

We will show that all the conditions (i)–(iii) of Lemma 3.1 are satisfied. In order to prove (i), we apply the Koul and Surgailis (1997) decomposition. We have

$$V_{u,n} := \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} \sum_{s=0}^{u-1} U_{j+s,s} + H_{u,n}. \tag{4.1}$$

Since, for some constant C, $\|U_{j,s}\|_2 \leqslant C$ (see Lemma 3.4) for $s = 0, 1, \ldots$ and any $j \in \boldsymbol{N}$, we have $\max_{j,s} \boldsymbol{E}U_{j,s}^2 \leqslant C^2$, and therefore

$$H_{u,n} = \mathcal{O}_{\boldsymbol{P}}(n^{-1/2}) \quad \text{for every } u \in \boldsymbol{N}.$$

Hence

(4.2) $$H_{u,n} \xrightarrow{P} 0 \text{ (in probability)} \quad \text{for every } u \in N.$$

Put

$$M_j(u) := \sum_{s=0}^{u-1} U_{j+s,s}.$$

Since $(M_j(u), \mathscr{F}_j)_j$ is a stationary, ergodic, centered sequence of the martingale differences, by Lemma 3.2 we obtain

(4.3) $$\frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} \sum_{s=0}^{u-1} U_{j+s,s} = \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} M_j(u) \Rightarrow \sigma_u \boldsymbol{B} \quad \text{as } n \to \infty,$$

where $\sigma_u^2 = \mathbf{Var}(M_1(u))$.

By (4.2), (4.3) and the decomposition (4.1), we get

$$V_{u,n} \Rightarrow \sigma_u \boldsymbol{B} \quad \text{as } n \to \infty.$$

Thus, the condition (i) of Lemma 3.1 is satisified.

The proof of condition (ii) follows immediately from Lemma 3.5.

Thus, it remains to prove the condition (iii). Notice that, by Chebyshev's inequality,

$$\boldsymbol{P}(|V_{u,n} - W_n| \geqslant \varepsilon) \leqslant \varepsilon^{-2} \boldsymbol{E}(V_{u,n} - W_n)^2 = \varepsilon^{-2} n^{-1} \boldsymbol{E}\Big(\sum_{j=1}^{n} \sum_{s=u}^{\infty} U_{j,s}\Big)^2.$$

From Lemma 3.3 and the Schwarz inequality we have

$$\begin{aligned} \boldsymbol{E}\Big(\sum_{j=1}^{n} \sum_{s=u}^{\infty} U_{j,s}\Big)^2 &= \sum_{i,j=1}^{n} \sum_{s,p=u}^{\infty} \boldsymbol{E}(U_{j,s} U_{i,p}) \\ &= \sum_{s,p=u}^{\infty} \sum_{\substack{i,j=1 \\ i-j=p-s}}^{n} \boldsymbol{E}(U_{j,s} U_{i,p}) \leqslant \sum_{s,p=u}^{\infty} \sum_{j=1}^{n} |\boldsymbol{E}(U_{j,s} U_{p-s+j,p})| \\ &\leqslant n \max_{1 \leqslant j \leqslant n} \sum_{s,p=u}^{\infty} |\boldsymbol{E}(U_{j,s} U_{p-s+j,p})| \leqslant n \max_{1 \leqslant j \leqslant n} \sum_{s,p=u}^{\infty} \|U_{j,s}\|_2 \|U_{p-s+j,p}\|_2. \end{aligned}$$

Hence, by Lemma 3.4, we infer that for some constant C

$$\boldsymbol{E}\Big(\sum_{j=1}^{n} \sum_{s=u}^{\infty} U_{j,s}\Big)^2 \leqslant nC \sum_{s,p=u}^{\infty} \|A_s\| \|A_p\| \leqslant Cn\Big(\sum_{s=u}^{\infty} \|A_s\|\Big)^2.$$

Finally, we have

$$\frac{1}{n} \boldsymbol{E}\Big(\sum_{j=1}^{n} \sum_{s=u}^{\infty} U_{j,s}\Big)^2 \leqslant C\Big(\sum_{s=u}^{n} \|A_s\|\Big)^2.$$

Hence, applying (a_0), we see that the condition (iii) of Lemma 3.1 holds. This completes the proof of Theorem 2.1. ∎

With reference to Theorem 2.1 the following property is worth proving:

PROPOSITION 4.1. *Let* Lip(G) *denote the Lipschitz constant of the function* G. *The following statements are true:*

(i) *If* G *is Lipschitz and* (b_1) *holds, then* (**Lip**) *is satisfied with constant* Lip(G_s) *bounded as follows*:

$$\mathrm{Lip}(G_s) \leqslant \mathrm{Lip}(G).$$

(ii) *If* G *is integrable, i.e.* $G \in L^1(R^m)$, *and* (b_1) *holds, then* (**Lip**) *is satisfied with constant* Lip(G_s) *bounded as follows*:

$$\mathrm{Lip}(G_s) \leqslant \mathrm{Lip}(f_1) \int |G(s)|\, ds,$$

where f_1 *is the density of* Z_1.

(iii) *If* $m = 1$ *and* G *has a bounded total variation, then* (**Lip**) *is satisfied with constant* Lip(G_s) *bounded as follows*:

$$\mathrm{Lip}(G_s) \leqslant \|G\|_{tv}\, \mathrm{Lip}(F_s) \leqslant C\, \|G\|_{tv},$$

where $\|G\|_{tv}$ *denotes the total variation of* G *on the real line and the following condition holds*:

(b_0) *there exists some constant* C *such that* $\sup_{s\in N} \mathrm{Lip}(F_s) \leqslant C$, *where* F_s *is the distribution function of* $\sum_{r=0}^{s-1} a_r Z_{1-r}$.

Proof. The condition (i) is clear. In order to prove (ii), let us notice that

$$|G_s(x) - G_s(y)| = \left|\int G(z+x)\, f_s(z)\, dz - \int G(z+y)\, f_s(z)\, dz\right|,$$

where f_s denotes the density of random vector $\sum_{r=0}^{s-1} A_r Z_{k-r}$. Since $f_s = g_s * f_1$, where $g_s * f_1$ denotes the convolution of the densities g_s and f_1, and g_s is the density of $\sum_{r=1}^{s-1} A_r Z_{k-r}$, f_1 is also the density of Z_1, so by (b_1) we obtain

$$\begin{aligned}
&\left|\int G(z+x)\, f_s(z)\, dz - \int G(z+y)\, f_s(z)\, dz\right| \\
&\quad = \left|\int G(z)\big(f_s(z-x) - f_s(z-y)\big)\, dz\right| \\
&\quad = \left|\int G(z)\big(g_s * f_1(z-x) - g_s * f_1(z-y)\big)\, dz\right| \leqslant |x-y|\, \mathrm{Lip}(f_1) \int |G(z)|\, dz.
\end{aligned}$$

In order to prove (iii), let us observe that from the elementary properties of the functions F_s and G, we have

$$G(z)\, F_s(z-x)\big|_{z=-\infty}^{z=\infty} - G(z)\, F_s(z-y)\big|_{z=-\infty}^{z=\infty} = 0 \qquad \text{for each } x,\ y.$$

Consequently, by the formula for the integration by parts for Stieltjes integrals, we obtain

$$\begin{aligned}
G_s(x) - G_s(y) &= \int G(z)\, dF_s(z-x) - \int G(z)\, dF_s(z-y) \\
&= \int \big(F_s(z-x) - F_s(z-y)\big)\, dG(z),
\end{aligned}$$

where F_s is the distribution function of $\sum_{r=0}^{s-1} a_r Z_{1-r}$.

Therefore, (b_0) yields $\sup_{s\in N}\operatorname{Lip}(F_s) \leqslant C$ and

$$|G_s(x)-G_s(y)| \leqslant \operatorname{Lip}(F_s)|x-y|\,\|G\|_{tv} \leqslant C\,\|G\|_{tv}|x-y|. \ \blacksquare$$

Now we prove our second main result, i.e. Theorem 2.2.

Proof of Theorem 2.2. Since the sequence $\mathscr{X}_i = (\ldots, Z_{i-1}, Z_i)$ is a stationary ergodic Markov chain, by the theorem of Maxwell and Woodroofe (Theorem 1 in Wu (2002)) it is sufficient to prove that

$$\|(V_n\tilde{G})(\mathscr{X}_0)\|_2 = \mathcal{O}(n^{\kappa}) \quad \text{for some } \kappa < \tfrac{1}{2}, \tag{4.4}$$

where $(V_n\tilde{G})(\mathscr{X}_0) = \sum_{i=1}^{n} E\{\tilde{G}(X_i)\,|\,\mathscr{F}_0\}$ and $\tilde{G}(X_i) = G(X_i)-EG(X_i)$.

Obviously,

$$E\{\tilde{G}(X_i)\,|\,\mathscr{F}_0\} = G_i\big(\sum_{r=i}^{\infty} A_r Z_{i-r}\big)-E\Big(G_i\big(\sum_{r=i}^{\infty} A_r Z_{i-r}\big)\Big),$$

because $EG(X_i) = E\big(E\{G(X_i)\,|\,\mathscr{F}_0\}\big) = E\big(G_i(\sum_{r=i}^{\infty} A_r Z_{i-r})\big)$.

From the triangle inequality we get

$$\begin{aligned} \Big\|\sum_{i=1}^{n} E\{\tilde{G}(X_i)\,|\,\mathscr{F}_0\}\Big\|_2 &\leqslant \sum_{i=1}^{n} \|E\{\tilde{G}(X_i)\,|\,\mathscr{F}_0\}\|_2 \\ &\leqslant \sum_{i=1}^{n} \Big\|G_i\big(\sum_{r=i}^{\infty} A_r Z_{i-r}\big)-G_i(0)\Big\|_2 \\ &\quad + \sum_{i=1}^{n} \Big\|G_i(0)-E\Big(G_i\big(\sum_{r=i}^{\infty} A_r Z_{i-r}\big)\Big)\Big\|_2. \end{aligned} \tag{4.5}$$

Notice that $G_i(0) = E\big(G(\sum_{r=0}^{i-1} A_r Z_{i-r})\big)$, and

$$\Big\|G_i(0)-E\Big(G_i\big(\sum_{r=i}^{\infty} A_r Z_{i-r}\big)\Big)\Big\|_2 = \Big\|E(G_i(0))-E\Big(G_i\big(\sum_{r=i}^{\infty} A_r Z_{i-r}\big)\Big)\Big\|_2.$$

Then from (4.5), using **(Lip*)**, we obtain

$$\Big\|\sum_{i=1}^{n} E\{\tilde{G}(X_i)\,|\,\mathscr{F}_0\}\Big\|_2 \leqslant 2\sup_i \operatorname{Lip}(G_i) \sum_{i=1}^{n} \Big\|\sum_{r=i}^{\infty} A_r Z_{i-r}\Big\|_2.$$

By (a_2) and (1.1), we have

$$\Big\|\sum_{r=i}^{\infty} A_r Z_{i-r}\Big\|_2 \leqslant \big(\sum_{r=i}^{\infty} \|A_r\|^2\big)^{1/2} = \mathcal{O}(i^{-t/2}) \quad \text{for some } t > 1,$$

which is the desired result (4.4). ■

5. APPENDIX

Proof of Lemma 3.4. The bound (3.1) follows immediately from the obvious fact:

$$\|G(X_k)-E\{G(X_k)\,|\,\mathscr{F}_{k-1}\}\|_2 \leqslant \|G(X_k)\|_2.$$

Thus, it remains to prove (3.2). By the definition of G_s, we obtain

$$U_{k,s} = E\{G(X_k) \mid \mathscr{F}_{k-s}\} - E\{G(X_k) \mid \mathscr{F}_{k-s-1}\} = G_s(R_{k,s}) - G_{s+1}(R_{k,s+1}),$$

where $R_{k,s} = X_k - \sum_{r=0}^{s-1} A_r Z_{k-r}$. By independence of $A_s Z_{k-s}$ and $R_{k,s+1}$, we have $G_{s+1}(x) = \int G_s(x+z)\,dF_{1,s}(z)$, where $F_{1,s}$ is the distribution function of $A_s Z_{k-s}$. Hence

$$G_{s+1}(R_{k,s+1}) = \int G_s(R_{k,s+1}+z)\,dF_{1,s}(z).$$

Consequently, applying the condition **(Lip)**, we get

$$\begin{aligned} |U_{k,s}| &\leqslant \int |G_s(R_{k,s}) - G_s(R_{k,s+1}+z)|\,dF_{1,s}(z) \\ &\leqslant \operatorname{Lip}(G_s) \int |R_{k,s} - R_{k,s+1} - z|\,dF_{1,s}(z) \\ &\leqslant \operatorname{Lip}(G_s)\big(|R_{k,s} - R_{k,s+1}| + \int |z|\,dF_{1,s}(z)\big) \leqslant \operatorname{Lip}(G_s)(|A_s Z_{k-s}| + E|A_s Z_{k-s}|). \end{aligned}$$

From (1.1) and (1.2) we obtain $E|A_s Z_{k-s}| \leqslant \|A_s\|$, and hence

$$(5.1) \qquad |U_{k,s}| \leqslant \operatorname{Lip}(G_s)\|A_s\|(1+|Z_{k-s}|)$$

and

$$\|U_{k,s}\|_2 \leqslant C\|A_s\| \quad \text{for some constant } C.$$

This completes the proof of Lemma 3.4. ■

Proof of Lemma 3.5. Let us notice that

$$\sigma_u^2 = \mathbf{Var}(M_1(u)) = E\Big(\sum_{s=0}^{u-1} U_{1+s,s}\Big)^2 = \sum_{s,p=0}^{u-1} E(U_{1+s,s}U_{1+p,p}).$$

Hence, by Lemma 3.3,

$$E(U_{1+s,s}U_{1+p,p}) = E(U_{1,s}U_{1+p-s,p}),$$

and, consequently,

$$\sigma_u^2 = \sum_{s,p=0}^{u-1} E(U_{1,s}U_{1+p-s,p}).$$

Clearly, we have

$$\begin{aligned} \sum_{s,p=0}^{\infty} |E(U_{1,s}U_{1+p-s,p})| &\leqslant |E(U_{1,0}^2)| + \sum_{p=1}^{\infty} |E(U_{1,0}U_{1+p,p})| \\ &\quad + \sum_{s=1}^{\infty} |E(U_{1,s}U_{1-s,0})| + \sum_{s,p=1}^{\infty} |E(U_{1,s}U_{1+p-s,p})|. \end{aligned}$$

Let C, C' and C'' be some constants. By the Schwarz inequality and Lemma 3.4, we have

$$\begin{aligned} |E(U_{1,0}U_{1+p,p})| &\leqslant \|U_{1,0}\|_2\|U_{1+p,p}\|_2 \leqslant C\|G(X_1)\|_2\|A_p\| \leqslant C'\|A_p\|, \\ |E(U_{1,s}U_{1-s,0})| &\leqslant C\|G(X_1)\|_2\|A_s\| \leqslant C'\|A_s\|, \\ |E(U_{1,s}U_{1+p-s,p})| &\leqslant C\|A_s\|\,\|A_p\| \leqslant C'\|A_s\|\,\|A_p\|. \end{aligned}$$

Thus

$$\sum_{s,p=0}^{\infty} |E(U_{1,s} U_{1+p-s,p})| \leqslant \|G(X_1)\|_2^2 + C' \sum_{s=1}^{\infty} \|A_s\| + C' \Big(\sum_{s=1}^{\infty} \|A_s\|\Big)^2 \leqslant C''.$$

Hence, by the condition (a_0), the sum $\sum_{s,p=0}^{\infty} E(U_{1,s} U_{1+p-s,p})$ is absolutely convergent and

$$\sigma_u^2 \to \sum_{s,p=0}^{\infty} E(U_{1,s} U_{1+p-s,p}) \quad \text{as } u \to \infty.$$

Since, by Lemma 3.3,

$$E(U_{i,s} U_{j,p}) = 0 \quad \text{for } i-s \neq j-p, \tag{5.2}$$

we have

$$\sum_{s,p=0}^{\infty} E(U_{1,s} U_{1+p-s,p}) = \sum_{j \in \mathbf{Z}} \sum_{s,p=0}^{\infty} E(U_{1,s} U_{1+j,p}).$$

From the fact that $\sum_{s=0}^{\infty} U_{k,s} \to Y_k$ in $L^2(P)$ we conclude

$$\sum_{j \in \mathbf{Z}} \sum_{s,p=0}^{\infty} E(U_{1,s} U_{1+j,p}) = \sum_{j \in \mathbf{Z}} E(Y_1 Y_{1+j}) = E(Y_1^2) + 2 \sum_{j=1}^{\infty} E(Y_1 Y_{1+j}).$$

Consequently,

$$\sigma_u^2 \to E(Y_1^2) + 2 \sum_{j=1}^{\infty} E(Y_1 Y_{1+j}) \quad \text{as } u \to \infty.$$

We now prove the second part of our lemma. It follows from Lemma 3.3 that

$$E(Y_1 Y_{1+j}) = \sum_{s,t=0}^{\infty} E(U_{1,s} U_{1+j,t}) = \sum_{s=0}^{\infty} E(U_{1,s} U_{1+j,j+s}).$$

Hence, by (5.1), we have

$$E(Y_1 Y_{1+j}) \leqslant \sup_{s \geqslant 0} \big(\mathrm{Lip}(G_s)\big)^2 E(1+|Z_1|)^2 \sum_{s=0}^{\infty} \|A_s\| \, \|A_{j+s}\|.$$

Finally, we get $\sum_{j=1}^{\infty} |E(Y_1 Y_{1+j})| < \infty$, which is a desired result. ■

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REFERENCES

[1] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York 1968.

[2] P. J. Brockwell and R. A. Davis, *Time Series: Theory and Methods*, Springer, New York–Berlin–Heidelberg 1987.

[3] Y. A. Davydov, *The invariance principle for stationary processes*, Theory Probab. Appl. 15 (1970), pp. 487–498.

[4] L. Giraitis, *Central limit theorem for functionals of linear process*, Liet. Mat. Rink. 25 (1986), pp. 43–57.

[5] L. Giraitis and D. Surgailis, *Multivariate Appel polynomials and the Central Limit Theorem*, in: *Dependence in Probability and Statistics: A Survey of Recent Results*, Birkhäuser, Boston, MA, 1986, pp. 21–71.

[6] C. Ho and T. Hsing, *On the asymptotic expansion of the empirical process of long memory moving averages*, Ann. Statist. 24 (1996), pp. 992–1024.

[7] C. Ho and T. Hsing, *Limit theorems for functional of moving averages*, Ann. Probab. 25 (1997), pp. 1636–1669.

[8] T. Hsing, *Linear processes, long-range dependence and asymptotic expansions*, Stat. Inference Stoch. Process. 3 (2000), pp. 19–29.

[9] H. L. Koul and D. Surgailis, *Asymptotic expansion of M-estimators with long-memory errors*, Ann. Statist. 25 (1997), pp. 818–850.

[10] H. L. Koul and D. Surgailis, *Asymptotics of empirical processes of long memory moving averages with infinite variance*, Stochastic Process. Appl. 91 (2001), pp. 309–336.

[11] M. Woodroofe, *A central limit theorem for functionals of a Markov chain with applications to shift*, Stochastic Process. Appl. 41 (1992), pp. 33–44.

[12] W. B. Wu, *Central limit theorems for functionals of linear processes and their applications*, Statist. Sinica 12 (2002), pp. 635–649.

[13] W. B. Wu, *Additive functionals of infinite-variance moving averages*, Statist. Sinica 13 (2003), pp. 1259–1267.

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