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METRIC ENTROPY AND THE SMALL DEVIATION PROBLEM FOR STABLE PROCESSES

BY

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Abstract. The famous connection between metric entropy and small deviation probabilities of Gaussian processes was discovered by Kuelbs and Li in [6] and completed by Li and Linde in [9]. The question whether similar connections exist for other types of processes has remained open ever since. In [10], Li and Linde propose a first approach to this problem for stable processes. The present article clarifies the question completely for symmetric stable processes.

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1. INTRODUCTION AND RESULTS

The small deviation problem and metric entropy. Let us recall the small deviation problem and the concept of metric entropy.

We start with the definition of the entropy numbers. For this purpose, let E and F be two quasi-Banach spaces and let $u: E \to F$ be a bounded linear operator. Then we define the *n-th entropy number* by

$$e_n(u) := \inf\{\varepsilon > 0 \mid \exists\, y_j \in F: u(B_E) \subseteq \bigcup_{j=1}^{2^{n-1}} \{y_j + \varepsilon B_F\}\}, \qquad n \geqslant 1,$$

where B_E and B_F denote the open unit ball in E and F, respectively.

Entropy numbers are used in many applications, in particular in functional analysis. We refer to [4] for an introduction. Metric entropy tools also have many applications in probability theory (cf. [8]), only one of which is the connection to the small deviation problem for Gaussian processes.

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The small deviation problem concerns the probability that a stochastic process X with paths almost surely in some normed space $(E, \|\cdot\|)$ has norm less than a positive number ε, i.e. if $\|X\|$ is measurable, the quantity

$$P(\|X\| \leqslant \varepsilon) \quad \text{as } \varepsilon \to 0+$$

is investigated. However, this is a hard question that can only be solved for very few processes. In many cases, it is only possible to estimate the rate of the logarithm of the above expression, the so-called small deviation function (or small ball function).

For technical reasons, we use the same setup as in [10], namely, we consider only stochastic processes X with paths almost surely in the dual space E' of a Banach space E endowed with the weak-$*$-topology. Then the small deviation function is defined as follows:

$$\phi(X, \varepsilon) := -\log P(\|X\|_{E'} \leqslant \varepsilon).$$

We investigate the rate of this quantity when $\varepsilon \to 0+$. By 'rate' we mean the behaviour in terms of weak asymptotics. Here, we use the symbol $f(\varepsilon) \lesssim g(\varepsilon)$ meaning $\limsup_{\varepsilon \to 0+} f(\varepsilon)/g(\varepsilon) < \infty$. Then $f(\varepsilon) \gtrsim g(\varepsilon)$ means $g(\varepsilon) \lesssim f(\varepsilon)$, whereas we write $f(\varepsilon) \approx g(\varepsilon)$ if $f(\varepsilon) \lesssim g(\varepsilon)$ and $f(\varepsilon) \gtrsim g(\varepsilon)$ hold. The notation is used analogously for sequences when n tends to infinity.

Gaussian processes. Let us recall the notation of a Gaussian process and the operator generating it. Let $(E, \|\cdot\|)$ be a Banach space. Then X is called a *centred Gaussian process* with values almost surely in the dual space E' if $\langle x, X \rangle$ is a centred Gaussian random variable for all $x \in E$. We say that X is *generated* by the operator $u: E \to \boldsymbol{H}$, where $\boldsymbol{H}$ is a separable Hilbert space, if

$$\mathbf{E} e^{i\langle x, X\rangle} = \exp(-\|u(x)\|_{\boldsymbol{H}}^2) \quad \text{for all } x \in E.$$

The analytic operator u contains all information on the stochastic process X. In particular, the small deviation behaviour of X is encoded by the behaviour of the entropy numbers of u, as we can see from the following result. Observe that, by Proposition 2.5 below, $e_n(u) \approx e_n(u')$, where $u': \boldsymbol{H} \to E'$ is the dual operator of u. Combining this with Theorem 1.1 in [9] we have

PROPOSITION 1.1. *Let a Gaussian process X be generated by the operator u. Let $\tau > 0$ and $\theta \in \boldsymbol{R}$. Then the following implications hold*:

(a) *We have*

$$e_n(u) \gtrsim n^{-1/2-1/\tau} (\log n)^{\theta/\tau}$$

if and only if

$$\phi(X, \varepsilon) \gtrsim \varepsilon^{-\tau} (-\log \varepsilon)^{\theta},$$

where for the 'if' part, the additional assumption $\phi(X, \varepsilon) \approx \phi(X, 2\varepsilon)$ is required.

(b) *We have*

$$e_n(u) \lesssim n^{-1/2-1/\tau} (\log n)^{\theta/\tau}$$

if and only if

$$\phi(X, \varepsilon) \preceq \varepsilon^{-\tau}(-\log \varepsilon)^{\theta}.$$

This result shows that the question of determining the small deviation rate of a Gaussian process is, in this polynomial case, fully equivalent to the investigation of the compactness properties of the operator generating it.

Stable processes. It seems natural to ask whether similar relations also hold for other classes of processes. Prime candidates for such a generalisation are symmetric α-stable processes.

The definition of the generating operator for stable processes for $0 < \alpha < 2$ is analogous. We say that X is a *symmetric α-stable process* with values almost surely in E' if $\langle x, X\rangle$ is a symmetric α-stable random variable for all $x \in E$. We say that X is *generated* by the operator $u: E \to L_\alpha(S, \sigma)$, where (S, σ) is some σ-finite measure space, if

$$\mathbf{E} e^{i\langle x, X\rangle} = \exp\left(-\|u(x)\|_{L_\alpha(S,\sigma)}^{\alpha}\right) \quad \text{for all } x \in E.$$

Further details on the existence of a generating operator and measurability questions can be found in [10]. The following result is Theorem 4.5 in [10]. Additionally, the recent work [1] shows that the assumption that the space satisfies property D from [10] can be dropped.

PROPOSITION 1.2. *Let a symmetric α-stable process X be generated by the operator u. Let $\tau > 0$ and $\theta \in \mathbf{R}$ be given, where additionally $\tau < \alpha/(1-\alpha)$ for $0 < \alpha < 1$. If*

$$e_n(u) \succeq n^{1/\alpha - 1/\tau - 1} (\log n)^{\theta/\tau},$$

then

$$\phi(X, \varepsilon) \succeq \varepsilon^{-\tau}(-\log \varepsilon)^{\theta}.$$

Note that Proposition 1.2 becomes the 'only if' part of (a) in Proposition 1.1 for the Gaussian case ($\alpha = 2$).

The restriction $\tau < \alpha/(1-\alpha)$ for $0 < \alpha < 1$ is natural and corresponds to the result in [12], where it was shown that $\phi(X, \varepsilon) \preceq \varepsilon^{-\alpha/(1-\alpha)}$ for any strictly α-stable process X with $0 < \alpha < 1$.

In this article, we prove the following result, which becomes the 'if' part of (b) in Proposition 1.1 for the Gaussian case ($\alpha = 2$).

THEOREM 1.1. *Let a symmetric α-stable process X be generated by the operator u. Let $\tau > 0$ and $\theta \in \mathbf{R}$. If*

$$\phi(X, \varepsilon) \preceq \varepsilon^{-\tau}(-\log \varepsilon)^{\theta},$$

then

$$e_n(u) \preceq n^{1/\alpha - 1/\tau - 1} (\log n)^{\theta/\tau}.$$

For the other two implications in Proposition 1.1 we shall construct counterexamples, which imply the following negative results.

THEOREM 1.2. *There are certain values of $\tau > 0$ and $\theta \in \mathbf{R}$ and a symmetric α-stable process X with generating operator u such that*

$$e_n(u) \preceq n^{1/\alpha - 1/\tau - 1} (\log n)^{\theta/\tau}, \quad \text{but} \quad \lim_{\varepsilon \to 0+} \varepsilon^{\tau} (-\log \varepsilon)^{-\theta} \phi(X, \varepsilon) = \infty.$$

On the other hand, there are certain values of $\tau > 0$ and $\theta \in \mathbf{R}$ and a symmetric α-stable process X with generating operator u such that $\phi(X, \varepsilon) \approx \phi(X, 2\varepsilon)$ and

$$\phi(X, \varepsilon) \succeq \varepsilon^{-\tau} (-\log \varepsilon)^{\theta}, \quad \text{but} \quad \lim_{n \to \infty} n^{-(1/\alpha - 1/\tau - 1)} (\log n)^{-\theta/\tau} e_n(u) = 0.$$

This theorem shows that the respective converse in Proposition 1.2 and Theorem 1.1 cannot hold in general. Thus, contrary to the Gaussian case, for stable processes only two of the implications in Proposition 1.1 are valid analogously.

Nevertheless, for many examples of stable processes still all the implications of Proposition 1.1 are satisfied. This will become clear from the examples below. Therefore, it is possible that the implications disproved in Theorem 1.2 do hold under additional assumptions. In particular, the examples used for Theorem 1.2 are somewhat borderline cases which are 'almost' unbounded. This observation suggests the conjecture that counterexamples only exist for $\tau = \alpha$ and that all implications are valid as long as the processes are sufficiently 'far away' from the region of unboundedness. It seems a natural and challenging problem to determine conditions under which all four implications from the Gaussian setup can be transferred to general (symmetric) stable processes.

This paper is structured as follows. In Section 2, we recall some technical results from [10] and mention some other elementary facts. Section 3 is dedicated to the proof of Theorem 1.1. A discussion of examples can be found in Section 4, where also the counterexamples mentioned in Theorem 1.2 are constructed.

2. TECHNICAL FACTS

In order to use the original formulation of the connection between metric entropy and Gaussian processes, we have to consider the 'inverse' concept of the entropy numbers, namely, the covering numbers of an operator $u: E \to F$:

$$H(u, \varepsilon) := \log \min \{ n \in N \mid \exists y_j \in F : u(B_E) \subseteq \bigcup_{j=1}^{n} \{ y_j + \varepsilon B_F \} \}.$$

One way to express that $H(u, \varepsilon)$ and $e_n(u)$ are inverse to each other is the following lemma, the proof of which is straightforward.

LEMMA 2.1. *Let $D > 0$ be a real number and denote by $\lfloor D \rfloor$ the integer part of D. If $H(u, \varepsilon) \leqslant D$, then $e_{\lfloor D/\log 2 \rfloor + 2}(u) \leqslant \varepsilon$.*

Furthermore, we need one of the main results of [6], namely Lemma 1, which was the starting point of the connection between metric entropy and small deviations for Gaussian processes.

PROPOSITION 2.1. *Let $v\colon \boldsymbol{H} \to E'$ be a bounded operator. Let Y be a Gaussian process generated by $v'\colon E \to \boldsymbol{H}$, and assume that Y is also a (strongly) measurable random variable with values in E'. Then, for all $\varepsilon, \mu > 0$,*

$$H\left(v, \frac{2\varepsilon}{\mu}\right) \leqslant \frac{\mu^2}{2} + \phi(Y, \varepsilon).$$

The main idea of the proof of the results in [10] is to represent the symmetric α-stable process as a 'mixture' of Gaussian processes. This is also the decisive idea in the proof of our main result. For this purpose, we quote the operator theoretic formulation of the 'mixture' from Proposition 2.1 in [10].

PROPOSITION 2.2. *Let $u\colon E \to L_\alpha(S, \sigma)$ be an operator that generates the symmetric α-stable process X with values almost surely in E'. Then there are a probability space $(\Delta, \boldsymbol{Q})$, a separable Hilbert space $\boldsymbol{H}$, and bounded linear operators $v_\delta\colon \boldsymbol{H} \to E'$, $\delta \in \Delta$, such that*

$$\exp(-\|u(x)\|_{L_\alpha(S,\sigma)}^\alpha) = \int_\Delta \exp\left(-\frac{1}{2}\|v_\delta'(x)\|_{\boldsymbol{H}}^2\right) d\boldsymbol{Q}(\delta) \tag{2.1}$$

for all $x \in E$, where $v_\delta'\colon E \to \boldsymbol{H}$ is the dual operator of v_δ.

Note that, by Proposition 4.4 in [10], the v_δ' generate Gaussian processes with values almost surely in E', which will be denoted by Y_δ. These processes are even (strongly) measurable random variables with values in E', cf. [10]. Moreover, we have

PROPOSITION 2.3. *Let X and Y_δ, $\delta \in \Delta$, be processes with generating operators u and v_δ', respectively, for which the relation* (2.1) *holds. Then*

$$\boldsymbol{P}(\|X\|_{E'} \leqslant \varepsilon) = \int_\Delta \boldsymbol{P}(\|Y_\delta\|_{E'} \leqslant \varepsilon)\, d\boldsymbol{Q}(\delta) \quad \text{for all } \varepsilon > 0. \tag{2.2}$$

From the 'mixture' (2.1) one can derive a relation between the entropy numbers of the operator generating the stable process and the entropy numbers of the operators generating the Gaussian processes. Here, from Proposition 3.1 in [10] the following is known:

PROPOSITION 2.4. *Let $u\colon E \to L_\alpha(S, \sigma)$ be an operator generating the symmetric α-stable process X and let $v_\delta\colon \boldsymbol{H} \to E'$, $\delta \in \Delta$, be operators such that* (2.1) *holds. Then there are universal constants $\rho, \kappa > 0$ such that, for all $m \geqslant 1$,*

$$\boldsymbol{Q}\left(\{\delta \in \Delta \mid \forall n \geqslant m\colon n^{1/\alpha - 1/2} e_n(v_\delta') \geqslant \rho e_n(u)\}\right) \geqslant 1 - \kappa e^{-m}.$$

Note that [1] allows us to replace v_δ' by v_δ. We are going to use the following version of the results in [1].

PROPOSITION 2.5. *There are integer constants $f, g \geqslant 1$ such that, for any operator $v\colon \boldsymbol{H} \to E'$, where E is a Banach space and $\boldsymbol{H}$ a separable Hilbert space, with dual operator $v'\colon E \to \boldsymbol{H}$, we have*

$$e_{gn}(v') \leqslant f e_n(v) \quad \textit{for all } n \geqslant 1.$$

3. PROOF OF THEOREM 1.1

We are going to show the following version of Theorem 1.1, which is slightly more precise and expresses the involved constant explicitly, as in Theorem 4.5 in [10]. Theorem 1.1 then follows immediately from Theorem 3.1.

THEOREM 3.1. *Let a symmetric α-stable process X be generated by the operator u. Let $\tau > 0$, $\theta \in \boldsymbol{R}$, and $C > 0$. Then*

$$\limsup_{\varepsilon \to 0+} \varepsilon^{\tau} (-\log \varepsilon)^{-\theta} \phi(X, \varepsilon) \leqslant C/2 < \infty$$

implies

$$\limsup_{n \to \infty} n^{-(1/\alpha - 1/\tau - 1)} (\log n)^{-\theta/\tau} e_n(u) \leqslant c_0 C^{1/\tau},$$

where $0 < c_0 < \infty$ is a constant only depending on τ and θ.

Proof. Step 1. The assumption implies that, for all $0 < \varepsilon < \varepsilon_1$, $\phi(X, \varepsilon) \leqslant C\varepsilon^{-\tau}(-\log \varepsilon)^{\theta}$. We use this and write (2.2) with the help of the small deviation functions in order to obtain

$$\begin{aligned} \exp\big(-C\varepsilon^{-\tau}(-\log \varepsilon)^{\theta}\big) &\leqslant \exp\big(-\phi(X, \varepsilon)\big) \\ &= \int_{\Delta} \exp\big(-\phi(Y_\delta, \varepsilon)\big)\, dQ(\delta) = \mathbf{E}_Q \exp\big(-\phi(Y_\delta, \varepsilon)\big) \end{aligned} \tag{3.1}$$

for all $0 < \varepsilon < \varepsilon_1$. Let furthermore $\lambda > 0$. Then

$$\begin{aligned} \mathbf{E}_Q \exp\big(-\phi(Y_\delta, \varepsilon)\big) &= \int_0^{e^{-\lambda\varepsilon}} Q\big(\exp(-\phi(Y_\delta, \varepsilon)) \geqslant t\big)\, dt \\ &\quad + \int_{e^{-\lambda\varepsilon}}^{1} Q\big(\exp(-\phi(Y_\delta, \varepsilon)) \geqslant t\big)\, dt \\ &\leqslant \int_0^{e^{-\lambda\varepsilon}} 1\,dt + \int_{e^{-\lambda\varepsilon}}^{1} Q\big(\exp(-\phi(Y_\delta, \varepsilon)) \geqslant e^{-\lambda\varepsilon}\big)\, dt \\ &= e^{-\lambda\varepsilon} + (1 - e^{-\lambda\varepsilon})\, Q\big(\phi(Y_\delta, \varepsilon) \leqslant \lambda\varepsilon\big). \end{aligned}$$

Combining this estimate with (3.1), we get

$$\exp\big(-C\varepsilon^{-\tau}(-\log \varepsilon)^{\theta}\big) - e^{-\lambda\varepsilon} \leqslant (1 - e^{-\lambda\varepsilon})\, Q\big(\phi(Y_\delta, \varepsilon) \leqslant \lambda\varepsilon\big).$$

Let $\lambda := 2C\varepsilon^{-1-\tau}(-\log \varepsilon)^{\theta}$. In the last estimate, we divide by $1 - e^{-\lambda\varepsilon}$ and use the result in the last step of the following calculation. This enables us to infer that, for all $0 < \varepsilon < \varepsilon_2 \leqslant \varepsilon_1$,

$$(3.2)\quad \begin{aligned} &\exp\left(-2C\varepsilon^{-\tau}(-\log\varepsilon)^{\theta}\right)\\ &\leqslant \exp\left(-C\varepsilon^{-\tau}(-\log\varepsilon)^{\theta}\right)\frac{1-\exp\left(-C\varepsilon^{-\tau}(-\log\varepsilon)^{\theta}\right)}{1-\exp\left(-2C\varepsilon^{-\tau}(-\log\varepsilon)^{\theta}\right)}\\ &=\frac{\exp\left(-C\varepsilon^{-\tau}(-\log\varepsilon)^{\theta}\right)-\exp\left(-2C\varepsilon^{-\tau}(-\log\varepsilon)^{\theta}\right)}{1-\exp\left(-2C\varepsilon^{-\tau}(-\log\varepsilon)^{\theta}\right)}\\ &\leqslant Q\left(\phi(Y_\delta,\varepsilon)\leqslant 2C\varepsilon^{-\tau}(-\log\varepsilon)^{\theta}\right).\end{aligned}$$

We observe that Proposition 2.1 yields that, for all ε, $\mu > 0$,

$$\phi(Y_\delta,\varepsilon)\leqslant 2C\varepsilon^{-\tau}(-\log\varepsilon)^{\theta} \Rightarrow H\left(v_\delta,\frac{2\varepsilon}{\mu}\right)\leqslant\frac{\mu^2}{2}+2C\varepsilon^{-\tau}(-\log\varepsilon)^{\theta}.$$

Therefore, with (3.2), we have, for all $0<\varepsilon<\varepsilon_2$ and $\mu>0$,

$$\exp\left(-2C\varepsilon^{-\tau}(-\log\varepsilon)^{\theta}\right)\leqslant Q\left(H\left(v_\delta,\frac{2\varepsilon}{\mu}\right)\leqslant\frac{\mu^2}{2}+2C\varepsilon^{-\tau}(-\log\varepsilon)^{\theta}\right).$$

We use $\mu := 2\left(C\varepsilon^{-\tau}(-\log\varepsilon)^{\theta}\right)^{1/2}$. Then

$$\exp\left(-2C\varepsilon^{-\tau}(-\log\varepsilon)^{\theta}\right)\leqslant Q\left(H\left(v_\delta,\frac{\varepsilon^{1+\tau/2}}{C^{1/2}(-\log\varepsilon)^{\theta/2}}\right)\leqslant 4C\varepsilon^{-\tau}(-\log\varepsilon)^{\theta}\right).$$

By Lemma 2.1, the latter implies

$$(3.3)\quad \begin{aligned}&\exp\left(-2C\varepsilon^{-\tau}(-\log\varepsilon)^{\theta}\right)\\ &\leqslant Q\left(e_{\lfloor(4C\varepsilon^{-\tau}(-\log\varepsilon)^{\theta})/\log 2\rfloor+2}(v_\delta)\leqslant\frac{\varepsilon^{1+\tau/2}}{C^{1/2}(-\log\varepsilon)^{\theta/2}}\right),\end{aligned}$$

which holds for all $0<\varepsilon<\varepsilon_2=\varepsilon_2(\varepsilon_1,\tau,\theta,C)$.

Step 2. Let $m\in N$ be large enough such that

$$m>\max\left(4,\frac{\log\kappa}{1-(\log 2)/2}\right),$$

where κ is the constant in Proposition 2.4, and such that there is an ε with $0<\varepsilon<\min(\varepsilon_2,\varepsilon_3)$ and

$$(3.4)\quad m=\left\lfloor\frac{4C}{\log 2}\varepsilon^{-\tau}(-\log\varepsilon)^{\theta}\right\rfloor+2,$$

where $\varepsilon_2=\varepsilon_2(\varepsilon_1,\tau,\theta,C)$ is from Step 1 and ε_3 is a number only depending on C, τ, and θ to be chosen presently. Namely, elementary estimates entail that, for (3.4) and $\varepsilon<\varepsilon_3=\varepsilon_3(C,\tau,\theta)$,

$$(3.5)\quad \frac{\varepsilon^{1+\tau/2}}{C^{1/2}(-\log\varepsilon)^{\theta/2}}\leqslant DC^{1/\tau}m^{-1/\tau-1/2}(\log m)^{\theta/\tau}$$

for some constant $D = D(\theta, \tau)$. Furthermore, (3.4) implies that

$$2C\varepsilon^{-\tau}(-\log\varepsilon)^{\theta} \leqslant \frac{\log 2}{2} m. \tag{3.6}$$

Using (3.6), (3.3), (3.4), and (3.5), we obtain

$$\begin{aligned}\exp\left(-\frac{\log 2}{2} m\right) &\leqslant \exp\left(-2C\varepsilon^{-\tau}(-\log\varepsilon)^{\theta}\right)\\ &\leqslant Q\left(e_{\lfloor (4C\varepsilon^{-\tau}(-\log\varepsilon)^{\theta})/\log 2\rfloor+2}(v_\delta) \leqslant \frac{\varepsilon^{1+\tau/2}}{C^{1/2}(-\log\varepsilon)^{\theta/2}}\right)\\ &\leqslant Q\left(e_m(v_\delta) \leqslant DC^{1/\tau} m^{-1/\tau-1/2}(\log m)^{\theta/\tau}\right).\end{aligned}$$

By Proposition 2.4, we have (since $m > (\log\kappa)/\lfloor 1-(\log 2)/2\rfloor$)

$$\begin{aligned}Q(\exists n \geqslant m\colon n^{1/\alpha-1/2} e_n(v'_\delta) \leqslant \rho e_n(u)) &\leqslant \kappa e^{-m}\\ &< \exp\left(-\frac{\log 2}{2} m\right) \leqslant Q\left(e_m(v_\delta) \leqslant DC^{1/\tau} m^{-1/\tau-1/2}(\log m)^{\theta/\tau}\right).\end{aligned}$$

Since $Q(A) < Q(B)$ implies $B \cap A^c \neq \emptyset$, there is a $\delta = \delta(m)$ such that, for all $n \geqslant m$,

$$n^{1/\alpha-1/2} e_n(v'_\delta) \geqslant \rho e_n(u) \quad \text{and} \quad e_m(v_\delta) \leqslant DC^{1/\tau} m^{-1/\tau-1/2}(\log m)^{\theta/\tau}.$$

Therefore, using Proposition 2.5,

$$\begin{aligned}e_{gm}(u) &\leqslant \rho^{-1}(gm)^{1/\alpha-1/2} e_{gm}(v'_\delta)\\ &\leqslant \rho' m^{1/\alpha-1/2} e_m(v_\delta) \leqslant \rho' m^{1/\alpha-1/2} DC^{1/\tau} m^{-1/\tau-1/2}(\log m)^{\theta/\tau}.\end{aligned}$$

Thus, we have shown that, for all $m \geqslant m_0 = m_0(C, \tau, \theta, \varepsilon_1, \kappa)$,

$$e_{gm}(u) \leqslant \rho' DC^{1/\tau} m^{1/\alpha-1/\tau-1}(\log m)^{\theta/\tau},$$

where $g \geqslant 1$ is the integer constant from Proposition 2.5, $D = D(\tau, \theta)$, and $\rho' = g^{1/\alpha-1/2} f/\rho$. This shows the assertion of Theorem 3.1. ∎

4. EXAMPLES AND APPLICATIONS

4.1. Example 1: Stable processes with integral representation. Let us consider symmetric α-stable processes given by their integral representation

$$X_t = \int_S K(t, s)\, dM(s), \qquad t \in T, \tag{4.1}$$

where M is an independently scattered symmetric α-stable random measure with control measure σ, (S, σ) is a σ-finite measure space, and (T, m) is some finite measure space. All important stable processes can be written in this form;

we refer to [14] for details and examples. Necessary and sufficient conditions for X to be almost surely in $L_p(T, m)$, $1 \leqslant p < \infty$, in terms of the kernel K, are given in Theorem 11.3.2 in [14]. Unfortunately, there is no universal condition ensuring that X is in $L_\infty(T)$ or $C(T)$ almost surely.

Let $E = L_{p'}(T, m)$. Then $E' = L_p(T, m)$ for $1 < p \leqslant \infty$ and $E' \supseteq L_1(T, m)$ for $p = 1$. Let us assume that X is almost surely in E'. Then it is easy to see with the help of Theorem 11.4.1 in [14] that the operator $\tilde{u}\colon L_{p'}(T, m) \to L_\alpha(S, \sigma)$ given by

$$\tilde{u}\colon x \mapsto \int_T K(t, \cdot)\, x(t)\, dm(t) \tag{4.2}$$

generates the process X. If $T = S = [0, 1]$ and $m = \sigma$ is the Lebesgue measure, the operator $\tilde{u}$ has the same entropy behaviour as the operator $u\colon L_{p'}[0, 1] \to L_\alpha[0, 1]$ given by

$$u\colon x \mapsto \int_0^1 K(1-t, 1-\cdot)\, x(t)\, dt; \tag{4.3}$$

cf. the argument at the bottom of p. 281 in [10]. The latter operator is usually easier to identify.

We apply Theorem 1 and obtain the analog of Theorem 5.2 in [10].

COROLLARY 4.1. *Let X be a symmetric α-stable process given by the integral representation* (4.1) *with values almost surely in $L_p(T, m)$, $1 \leqslant p \leqslant \infty$. Let $\tau > 0$ and $\theta \in \mathbf{R}$.*

If $\phi(X, \varepsilon) \lesssim \varepsilon^{-\tau}(-\log\varepsilon)^\theta$, then $e_n(\tilde{u}) \lesssim n^{1/\alpha - 1/\tau - 1}(\log n)^{\theta/\tau}$, where $\tilde{u}$ is as in (4.2).

If additionally $T = S = [0, 1]$ and $m = \sigma$ is the Lebesgue measure, then $\phi(X, \varepsilon) \lesssim \varepsilon^{-\tau}(-\log\varepsilon)^\theta$ also implies $e_n(u) \lesssim n^{1/\alpha - 1/\tau - 1}(\log n)^{\theta/\tau}$, where u is as in (4.3).

The probably most important symmetric α-stable process is the so-called symmetric α-stable Lévy motion $(X_t)_{t \geqslant 0}$, which is a symmetric α-stable stochastic process with $X_0 = 0$ almost surely, having independent increments, and $X_t - X_s$ being a symmetric α-stable random variable with scale parameter $|t-s|^{1/\alpha}$. It is well-known that X has the integral representation $X_t = \int_0^t dM(s)$ on $T = S = [0, 1]$ using the Lebesgue measure for σ and m. Thus the operator u in (4.3) is the usual integration operator: $(ux)(s) = \int_0^s x(t)\, dt$. The behaviour of the entropy numbers of this operator is well-known to be $e_n(u) \approx n^{-1}$. On the other hand, the small deviation probabilities of X with respect to the $L_p[0, 1]$ norms are well-known to be $\phi(X, \varepsilon) \approx \varepsilon^{-\alpha}$ (cf. Section 6.3 in [11] for a historical overview of this result). This shows that Corollary 4.1 is sharp in this case.

4.2. Example 2: Fractional stable processes. Let us consider the examples of the Riemann–Liouville process

$$R_t^H := \int_0^t (t-s)^{H-1/\alpha}\, dM(s), \qquad t \in [0, 1],$$

with Hurst parameter $H > 0$ and the linear α-stable fractional motion

$$X_t^H := \int_{-\infty}^{t} [(t-s)^{H-1/\alpha} - \max(-s, 0)^{H-1/\alpha}]\, dM(s), \qquad t \in [0, 1],$$

with Hurst parameter $1/\alpha < H < 1$, where M is as in Example 1. Note that these processes are special cases of Example 1.

The processes R^H and X^H have been under investigation for a long time. It is well-known that both processes are closely related, in particular in the Gaussian case ($\alpha = 2$), where X^H is a fractional Brownian motion. For the non-Gaussian case, X^H is one possible generalisation of the fractional Brownian motion.

Recently, a detailed study on the small deviation problem for R^H and X^H under a wide class of norms has been carried out in [11].

However, it does not seem possible to convert all of these results into entropy estimates via Theorem 1.1. The reason for this is that not all the spaces considered in [11] can be represented as dual spaces of a suitable Banach space E. In particular, this seems to be the case with the p-variation norm, which was also the topic of [15].

Because of this, let us concentrate on the case $E = L_{p'}[0, 1]$ with $E' \supseteq L_p[0, 1]$ already considered in Example 1. The main result of [11] states that if $H > 1/\alpha$, then

$$\phi(R^H, \varepsilon) \approx \phi(X^H, \varepsilon) \approx \varepsilon^{-1/H}. \tag{4.4}$$

Note that in the case $H = 1/\alpha$ the process $R^{1/\alpha}$ is exactly the symmetric α-stable Lévy motion considered above.

The operator u in (4.3) having the same entropy behaviour as the operator $\tilde{u}$ generating R^H is given by $u = \Gamma(H - 1/\alpha + 1) R_{H-1/\alpha+1}$, where

$$(R_\beta x)(s) := \frac{1}{\Gamma(\beta)} \int_0^s (s-t)^{\beta-1} x(t)\, dt, \qquad s \in [0, 1],$$

is the so-called Riemann–Liouville operator.

The entropy numbers of the Riemann–Liouville operator can be calculated with the help of the methods in [5]; also cf. Proposition 6.1 in [9]. The result is

$$e_n(R_{H-1/\alpha+1}) \approx n^{1/\alpha - H - 1}. \tag{4.5}$$

Here, (4.4) and (4.5) agree in the sense of Proposition 1.2 and Theorem 1.1, i.e. showing their respective sharpness. The situation is likewise for X^H in place of R^H.

Certainly, a similar comparison can be carried out for Sobolev and Besov spaces with the help of the small deviation results in [11] and the entropy results in [5].

Nevertheless, we concentrate on a different aspect, namely unbounded Riemann–Liouville processes. It is known that $R^H \in L_p[0, 1]$ almost surely if and only if $H > \max(1/\alpha - 1/p, 0)$ (cf. Section 6 in [11]). For $H < 1/\alpha$, the above-mentioned results from [11] do not apply in their full extent. Theorem 4 from [11] or, alternatively, the result of Li and Linde quoted above as Proposition 1.2 yields

$$\phi(R^H, \varepsilon) \succeq \varepsilon^{-1/H}.$$

An upper bound for ϕ is not known, cf. the remark at the very end of Section 6.4 in [11]. However, if it turns out that $\phi(R^H, \varepsilon) \succeq \varepsilon^{-\tau}$ with $\tau > 1/H$, we would have found another counterexample as those mentioned in Theorem 1.2.

4.3. Example 3: H-sssi processes. Now we are going to consider H-self-similar processes with stationary increments (H-sssi processes), i.e. let $(X_t)_{t \geq 0}$ be a symmetric α-stable process satisfying

$$(X_{ct})_{t \geq 0} \stackrel{d}{=} (c^H X_t)_{t \geq 0}, \qquad (X_{t+c} - X_c)_{t \geq 0} \stackrel{d}{=} (X_t - X_0)_{t \geq 0}$$

for all $c > 0$, where $\stackrel{d}{=}$ means that the finite-dimensional distributions coincide.

The self-similarity property makes these processes very important in applications. Further properties and examples of H-sssi processes as well as literature links to applications are given in Chapters 7 and 8 of [14].

The small deviation problem for this class of processes in $L_\infty[0, 1]$ was studied thoroughly in [13], and a recent improvement was obtained in [3]. It can be shown that if X is almost surely bounded, then

$$\phi(X, \varepsilon) \preceq \varepsilon^{-1/H} \quad \text{for } 1 < \alpha < 2,\ 1/\alpha < H \leqslant 1.$$

Let us remark that [13] also proves lower bounds. The bound for $0 < \alpha < 1$ is just the result from [12] we have already commented on above, which is not interesting for us. However, the result just quoted for $1 < \alpha < 2$ (Theorem 3.2 in [13]) combined with Theorem 1.1 yields the following result.

COROLLARY 4.2. *Let $1 < \alpha < 2$ and assume that X is a symmetric α-stable H-sssi process that is almost surely bounded. Assume that there is an operator $u\colon L_1[0, 1] \to L_\alpha(S, \sigma)$, where (S, σ) is some σ-finite measure space that generates X. Then*

$$e_n(u) \preceq n^{1/\alpha - H - 1}.$$

There are several important examples of stable H-sssi processes such as the α-stable Lévy motion (mentioned already at the end of Example 1), the linear α-stable fractional motion (mentioned in Example 2), or the sub-fractional Brownian motion, defined by $X_t = A^{1/2} Y_t$, where A is an $\alpha/2$-stable, non-negative random variable that is totally skewed to the right and Y is a fractional Brownian motion with Hurst parameter H. The small deviation problem for the latter was solved in [13] to the end that

$$\phi(X, \varepsilon) \approx \varepsilon^{-2\alpha/(2-\alpha+2\alpha H)}.$$

By Theorem 1.1, this shows that $e_n(u) \preceq n^{-(H+1/2)}$, which is not surprising. It can be derived by different methods that $e_n(u) \approx n^{-(H+1/2)}$, which again shows that in this case Proposition 1.2 and Theorem 1.1 are sharp.

4.4. Example 4: Sequences of independent random variables. Let us consider $\xi_1, \xi_2, \ldots$ to be independent, identically distributed symmetric α-stable random variables, and let $\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant 0$ be a sequence of real numbers that tends to zero. Then we consider the symmetric α-stable process

$$X = (\sigma_1 \xi_1, \sigma_2 \xi_2, \ldots) \in l_p,$$

where $1 \leqslant p \leqslant \infty$. The small deviation probabilities of this class of processes are included in the results of [2].

On the other hand, it is easy to see that the diagonal operator

$$u: l_{p'} \to l_\alpha \quad \text{with } u(x_1, x_2, \ldots) := (\sigma_1 x_1, \sigma_2 x_2, \ldots)$$

generates the process X. Entropy numbers of diagonal operators have been studied by many authors; we refer to [7] for a recent survey.

Let us compare the results on entropy numbers and small deviations in the case that there are $\mu > 0$ and $\nu \in \boldsymbol{R}$ such that

$$\frac{\sigma_n}{n^{-\mu} (\log n)^{-\nu}} \to 1.$$

For $\mu > \max(1/\alpha, 1/p)$, Corollary 4.1 in [2] yields

$$\lim_{\varepsilon \to 0+} \varepsilon^{1/(\mu - 1/p)} (-\log \varepsilon)^{\nu/(\mu - 1/p)} \phi(X, \varepsilon) = C_p, \tag{4.6}$$

where C_p is a finite, positive constant. If $\mu \leqslant \max(1/\alpha, 1/p)$, then (4.6) holds with $C_p = \infty$.

On the other hand, Theorem 2.2 in [7] yields (for $\mu > \max(1/\alpha - 1/p', 0)$)

$$e_n(u) \approx n^{1/\alpha - 1/p' - \mu} (\log n)^{-\nu} = n^{1/\alpha - (\mu - 1/p) - 1} (\log n)^{-\nu}. \tag{4.7}$$

Note that (4.6) and (4.7) agree for $\mu > \max(1/p, 1/\alpha)$ in the sense of Proposition 1.2 and Theorem 1.1, showing their respective sharpness. However, note that they do not agree for those cases when X is in l_p almost surely and $\mu \leqslant \max(1/p, 1/\alpha)$, which can be used to prove Theorem 1.2.

Proof of Theorem 1.2. Let $\xi_1, \xi_2, \ldots$ be as above and let us consider the example $\sigma_n = n^{-1/\alpha} (\log n)^{-b/\alpha}$ with $b > 1$. Then we study the symmetric α-stable process

$$X = (\sigma_1 \xi_1, \sigma_2 \xi_2, \ldots) \in l_\infty = l_1'.$$

It was shown in [2] that in the case $\sigma_n = n^{-1/\alpha} (\log n)^{-b/\alpha}$ we have

$$\phi(X, \varepsilon) \approx \varepsilon^{-\alpha} (-\log \varepsilon)^{-b+1}.$$

The above-mentioned estimate (4.7) for entropy numbers, however, states that (since $\mu = 1/\alpha > \max(1/\alpha - 1, 0)$)

$$e_n(u) \approx n^{-1} (\log n)^{-b/\alpha}.$$

This example serves as the counterexample asserted in Theorem 1.2, since

$$e_n(u) \preceq n^{-1} (\log n)^{-b/\alpha}, \quad \text{but} \quad \lim_{\varepsilon \to 0+} \varepsilon^\alpha (-\log \varepsilon)^b \phi(X, \varepsilon) = \infty$$

and $\phi(X, \varepsilon) \approx \phi(X, 2\varepsilon)$,

$$\phi(X, \varepsilon) \succeq \varepsilon^{-\alpha} (-\log \varepsilon)^{-b+1}, \quad \text{but} \quad \lim_{n \to \infty} n (\log n)^{-(-b+1)/\alpha} e_n(u) = 0. \ \blacksquare$$

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REFERENCES

[1] S. Artstein, V. Milman and S. J. Szarek, *Duality of metric entropy*, Ann. of Math. (2) 159 (2004), pp. 1313–1328.
[2] F. Aurzada, *On the lower tail probabilities of some random sequences in l_p*, preprint (2005), to appear in: J. Theoret. Probab., available at: http://www.springerlink.com/content/2487424381403th0/?p = 018780b6b8e34648916c47dc98cc8601&pi = 0
[3] F. Aurzada, *Small deviations for stable processes via compactness properties of the parameter set*, preprint (2006), to appear in: Statist. Probab. Lett., available at: http://www.math.tu-berlin.de/~ aurzada/sdviadudleymetric.pdf
[4] B. Carl and I. Stephani, *Entropy, Compactness and the Approximation of Operators*, Cambridge University Press, Cambridge 1990.
[5] D. E. Edmunds and H. Triebel, *Function Spaces, Entropy Numbers, Differential Operators*, Cambridge University Press, Cambridge 1996.
[6] J. Kuelbs and W. V. Li, *Metric entropy and the small ball problem for Gaussian measures*, J. Funct. Anal. 116 (1993), pp. 133–157.
[7] T. Kühn, *Entropy numbers of general diagonal operators*, Rev. Mat. Complut. 18 (2005), pp. 479–491.
[8] M. Ledoux and M. Talagrand, *Probability in Banach Spaces*, Springer, Berlin 1991.
[9] W. V. Li and W. Linde, *Approximation, metric entropy and small ball estimates for Gaussian measures*, Ann. Probab. 27 (1999), pp. 1556–1578.
[10] W. V. Li and W. Linde, *Small deviations of stable processes via metric entropy*, J. Theoret. Probab. 17 (2004), pp. 261–284.
[11] M. A. Lifshits and T. Simon, *Small deviations for fractional stable processes*, Ann. Inst. H. Poincaré Probab. Statist. 41 (2005), pp. 725–752.
[12] M. Ryznar, *Asymptotic behaviour of stable seminorms near the origin*, Ann. Probab. 14 (1986), pp. 287–298.
[13] G. Samorodnitsky, *Lower tails of self-similar stable processes*, Bernoulli 4 (1998), pp. 127–142.
[14] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes*, Chapman and Hall, New York 1994.

[15] T. Simon, *Small ball estimates in p-variation for stable processes*, J. Theoret. Probab. 17 (2004), pp. 979–1002.

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