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FINE STRUCTURE OF THE COMPLEX HYPERBOLIC BROWNIAN MOTION AND RUDIN'S QUESTION

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Abstract. We investigate the fine structure of the complex hyperbolic Brownian motion in the unit ball of $\mathbb{C}^n$. It turns out that the generator of the process is locally very close to the generator of some simple transformation of the classical Brownian motion. This fact helps us to give an intuitive explanation why the invariant Laplace operator in the unit ball of $\mathbb{C}^n$ is a difference of two ordinary Laplace operators – the question set by W. Rudin in his monograph *Function Theory in the Unit Ball of* $\mathbb{C}^n$.

In the second part of the paper we find stochastic differential equations for the complex hyperbolic Brownian motion on the ball model of the complex hyperbolic space and furnish in this way an important tool in a further investigation of this process.

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1. INTRODUCTION

Let $B \subset \mathbb{C}^n$ be the open unit ball equipped with the Bergman metric. B is a model of the complex hyperbolic space $\mathbb{H}^n(\mathbb{C})$. Analysis on the ball B is deeply studied in [4] and in the sequel we use numerous facts from that monograph.

Let $Aut(B)$ be the Moebius group of B, i.e. the group of biholomorphic maps of B onto B. These maps from $Aut(B)$ are isometries of B and they are all generated by unitary linear transformations from $U(n)$ and by the maps φ_a, $a \in B$, generalizing homographical maps in the unit disc of $\mathbb{C}$. The definition of φ_a is the following: fix $a \in B$, $a \neq 0$, denote by P_a the orthogonal projection of $\mathbb{C}^n$ on the complex line lin$[a]$ and by $Q_a = I - P_a$ the orthogonal projection of $\mathbb{C}^n$ on the orthogonal complement to lin$[a]$. Then

$$\varphi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle}, \tag{1.1}$$

where $s_a = (1 - |a|^2)^{1/2}$. Observe that $\varphi_a(0) = a$. Now, if $\psi \in Aut(B)$ and $a =$

$\psi^{-1}(0)$, then there exists a unitary operator $g \in U(n)$ such that $\psi = g\varphi_a$ (Theorem 2.2.5 in [4]).

2. RUDIN'S QUESTION

The invariant Laplace operator $\tilde{\Delta}$ (invariant with respect to the action of the group $Aut(B)$) is defined as follows: for $a \in B$ and $f \in C^2(B)$

$$(\tilde{\Delta} f)(a) = \Delta^{\mathbb{C}^n}(f \circ \varphi_a)(0), \tag{2.1}$$

where $\Delta^{\mathbb{C}^n}$ is the ordinary Laplace operator in $\mathbb{C}^n$. This is the Laplace–Beltrami operator for B equipped with Bergman metric. It can be computed ([4], Theorem 4.1.3 (ii)) that

$$(\tilde{\Delta} f)(a) = (1 - |a|^2)[(\Delta^{\mathbb{C}^n} f)(a) - (\Delta^{\mathbb{C}} f_a)(1)],$$

where $f_a(\lambda) = f(\lambda a)$.

In the last section of his monograph, Rudin asks the following ([4], 19.3.10):

Is there a more intuitive (geometric? group-theoretic?) way of seeing why, except for the factor $1 - |a|^2$, $\tilde{\Delta} f$ is a difference of two ordinary Laplacians?

We answer this question by investigating the fine structure of the complex hyperbolic Brownian motion.

3. THE COMPLEX HYPERBOLIC BROWNIAN MOTION

The operator $\frac{1}{2}\tilde{\Delta}$, one half of the Laplace–Beltrami operator on a Riemannian manifold, is the generator of a diffusion process X_t – the Brownian motion on this manifold.

Observe that, in the particular case when $a = (a_1, 0, \ldots, 0)$, the formula (1.1) reads as follows

$$\begin{aligned} \varphi_a(z) &= \left(\frac{a_1 - z_1}{1 - \bar{a}_1 z_1}, -\frac{\sqrt{1-|a|^2}}{1 - \bar{a}_1 z_1} z_2, \ldots, -\frac{\sqrt{1-|a|^2}}{1 - \bar{a}_1 z_1} z_n \right) \\ &= \big(h_a(z_1),\, c_a(z_1) z_2, \ldots,\, c_a(z_1) z_n \big), \end{aligned} \tag{3.1}$$

where $h_a(z_1) = (a_1 - z_1)/(1 - \bar{a}_1 z_1)$ is a homographical map of the unit disc of $\mathbb{C}$, and $c_a(z_1) = -\sqrt{1-|a|^2}/(1 - \bar{a}_1 z_1)$ is a factor depending only on the first coordinate z_1.

Let $B_t = (B_t^{(1)}, \ldots, B_t^{(n)})$ be a standard complex Brownian motion in $\mathbb{C}^n$, starting from the origin, and let $W_t = (W_t^{(1)}, \ldots, W_t^{(n)})$ be a standard complex Brownian motion in $\mathbb{C}^n$, starting from 0 and killed when exiting B. We denote by τ_B the first exit time of B_t from the ball B. For $t < \tau_B$ the processes W_t and B_t are equal, next W_t goes to a "cemetery" state. The generator of W_t is equal to $\frac{1}{2}\Delta^{\mathbb{C}^n}|_B$ and the generator of B_t is $\frac{1}{2}\Delta^{\mathbb{C}^n}$, half of the Euclidean Laplacian $\Delta^{\mathbb{C}^n}$. Define

$$Y_t = \varphi_a(W_t) = \big(h_a(W_t^{(1)}),\, c_a(W_t^{(1)})W_t^{(2)}, \dots,\, c_a(W_t^{(1)})W_t^{(n)}\big). \tag{3.2}$$

REMARK 3.1. This definition does not make sense for the Brownian motion B_t on $\mathbb{C}^n$ because it may happen that $\varphi_a(B_t)$ is not defined. We will study the process W_t for $t < \tau_B$.

Denote by L^Y the generator of the diffusion process Y_t. By the invariance (2.1) of the operator $\tilde{\Delta}$ it follows that

$$L^Y|_a = \frac{1}{2}\tilde{\Delta}|_a,$$

so the generators of the hyperbolic Brownian motion X_t and the process Y_t coincide at the point a. One can interpret this fact saying that the hyperbolic Brownian motion X_t is "locally" an image of the standard Euclidean Brownian motion. Rigorously, these processes are similar in the sense of comparison of their transition probabilities

$$p^X(t,a,y) \asymp p^Y(t,a,y)$$

for $t \to 0$ and y near to a; see [1]. Recall that $f \asymp g$ means that there exists $C > 0$ such that $C^{-1}f \leqslant g \leqslant Cf$.

4. THE INTUITIVE EXPLANATION

We answer Rudin's question showing that the form of the isometry φ_a implies that the Brownian motion X_t generated by one half of the invariant Laplace operator $\frac{1}{2}\tilde{\Delta}$ can be (locally) decomposed into two parts and both parts are (locally) time-changed classical Brownian motions. Instead of the process X_t we look at the process Y_t. As we explained before, they are very similar "locally" at the point a, their generators are equal at a.

The isometry $\varphi_a(z) = \big(h_a(z_1),\, c_a(z_1)z_2, \dots,\, c_a(z_1)z_n\big)$ has a very particular form: on the first coordinate it is a homography (hence a conformal mapping) while on all the remaining coordinates it is a dilation by the same factor, depending on z_1. Now the following facts are crucial for our reasoning:

A. *Generator of a "product" of two independent processes is the sum of their generators*: If $V_1(t), V_2(t)$ are two independent processes with generators L_1, L_2, respectively, then the generator of the process $(V_1, V_2)_{t\geqslant 0}$ is equal to $Lf(x,y) = L_1 f(\cdot,y)|_x + L_2 f(x,\cdot)|_y$.

B. *Standard Brownian motion after a dilation by* $c \neq 0$ *is a time-dilated Brownian motion and its generator is multiplied by a constant*: The process $(cB_t)_{t\geqslant 0}$ is equal in law to the process $(B_{c^2 t})_{t\geqslant 0}$ and has the generator $(c^2/2)\Delta$.

C. *Conformal image of a Brownian motion in $\mathbb{C}$ is a time-changed Brownian motion*. This conformal invariance of complex Brownian motion is a fundamental result of Paul Lévy (see [3], V (2.5)), which reads as follows:

If F is an entire and non-constant function, $F(B_t)$ is a time-changed Brownian motion. More precisely, on the probability space of B_t there exists a complex Brownian motion $\tilde{B}_t$ such that

$$F(B_t) = F(B_0) + \tilde{B}_{\langle X,X\rangle_t},$$

where $\langle X, X\rangle_t = \int_0^t |F'(B_s)|^2\,ds$ is strictly increasing and $\langle X, X\rangle_\infty = \infty$.

Our objective is to describe the generator $L^Y|_a$ of the diffusion Y_t, $Y_0 = a$, given by the formula

$$Y_t = \big(h_a(U_t), c_a(U_t)V_t\big),$$

where $W_t = (U_t, V_t) = (U_t, V_t^{(1)}, \ldots, V_t^{(n-1)})$ is a standard complex Brownian motion in $\mathbb{C}^n$ starting from 0 and killed when exiting B. For $u \in \mathbb{C}$ and $v \in \mathbb{C}^{n-1}$ let us consider a function $G(u, v) = c_a(u)v = \big(c_a(u)v_1, \ldots, c_a(u)v_{n-1}\big)$ and let $G_k(u, v_k) = c_a(u)v_k$ be the k-th coordinate of G. By the Taylor formula at 0 we have

$$G_k(u, v_k) = c_a(u)v_k \asymp G_k(0, 0) + \nabla G_k|_0 \cdot (u, v_k) = c_a(0)v_k.$$

We would like to apply such an expansion to the diffusion $c_a(U_t)V_t$. In general, the Itô formula implies that the deterministic differential is different from the stochastic one. But as the function G is holomorphic, the Itô and the Taylor formulas coincide (the terms with second order derivatives of G cancel because G is harmonic). Thus, in a stochastic differential sense we have

$$c_a(U_t)V_t \asymp c_a(0)V_t,$$

when t is small. Hence we can intuitively expect that the generator $L^Y|_a$ is the same as for the process

$$Z_t = \big(h_a(U_t), c_a(0)V_t\big).$$

The facts A, B and C imply that the generator of Z_t equals

$$L^Z = A(a, z_1)\Delta_1 + \frac{c_a(0)^2}{2}\Delta_{2,\ldots,n},$$

where Δ_1 is defined to be equal to $\Delta^{\mathbb{C}}$ acting in z_1 and $\Delta_{2,\ldots,n} = \Delta^{\mathbb{C}^{n-1}}$ acting in $z_2, \ldots, z_n$. It is easy to see that L^Z can be written in the form

$$L^Z = F_1(a)\Delta^{\mathbb{C}^n} + F_2(a, z_1)\Delta_1.$$

As $\frac{1}{2}\tilde{\Delta}|_a = L^Y|_a = L^Z|_a$ (the last equality is based on stochastic intuition), we get a simple answer to Rudin's question. The exact values of the coefficients $F_1(a)$

and $F_2(a, a_1)$ will be determined by calculations, made in the next section (see Proposition 5.2).

We summarize our answer to Rudin's question in the following way: The hyperbolic Brownian motion generated by the invariant Laplacian $\frac{1}{2}\tilde{\Delta}$ is locally comparable near the point a to the "product" of two independent time-changed Brownian motions $\big(h_a(U_t), c_a(0)V_t\big)$. Indeed, the first of these motions is the conformal image of a two-dimensional classical Brownian motion and the second one is a dilation of classical Brownian motion in $\mathbb{C}^{n-1}$. Hence $\tilde{\Delta}|_a$ is a sum of two ordinary Laplacians, multiplied by some functions of a.

5. FORMAL COMPUTATIONS

Using stochastic calculus, in this section we give proofs of all assertions of the intuitive reasoning from the previous section.

For the complex Brownian motion W_t we can apply the "complex" Itô formula (see [3], Proposition V (2.3)). If F is a holomorphic function and W_t is a complex Brownian motion on $\mathbb{C}$, we have

$$F(W_t) = F(W_0) + \int_0^t F'(W_s)dW_s.$$

This formula applies directly to the first coordinate $Y_t^{(1)} = h_a(W_t^{(1)})$ of the process Y_t. For other coordinates $Y_t^{(k)} = c_a(W_t^{(1)})W_t^{(k)}$, $k \geqslant 2$, we write the complex Itô formula for the space $\mathbb{C}^2$ and the holomorphic function $G(z_1, z_k) = c_a(z_1)z_k$:

$$\begin{aligned} G(W_t^{(1)}, W_t^{(k)}) &= G(W_0^{(1)}, W_0^{(k)}) \\ &\quad + \int_0^t \frac{\partial G}{\partial z_1}(W_t^{(1)}, W_t^{(k)})dW_s^{(1)} + \int_0^t \frac{\partial G}{\partial z_k}(W_t^{(1)}, W_t^{(k)})dW_s^{(k)}. \end{aligned}$$

Note that in both complex Itô formulas given above the terms with second derivatives vanish because the holomorphic functions F, G are harmonic.

PROPOSITION 5.1. *The diffusion Y_t satisfies for $t < \tau_B$ the following system of stochastic differential equations*:

$$\begin{aligned} dY_t^{(1)} &= A(Y_t^{(1)})dW_t^{(1)}, \\ dY_t^{(k)} &= \beta(Y_t^{(1)})Y_t^{(k)}dW_t^{(1)} + C(Y_t^{(1)})dW_t^{(k)}, \quad k \geqslant 2, \end{aligned} \tag{5.1}$$

where $A = h'_a \circ h_a^{-1}$, $\beta = (c'_a/c_a) \circ h_a^{-1}$ and $C = c_a \circ h_a^{-1}$.

Proof. By the complex Itô formula we have

$$\begin{aligned} dY_t^{(1)} &= h'_a(W_t^{(1)})dW_t^{(1)}, \\ dY_t^{(k)} &= c'_a(W_t^{(1)})W_t^{(k)}dW_t^{(1)} + c_a(W_t^{(1)})dW_t^{(k)}, \quad k \geqslant 2. \end{aligned}$$

We need only to express the process W_t, appearing in the coefficients of these equations, by Y_t. We have

$$W_t^{(1)} = h_a^{-1}(Y_t^{(1)}) \quad \text{and} \quad W_t^{(k)} = \frac{Y_t^{(k)}}{c_a(W_t^{(1)})} = \frac{Y_t^{(k)}}{c_a\big(h_a^{-1}(Y_t^{(1)})\big)}. \quad \blacksquare$$

Now we want to deduce the generator L_Y of the process Y_t from the system (5.1). Recall that if a d-dimensional real diffusion X_t satisfies the SDE

$$dX_t = s(X_t)d\beta_t + b(X_t)dt,$$

where β_t is a d-dimensional real Brownian motion, s is a $d\times d$ real matrix valued Borel locally bounded function, and b an $\mathbb{R}^d$-valued Borel locally bounded function, then the generator of X_t is

$$L_X = \frac{1}{2}\sum_{j,k=1}^{d} m_{jk}(x)\frac{\partial^2}{\partial x_j \partial x_k} + \langle b, \nabla\rangle, \qquad m = (m_{jk}) = ss^T.$$

In the complex setting, we identify a complex number $z = x + iy$ with the real matrix

$$\tilde{z} = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}.$$

With this convention, the stochastic differential equation

$$dY_t = zdB_t = zd(U_t + iV_t),$$

where $B = U + iV$ is a Brownian motion in $\mathbb{C}$, U, V are 1-dimensional real independent Brownian motions, $z = x + iy \in \mathbb{C}$, becomes

$$\begin{aligned} d(\mathrm{Re}Y) &= xdU - ydV, \\ d(\mathrm{Im}Y) &= ydU + xdV, \end{aligned}$$

i.e., if $\tilde{Y} = (\mathrm{Re}Y, \mathrm{Im}Y)^T$ and $\beta = (U, V)^T$, then

$$d\tilde{Y}_t = \tilde{z}d\beta_t,$$

so the coefficient z in the complex SDE is replaced by $\tilde{z}$ in the equivalent real one.

We have $\widetilde{zw} = \tilde{z}\tilde{w}$ and $(\tilde{z})^T = \tilde{\bar{z}}$. If $\sigma \in M_{\mathbb{C}}(n)$ is a complex square matrix, then $\tilde{\sigma} \in M_{\mathbb{R}}(2n)$ is obtained by replacing each coefficient σ_{kl} by the matrix $\tilde{\sigma}_{kl} \in M_{\mathbb{R}}(2)$. We have $(\tilde{\sigma})^T = (\sigma^*)\tilde{}$ and $\tilde{\sigma}(\tilde{\sigma})^T = (\sigma\sigma^*)\tilde{}$. It follows that if Y_t is a diffusion in $\mathbb{C}^n$ satisfying the SDE

$$dY_t = \sigma(Y_t)dB_t + b(Y_t)dt,$$

where B_t is an n-dimensional complex Brownian motion, $\sigma \in M_{\mathbb{C}}(n)$, $b \in \mathbb{C}^n$ are Borel locally bounded functions, then the generator of Y_t on $\mathbb{R}^{2n} = \mathbb{C}^n$ is

$$L_Y = \frac{1}{2} \sum_{j,k=1}^{2n} M_{jk}(x) \frac{\partial^2}{\partial x_j \partial x_k} + \langle b, \nabla \rangle, \qquad M = (M_{jk}) = (\sigma\sigma^*)\tilde{}. \tag{5.2}$$

In the following formulas we identify $a = (a_1, 0, \ldots, 0)$ with its first coordinate, i.e. we write a instead of a_1.

PROPOSITION 5.2. *Let Δ_k be the Laplace operator in the k-th complex variable z_k on $\mathbb{C}^n$. The generator L of the process Y_t at the point a has the following form*:

$$L|_a = \frac{1}{2}|A(a)|^2 \Delta_1 + \frac{1}{2}|C(a)|^2 \sum_{k=2}^{n} \Delta_k = \frac{1}{2}|C(a)|^2 \Delta^{\mathbb{C}^n} - \frac{1}{2} D(a) \Delta_1,$$

where $D(a) = |C(a)|^2 - |A(a)|^2$.

Proof. From (5.1) we deduce that the matrix σ equals

$$\sigma(z) = \begin{bmatrix} A(z_1) & 0 & \ldots & \ldots & 0 \\ z_2\beta(z_1) & C(z_1) & 0 & \ldots & 0 \\ z_3\beta(z_1) & 0 & C(z_1) & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_n\beta(z_1) & 0 & \ldots & \ldots & C(z_1) \end{bmatrix}.$$

When $z = a = (a_1, 0, \ldots, 0)$, we have $z_2 = \ldots = z_n = 0$, so the matrix $\sigma(a)$ is diagonal and

$$\sigma\sigma^*(a) = \begin{bmatrix} |A(a)|^2 & 0 & \ldots & \ldots & 0 \\ 0 & |C(a)|^2 & 0 & \ldots & 0 \\ 0 & 0 & |C(a)|^2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \ldots & |C(a)|^2 \end{bmatrix}.$$

The proposition now follows by the generator formula (5.2). ∎

COROLLARY 5.1. *The Laplace–Beltrami operator has the following form*:

$$\tilde{\Delta}|_a = C^2(a) \Delta^{\mathbb{C}^n} - D(a) \Delta_1.$$

Proof. The formula follows directly from the last proposition and the fact that $L|_a = \frac{1}{2}\tilde{\Delta}|_a$. ∎

Coefficients $C(a)$ and $D(a)$. Now we compute the exact values of the coefficients in $L|_a$ and $\tilde{\Delta}|_a$:

$$C(a) = c_a(0) = \sqrt{1-|a|^2}, \quad A(a) = h'_a(0) = |a|^2 - 1,$$

$$D(a) = C^2(a) - A^2(a) = |a|^2(1-|a|^2).$$

Finally,

$$\tilde{\Delta}|_a = (1-|a|^2)[\Delta^{\mathbb{C}^n}|_a - |a|^2\Delta_1|_a].$$

Recall that $f_a(\lambda) = f(\lambda a)$. The operator $|a|^2\Delta_1|_a$ may be identified as an Euler type second order operator $Ef(a) = \Delta^{\mathbb{C}} f_a(1)$ (Rudin [4] uses this last notation). In fact, by the chain rule ([4], (1.3.3))

$$|a|^2\Delta_1 f(a) = 4\frac{\partial^2}{\partial z\partial \overline{z}}\Big|_{z=1}[f(za)] = \Delta^{\mathbb{C}} f_a(1).$$

Consequently,

$$\tilde{\Delta}|_a = (1-|a|^2)[\Delta^{\mathbb{C}^n}|_a - E|_a].$$

COROLLARY 5.2. *For an arbitrary $b \in B$ the invariant Laplacian is equal to*

$$\tilde{\Delta}|_b = (1-|b|^2)[\Delta^{\mathbb{C}^n}|_b - E|_b].$$

Proof. If $b \in B$ and $\tilde{b} = (|b|, 0, \ldots, 0)$, we consider a unitary transformation u such that $u(\tilde{b}) = b$. All the three operators $\tilde{\Delta}$, $\Delta^{\mathbb{C}^n}$ and E are invariant with respect to the action of u. ■

REMARK 5.1. The operator E is degenerated elliptic. If the starting point is $z \neq 0$, then the diffusion process Z_t generated by E takes only the values from the complex line lin$[z]$. On such a complex line, Z_t is a time-changed 1-dimensional complex Brownian motion, with generator $|z|^2\Delta^{\mathbb{C}}$, the solution of the equation $dZ_t = Z_t dB_t$, where B_t is a complex planar Brownian motion.

By Corollary 5.2, up to a change of time, the complex hyperbolic Brownian motion can be interpreted as a Euclidean Brownian motion in $\mathbb{C}^n$ perturbed by Z_t generated by the operator E.

6. SDE FOR THE BROWNIAN MOTION ON B

Stochastic differential equations for the complex hyperbolic Brownian motion X_t on the upper half space model of $\mathbb{H}^n(\mathbb{C})$ have been given in [2]. In this section we find stochastic differential equations for X_t on the ball model B. This is an important tool for a further study of the process X_t. Let $\mathrm{Herm}^+(n)$ denote the cone of positive definite Hermitian matrices of order n.

LEMMA 6.1. *Let $A(z)$ be a locally bounded Borel function with values in* $\mathrm{Herm}^+(n)$ *and let L be a diffusion generator of the form*

$$L = \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2}{\partial z_j \partial \bar{z}_k}.$$

Let $z_j = u_j + iv_j$ and let the sequence (x_j) be defined as $u_1, v_1, u_2, v_2, \ldots, u_n, v_n$. Then, with respect to the real partial derivatives $\partial/\partial x_j$, the generator L has the following form:

$$L = \frac{1}{4} \sum_{j,k=1}^{2n} \gamma_{jk} \frac{\partial^2}{\partial x_j \partial x_k},$$

where the $2n \times 2n$ real matrix $(\gamma_{jk}) = \Gamma = \tilde{A}$ (defined in the previous section) is symmetric and positive definite.

Proof. When $j = k$, we have

$$\frac{\partial^2}{\partial z_j \partial \bar{z}_j} = \frac{1}{4}\left(\frac{\partial^2}{\partial u_j^2} + \frac{\partial^2}{\partial v_j^2}\right), \qquad a_{jj} \in \mathbb{R} \text{ and } \tilde{a}_{jj} = a_{jj} I_2.$$

The lemma is then obvious.

When $j < k$, we put $a_{jk} = c + id$ and $D_m = \partial/\partial z_m, \bar{D}_m = \partial/\partial \bar{z}_m$. Compute

$$\begin{aligned} 4[a_{jk} D_j \bar{D}_k + \bar{a}_{jk} \bar{D}_j D_k] &= 2\mathrm{Re}\Big((c+id)\big(\mathrm{Re}(D_j \bar{D}_k) + i\mathrm{Im}(D_j \bar{D}_k)\big)\Big) \\ &= 2\big(c\mathrm{Re}(D_j \bar{D}_k) - d\mathrm{Im}(D_j \bar{D}_k)\big). \end{aligned}$$

Since

$$\mathrm{Re}(D_j \bar{D}_k) = \frac{\partial^2}{\partial u_j \partial u_k} + \frac{\partial^2}{\partial v_j \partial v_k} \quad \text{and} \quad \mathrm{Im}(D_j \bar{D}_k) = \frac{\partial^2}{\partial u_j \partial v_k} - \frac{\partial^2}{\partial v_j \partial u_k},$$

it follows that in the matrix Γ of coefficients of second partial derivatives of L the 2×2 block $\Gamma_{(u_j, v_j),(u_k, v_k)}$ corresponding to the rows u_j, v_j and the columns u_k, v_k can be chosen equal to

$$\begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \tilde{a}_{jk} \quad \text{and} \quad \Gamma_{(u_k, v_k),(u_j, v_j)} = \tilde{a}_{jk}^T = \tilde{a}_{kj}. \ \blacksquare$$

COROLLARY 6.1. *Let X_t be a diffusion on $\mathbb{C}^n$ being a solution of the stochastic differential equation $dX_t = \sigma(X_t) dB_t$, where σ is a locally bounded Borel function with values in $M_{\mathbb{C}}(n)$. Set $\sigma\sigma^* = A$. Then the generator of X_t is equal to*

$$L = \sum_{j,k=1}^{n} a_{jk}(z) \frac{\partial^2}{\partial z_j \partial \bar{z}_k}.$$

Proof. By (5.2), the matrix of second order coefficients of L on $\mathbb{R}^{2n}$ is equal to $M = (\sigma\sigma^*)\breve{} = \tilde{A}$. We apply Lemma 6.1. ∎

Let B_t be a standard Brownian motion on $\mathbb{C}^n$ and X_t the complex hyperbolic Brownian motion. Recall ([4], 4.1.3 (ii)) that the generator of X_t is equal to

$$L_X = \frac{1}{2}\tilde{\Delta} = 2(1-|z|^2)\sum_{j,k=1}^{n}(\delta_{jk} - z_j\bar{z}_k)D_j\bar{D}_k.$$

Let us put

$$\lambda(z) = \sqrt{1-|z|^2}, \qquad Z(z) = (z_j\bar{z}_k)_{jk} = zz^*,$$

where z is a column vector $(z_1,\dots,z_n)^T$ and $z^* = (\bar{z}_1,\dots,\bar{z}_n)$. The matrix $A(z)$ of coefficients of $D_j\bar{D}_k$ in this generator can be written in the form

$$A(z) = 2\lambda^2(z)\,(I - Z)\,.$$

In the next theorem we determine two simple systems of SDE verified by the process X_t.

THEOREM 6.1. *The complex hyperbolic Brownian motion X_t is a solution of the following SDE in the matrix form*:

$$dX_t = \sqrt{2}\lambda(X_t)\left(I - \frac{Z(X_t)}{1+\lambda(X_t)}\right)dB_t.$$

It is also a solution of the equation

$$dX_t = \sqrt{2}\lambda(X_t)\left(I - \frac{Z(X_t)}{1-\lambda(X_t)}\right)dB_t.$$

Proof. According to Corollary 6.1, we search for $\sigma \in M_n(\mathbb{C})$ such that $\sigma\sigma^* = A(z)$. It is sufficient to find $s \in M_n(\mathbb{C})$ such that $ss^* = I - Z$.

Observe that $Z^2 = zz^*zz^* = |z|^2 Z$, so the matrices Z^2 and Z are collinear. In such a case it is natural to look for s of the form $s = I - cZ$. Then

$$s^2 = I - 2cZ + c^2|z|^2 Z = I - (2c - |z|^2c^2)Z$$

and we determine c such that $2c - |z|^2c^2 = 1$. There are two solutions

$$c = \frac{1}{|z|^2}(1 \pm \lambda) = \frac{1}{1 \mp \lambda}. \quad ∎$$

Observe that the matrix $\sigma_1 = I - Z/(1+\lambda)$ is Hermitian positive definite, with an eigenvalue $\lambda > 0$ and the eigenvalue 1 of multiplicity $n-1$. This follows from the facts that $Zz = |z|^2 z$ and that the matrix Z is of rank 1. In particular, the matrix $I - Z$ has an eigenvalue $1 - |z|^2 = \lambda^2 > 0$ and the eigenvalue 1 of multiplicity $n-1$. So $\sigma_1 = \sqrt{A}$, the Hermitian positive definite square root of A.

The matrix $\sigma_2 = I - Z/(1-\lambda)$ is also Hermitian, but not positive definite. The matrices σ_1, σ_2 are all Hermitian matrices such that X_t satisfies $dX_t = \sigma dB_t$.

Another form of the matrix σ, important in applications, is triangular. The next theorem gives a lower triangular system of SDE verified by the process X_t.

THEOREM 6.2. *The complex hyperbolic Brownian motion X_t is a solution of the following SDE in the matrix form*:

$$dX_t = \sqrt{2}\,\lambda(X_t)\,\Sigma(X_t)\,dB_t,$$

where $\Sigma(z) = [\sigma_{ij}(z)]_{i,j=1,\ldots,n}$ *with*

(1) $\sigma_{ij}(z) = 0,\ 1 \leqslant i < j \leqslant n$ (*by the definition of the lower triangular matrix*),

(2) $\sigma_{ii}(z) = \dfrac{\sqrt{1 - |z_1|^2 - \ldots - |z_i|^2}}{\sqrt{1 - |z_1|^2 - \ldots - |z_{i-1}|^2}},\ \ i = 1, \ldots, n,$

(3) $\sigma_{ij}(z) = \dfrac{-\bar{z}_j z_i}{\sqrt{(1 - |z_1|^2 - \ldots - |z_{i-1}|^2)(1 - |z_1|^2 - \ldots - |z_i|^2)}},$
$1 \leqslant j < i \leqslant n.$

If $i = 1$, *then in* (2) *we put* $1 - |z_1|^2 - \ldots - |z_{i-1}|^2 = 1$.

P r o o f. According to Corollary 6.1, it is enough to show that $\Sigma\Sigma^* = I - Z$. In order to compute the entries of $\Sigma\Sigma^*$ we need the following lemma:

LEMMA 6.2. *For $m \geqslant 2$ the following identity holds*:

$$\sum_{i=1}^{m-1} \frac{|z_i|^2}{(1 - |z_1|^2 - \ldots - |z_{i-1}|^2)(1 - |z_1|^2 - \ldots - |z_i|^2)} = \frac{|z_1|^2 + \ldots + |z_{m-1}|^2}{1 - |z_1|^2 - \ldots - |z_{m-1}|^2},$$

where for $i = 1$ we put $1 - |z_1|^2 - \ldots - |z_{i-1}|^2 = 1$.

P r o o f. The proof of the lemma follows immediately by an easy observation:

$$\frac{1}{1 - |z_1|^2 - \ldots - |z_i|^2} - \frac{1}{1 - |z_1|^2 - \ldots - |z_{i-1}|^2} = \frac{|z_i|^2}{(1 - |z_1|^2 - \ldots - |z_{i-1}|^2)(1 - |z_1|^2 - \ldots - |z_i|^2)}. \quad \blacksquare$$

Let us compute first the diagonal entries b_{mm} of $\Sigma\Sigma^*$. Using Lemma 6.2, we get for $m = 1, 2, \ldots, n$

$$\begin{aligned} b_{mm} &= \sum_{i=1}^{n} \sigma_{mi}\bar{\sigma}_{mi} = \sum_{i=1}^{m} \sigma_{mi}\bar{\sigma}_{mi} = \sum_{i=1}^{m-1} |\sigma_{mi}|^2 + |\sigma_{mm}|^2 \\ &= \frac{(|z_1|^2 + \ldots + |z_{m-1}|^2)|z_m|^2}{1 - |z_1|^2 - \ldots - |z_{m-1}|^2} + \frac{1 - |z_1|^2 - \ldots - |z_m|^2}{1 - |z_1|^2 - \ldots - |z_{m-1}|^2} = 1 - |z_m|^2. \end{aligned}$$

Now we deal with the off-diagonal entries. By the Hermitian symmetry of $I - Z$, it is enough to consider the case $1 \leqslant j < k \leqslant n$. Using Lemma 6.2 again, we get for such j, k:

$$\begin{aligned} b_{jk} &= \sum_{i=1}^{n} \sigma_{ji}\bar{\sigma}_{ki} = \sum_{i=1}^{j} \sigma_{ji}\bar{\sigma}_{ki} \\ &= \sum_{i=1}^{j-1} \frac{|z_i|^2 z_j \bar{z}_k}{(1 - |z_1|^2 - \ldots - |z_{i-1}|^2)(1 - |z_1|^2 - \ldots - |z_i|^2)} \\ &\quad + \frac{-z_j \bar{z}_k}{1 - |z_1|^2 - \ldots - |z_{j-1}|^2} \\ &= z_j \bar{z}_k \cdot \frac{|z_1|^2 + \ldots + |z_{j-1}|^2}{1 - |z_1|^2 - \ldots - |z_{j-1}|^2} - \frac{z_j \bar{z}_k}{1 - |z_1|^2 - \ldots - |z_{j-1}|^2} = -z_j \bar{z}_k. \ \blacksquare \end{aligned}$$

EXAMPLE. As an ilustration we write down two systems of stochastic differential equations for the case $n = 2$. Let us put $|X(t)|^2 = |X_1(t)|^2 + |X_2(t)|^2$. The first system is the one described in Theorem 6.1:

$$\begin{aligned} \frac{dX_1(t)}{2\sqrt{1 - |X(t)|^2}} &= \left(1 - \frac{|X_1(t)|^2}{1 + \sqrt{1 - |X(t)|^2}}\right) dB_1(t) + \frac{-X_1(t)\bar{X}_2(t)}{1 + \sqrt{1 - |X(t)|^2}} dB_2(t), \\ \frac{dX_2(t)}{2\sqrt{1 - |X(t)|^2}} &= \frac{-\bar{X}_1(t)X_2(t)}{1 + \sqrt{1 - |X(t)|^2}} dB_1(t) + \left(1 - \frac{|X_2(t)|^2}{1 + \sqrt{1 - |X(t)|^2}}\right) dB_2(t). \end{aligned}$$

The second system (triangular) is the one from Theorem 6.2:

$$\begin{aligned} \frac{dX_1(t)}{2\sqrt{1 - |X(t)|^2}} &= \sqrt{1 - |X_1(t)|^2}\, dB_1(t), \\ \frac{dX_2(t)}{2\sqrt{1 - |X(t)|^2}} &= \frac{-\bar{X}_1(t)X_2(t)}{\sqrt{1 - |X_1(t)|^2}}\, dB_1(t) + \frac{\sqrt{1 - |X(t)|^2}}{\sqrt{1 - |X_1(t)|^2}}\, dB_2(t). \end{aligned}$$

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