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OCCUPATION TIME FLUCTUATIONS OF POISSON AND EQUILIBRIUM BRANCHING SYSTEMS IN CRITICAL AND LARGE DIMENSIONS

BY

PIOTR MIŁOŚ* (WARSZAWA)

Abstract. Limit theorems are presented for the rescaled occupation time fluctuation process of a critical finite variance branching particle system in $\mathbb{R}^d$ with symmetric α-stable motion starting off from either a standard Poisson random field or the equilibrium distribution for critical $d = 2\alpha$ and large $d > 2\alpha$ dimensions. The limit processes are generalised Wiener processes. The obtained convergence is in space-time and finite-dimensional distributions sense. Under the additional assumption on the branching law we obtain functional convergence.

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1. INTRODUCTION

The basic object of our investigation is a branching particle system. It consists of particles evolving independently in $\mathbb{R}^d$ according to a spherically symmetric α-stable Lévy process (called a *standard α-stable process*), $0 < \alpha \leqslant 2$. The system starts off at time 0 from a random point measure M. The lifetime of a particle is an exponential random variable with parameter V. After that time the particle splits according to the law determined by a generating function F. We always assume that the branching is critical, i.e., $F'(0) = 1$. Each of the new-born particles undertakes the α-stable movement independently of the others, and so on. The evolution of the system is described by (and in fact can be identified with) the empirical (measure-valued) process N, where $N_t(A)$ denotes the number of particles in the set $A \subset \mathbb{R}^d$ at time t. We define the rescaled occupation time fluctuation

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process by

$$X_T(t) = \frac{1}{F_T} \int_0^{Tt} (N_s - \mathbb{E}N_s)\, ds, \qquad t \geqslant 0, \tag{1.1}$$

where T is a scaling parameter which accelerates time ($T \to +\infty$) and F_T is a proper deterministic norming. X_T is a signed-measure-valued process but it is convenient to regard it as a process in the tempered distributions space $S'(\mathbb{R}^d)$. The objectives are to find suitable F_T such that X_T converges in law as $T \to +\infty$ to a non-trivial limit and to identify this limit. This problem, or its modifications (e.g. its superprocess or discrete versions), has been studied in several papers ([2], [3], [11], [12], the list is not complete). The papers [2] and [3] are of special interest since they cope with a discrete space model similar to ours. In particular, the above papers study the fluctuations of the occupation time at the origin for a critical branching random walk on the d-dimensional lattice, $d \geqslant 3$, also in the equilibrium case. The convergence results drawn by our work are analogous to [2], [3].

Typically, the initial configuration M was a Poisson measure, in most cases a homogeneous one, i.e. with the intensity measure λ being the Lebesgue measure, and the branching law was either binary or of a special form, belonging to the domain of attraction of a $(1+\beta)$ stable distribution $(0 < \beta \leqslant 1)$. We consider a general branching law with finite variance and the initial measure M is either Poisson homogeneous or is the equilibrium measure of the system. In what follows, we will use superscripts $Poiss$ (e.g., N^{Poiss}) or eq (e.g., X_T^{eq}) to indicate which model we are dealing with.

It is known [10] that an equilibrium measure M^{eq} of our branching system exists provided that $d > \alpha$. In [12] the case of intermediate dimensions $\alpha < d < 2\alpha$ was considered. It was shown that the limits (in the sense of the convergence in law in $C\big([0,\tau], S'(\mathbb{R}^d)\big), \tau > 0$) of X_T^{Poiss} and X_T^{eq} are different; they have the form $K\lambda\xi$, where K is a constant and ξ is a real Gaussian process which in the Poisson case is a sub-fractional Brownian motion, while in the equilibrium case it is a fractional Brownian motion (see [5] for the definition and properties of the sub-fractional motion).

This paper may be regarded as an extension of [9]. While both papers consider the case of critical ($d = 2\alpha$) and large ($d > 2\alpha$) dimensions, the presented work considers more general branching law and also studies an equilibrium-starting system. It turns out that now the limits of X^{Poiss} and X^{eq} coincide, for $d = 2\alpha$ the limit is $K\lambda\beta$, where β is the standard Brownian motion, and if $d > 2\alpha$, then the limit is an $S'(\mathbb{R}^d)$-valued Wiener process. Moreover, these limits are, up to a constant, the same as those obtained in [9] for the Poisson system with binary branching. The proof method is based on the so-called space-time approach, similar to that employed in [9], though with some extra technical difficulties. For the sake of brevity we omit most of the calculations. Terms resulting for an equilibrium-starting system were generally more cumbersome (especially for the critical dimension $d = 2\alpha$) and required more careful analysis. Examples of such terms are given in Section 3.2. The finiteness of integrals was proved by using some delicate estimates employing e.g. Young's inequality. Additionally, in Section 3.3 we developed subtle inequalities using e.g. l'Hôpital's rule. The number of terms arising in the proof of tightness (Section 3.1.3) was also a considerable difficulty (see Remark 3.1).

2. RESULTS

As mentioned in the Introduction our state space is the space $S'(\mathbb{R}^d)$ of tempered distributions, dual to the space $S(\mathbb{R}^d)$ of smooth rapidly decreasing functions. Duality in the appropriate spaces is denoted by $\langle\cdot,\cdot\rangle$. Three kinds of convergence are used. Firstly, the *convergence of finite-dimensional distributions*, denoted by $\Rightarrow_f$. For a continuous $S'(\mathbb{R}^d)$-valued process $X=(X_t)_{t\geqslant 0}$ and any $\tau>0$ one can define an $\mathcal{S}'(\mathbb{R}^{d+1})$-valued random variable

$$\langle \tilde{X},\Phi\rangle = \int_0^\tau \langle X_s,\Phi(\cdot,s)\rangle\, ds, \qquad \Phi\in\mathcal{S}(\mathbb{R}^{d+1}). \tag{2.1}$$

If for any $\tau>0$ it follows that $\tilde{X}_n\to\tilde{X}$ in distribution, then we say that the *convergence in the space-time sense* holds and denote this fact by $\Rightarrow_i$. Finally, we consider the *functional weak convergence* denoted by $X_n\Rightarrow_c X$. It holds if for any $\tau>0$ processes $X_n=\big(X_n(t)\big)_{t\in[0,\tau]}$ converge to $X=\big(X(t)\big)_{t\in[0,\tau]}$ weakly in $C\big([0,\tau],\mathcal{S}'(\mathbb{R}^d)\big)$. It is known that $\Rightarrow_i$ and $\Rightarrow_f$ do not imply each other, but either of them together with tightness implies $\Rightarrow_c$ (see [4]). Conversely, $\Rightarrow_c$ implies both $\Rightarrow_i$ and $\Rightarrow_f$.

Consider a branching particle system described in the Introduction. Let us put (recall that F is the generating function of the branching law)

$$m=F''(1). \tag{2.2}$$

We start with the large dimension case.

THEOREM 2.1. *Assume that $d>2\alpha$ and let $F_T=T^{1/2}$. Assume that the initial configuration of the system is given either by a Poisson homogeneous measure or by the equilibrium measure and let X_T be defined by* (1.1)*, i.e. $X_T=X_T^{Poiss}$ or $X_T=X_T^{eq}$. Then*:

(1) *$X_T\Rightarrow_f X$ and $X_T\Rightarrow_i X$ as $T\to+\infty$, where X is a centered $\mathcal{S}'$-valued Gaussian process with the covariance function*

$$\mathrm{Cov}\left(\langle X_s,\varphi_1\rangle,\langle X_t,\varphi_2\rangle\right)=(s\wedge t)\frac{1}{2\pi}\int_{\mathbb{R}^d}\left(\frac{2}{|z|^\alpha}+\frac{Vm}{2|z|^{2\alpha}}\right)\widehat{\varphi_1}(z)\overline{\widehat{\varphi_2}(z)}dz,$$

where $\varphi_1,\varphi_2\in\mathcal{S}\left(\mathbb{R}^d\right)$.

(2) *If, additionally, the branching law has finite fourth moment, then*

$$X_T \Rightarrow_c X \qquad \textit{as } T \to +\infty.$$

For the critical dimension we have the following theorem:

THEOREM 2.2. *Assume that* $d = 2\alpha$ *and let* $F_T = (T \log T)^{1/2}$. *Assume that the initial configuration of the system is given either by a Poisson homogeneous measure or by the equilibrium measure and let* X_T *be defined by* (1.1), *i.e.* $X_T = X_T^{Poiss}$ *or* $X_T = X_T^{eq}$. *Then*:

(1) $X_T \Rightarrow_f X$ *and* $X_T \Rightarrow_i X$ *as* $T \to +\infty$, *where*

$$X = \left(\frac{mV}{2}\right)^{1/2} C_d \lambda \beta, \qquad C_d = \left(2^{d-2}\pi^{d/2} d\Gamma\left(\frac{d}{2}\right)\right)^{-1/2},$$

and β *is a standard Brownian motion.*

(2) *If, additionally, the branching law has finite fourth moment, then*

$$X_T \Rightarrow_c X \qquad \textit{as } T \to +\infty.$$

REMARK 2.1. (a) It is unclear if the assumption of the existence of the fourth moment is necessary for the functional convergence to hold. One can see that only the second moment influences the result. In the proof below the assumption is only used in the proof of tightness of the family X_T (see also Remark 3.1).

(b) The limit process X in Theorem 2.1 is an $S'(\mathbb{R}^d)$-valued homogeneous Wiener process.

3. PROOFS

3.1. General scheme.

3.1.1. *Space-time convergence.* We present a general scheme which will be used in the proofs of both theorems. It is similar to the one employed in [12] and [9]. Many parts of the proofs are the same for N^{Poiss} (the system starting from a Poisson field) and N^{eq} (the system starting from the equilibrium distribution), so we will omit superscripts when a formula holds for both of them. Let X_T be the occupation time fluctuation process defined by (1.1). Firstly we establish the convergence in the space-time sense. Let us consider $\tilde{X}_T$ defined according to (2.1) ($\tau = 1$). We will show the convergence of the Laplace transforms

$$\lim_{T \to +\infty} \mathbb{E}\mathrm{exp}(-\langle \tilde{X}_T, \Phi\rangle) = \mathbb{E}\mathrm{exp}(-\langle \tilde{X}, \Phi\rangle), \qquad \Phi \in \mathcal{S}(\mathbb{R}^{d+1}),\ \Phi \geqslant 0, \tag{3.1}$$

where X is the corresponding limit process. This will imply the weak convergence of $\tilde{X}_T$ since the limit processes are Gaussian ones (see the detailed explanation in [8]). The purpose of the rest of this section is to gather facts used to calculate the Laplace transforms and to show the convergence (3.1). To make the proof shorter we will consider Φ of the special form:

$$\Phi(x,t) = \varphi(x)\psi(t), \quad \varphi \in \mathcal{S}(\mathbb{R}^d), \psi \in \mathcal{S}(\mathbb{R}^+), \varphi \geqslant 0, \psi \geqslant 0.$$

We also put

$$\varphi_T = \frac{1}{F_T}\varphi, \quad \chi(t) = \int_t^1 \psi(s)ds, \quad \chi_T(t) = \chi\left(\frac{t}{T}\right). \tag{3.2}$$

We write

$$\Psi(x,t) = \varphi(x)\chi(t), \tag{3.3}$$

$$\Psi_T(x,t) = \varphi_T(x)\chi_T(t); \tag{3.4}$$

note that Ψ and Ψ_T are positive functions. For a generating function F we define

$$G(s) = F(1-s) - 1 + s. \tag{3.5}$$

We will need the following properties of G (we omit straightforward proofs):

FACT 3.1. 1. $G(0) = F(1) - 1 = 0$.
2. $G'(0) = -F'(1) + 1 = 0$.
3. $G''(0) = F''(1) < +\infty$.
4. $G(v) = (m/2)v^2 + g(v)v^2$, *where the parameter* m *is defined by* (2.2) *and* $\lim_{v\to 0} g(v) = 0$.
5. $G'''(0) < +\infty$ *and* $G^{\mathrm{IV}}(0) < +\infty$ *if the law determined by* F *has finite fourth moment.*

Let us recall the classical Young's inequality

$$\|f * g\|_p \leqslant \|f\|_{q_1}\|g\|_{q_2}, \tag{3.6}$$

which holds when $1/p = 1/q_1 + 1/q_2 - 1,\ q_1, q_2 \geqslant 1$.

Now we introduce an important function used throughout the rest of the paper:

$$v_\Psi(x,r,t) = 1 - \mathbb{E}\exp\Big\{-\int_0^t \langle N_s^x, \Psi(\cdot, r+s)\rangle\, ds\Big\}, \tag{3.7}$$

where N_s^x denotes the empirical measure of the particle system with the initial condition $N_0^x = \delta_x$. The function v_Ψ satisfies the equation

$$\begin{aligned} v_\Psi(x,r,t) = \int_0^t \mathcal{T}_{t-s}\big[&\Psi(\cdot, r+t-s)\big(1 - v_\Psi(\cdot, r+t-s, s)\big) \\ &- VG\big(v_\Psi(\cdot, r+t-s, s)\big)\big](x)\, ds. \end{aligned} \tag{3.8}$$

The equation can be proved by using the Feynman–Kac formula in the same way as Lemma 3.4 in [12]. We also define

$$n_{\Psi}(x,r,t) = \int_0^t \mathcal{T}_{t-s}\Psi(\cdot, r+t-s)(x)\,ds. \tag{3.9}$$

Since we consider only positive Ψ, so (3.7) and (3.8) yield

$$0 \leqslant v_T(x,r,t) \leqslant n_T(x,r,t), \tag{3.10}$$

where, for simplicity of the notation, we write

$$v_T(x,r,t) := v_{\Psi_T}(x,r,t), \tag{3.11}$$

$$n_T(x,r,t) := n_{\Psi_T}(x,r,t), \tag{3.12}$$

$$v_T(x) := v_T(x,0,T), \tag{3.13}$$

$$n_T(x) := n_T(x,0,T), \tag{3.14}$$

when no confusion can arise.

FACT 3.2. *It follows that* $n_T(x, T-s, s) \to 0$ *uniformly in* $x \in \mathbb{R}^d$, $s \in [0,T]$, *as* $T \to +\infty$.

The proof is the same as that of Fact 3.7 in [12].

We also introduce a function V_T which is defined by

$$V_T(x,l) = 1 - \mathbb{E}\exp\big(\langle N_l^x, \ln(1-v_T)\rangle\big) \tag{3.15}$$

and fulfills the equation

$$V_T(x,l) = \mathcal{T}_l v_T(x) - V\int_0^l \mathcal{T}_{l-s}G\big(V_T(\cdot,s)\big)(x)\,ds. \tag{3.16}$$

It satisfies (details can be found in [12], Section 3.2.2)

$$0 \leqslant V_T(x,l) \leqslant \mathcal{T}_l v_T(x) \quad \text{for all } x \in \mathbb{R}^d, l \geqslant 0. \tag{3.17}$$

Now we can write the Laplace transforms (see [12], Sections 3.1.2 and 3.2.2 for calculations)

$$\mathbb{E}\exp(-\langle \tilde{X}_T^{Poiss}, \Phi\rangle) = \exp\big(A(T)\big) \tag{3.18}$$

and

$$\mathbb{E}\exp(-\langle \tilde{X}_T^{eq}, \Phi\rangle) = \exp\big(A(T) + B(T)\big), \tag{3.19}$$

where

$$A(T) = \int_{\mathbb{R}^d} \int_0^T \Psi_T(x, T-s)\, v_T(x, T-s, s) + VG\big(v_T(x, T-s, s)\big) ds dx, \tag{3.20}$$

$$B(T) = V \int_0^{+\infty} \int_{\mathbb{R}^d} G\big(V_T(x,t)\big) dx dt. \tag{3.21}$$

We consider the following decomposition of $A(T)$:

$$A(T) = \exp\big\{V\big(I_1(T) + I_2(T)\big) + I_3(T)\big\}, \tag{3.22}$$

where

$$I_1(T) = \int_0^T \int_{\mathbb{R}^d} \frac{m}{2}\Big(\int_0^s \mathcal{T}_u \Psi_T(\cdot, T+u-s)(x)\, du\Big)^2 dx ds, \tag{3.23}$$

$$I_2(T) = \int_0^T \int_{\mathbb{R}^d} \Big[G\big(v_T(x, T-s, s)\big) - \frac{m}{2}\Big(\int_0^s \mathcal{T}_u \Psi_T(\cdot, T+u-s)(x)\, du\Big)^2\Big] dx ds, \tag{3.24}$$

$$I_3(T) = \int_0^T \int_{\mathbb{R}^d} \Psi_T(x, T-s)\, v_T(x, T-s, s)\, dx ds. \tag{3.25}$$

We claim that in the case of large dimensions ($d > 2\alpha$) we have

$$I_1(T) \to \frac{m}{2(2\pi)^d} \int_0^1 \int_0^1 (r \wedge r')\psi(r)\psi(r') dr dr' \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z)|^2}{|z|^{2\alpha}} dz, \tag{3.26}$$

$$I_2(T) \to 0, \tag{3.27}$$

$$I_3(T) \to \frac{1}{(2\pi)^d} \int_0^1 \int_0^1 (r \wedge r')\psi(r)\psi(r') dr dr' \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z)|^2}{|z|^{\alpha}} dz. \tag{3.28}$$

Using the decomposition (3.22) we obtain the limit of $A(T)$ and, consequently, the one for the Laplace transform (3.18). This establishes the space-time convergence of the Poisson-starting system X_T^{Poiss} considered in (1) of Theorem 2.1. Analogously, in the critical case ($d = 2\alpha$), we obtain the corresponding convergence considered in (1) of Theorem 2.2 once we show

$$I_1(T) \to \frac{m}{2} C_d^2 \int_0^1 \int_0^1 (r \wedge r')\psi(r)\psi(r') \Big(\int_{\mathbb{R}^d} \varphi(x) dx\Big)^2 \tag{3.29}$$

and

$$I_2(T), I_3(T) \to 0. \tag{3.30}$$

The limits (3.26)–(3.30) will be obtained in Sections 3.2 and 3.3.

Now we proceed to the case of the equilibrium-starting system X_T^{eq}. In both Theorems 2.1 and 2.2 the limits are the same as in the X_T^{Poiss} case. It follows immediately from (3.19) that it will be proved when we show

$$B(T) \to 0.$$

Let us first observe an elementary fact that the uniform convergence $V_T(\cdot,\cdot) \to 0$ as $T \to +\infty$ holds. It is a direct consequence of Fact 3.2 and the combination of inequalities (3.17) and (3.10). This together with Fact 3.1 yields

$$B(T) \leqslant c \int_0^{+\infty} \int_{\mathbb{R}^d} \big(\mathcal{T}_t n_T(x)\big)^2 dx dt. \tag{3.31}$$

Let us denote the right-hand side of (3.31) by $B_1(T)$. Now we need to obtain

$$\lim_{T\to+\infty} B_1(T) = 0, \tag{3.32}$$

which is put off to Sections 3.2 and 3.3.

3.1.2. *Finite dimensional convergence.* A similar method, based on the Laplace transform, can be applied to prove the finite distributions convergence. Indeed, for a sequence $0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n \leqslant \tau$ and functions $\varphi_1, \varphi_2, \ldots, \varphi_n \in \mathcal{S}(\mathbb{R}^d)$, $\varphi_i \geqslant 0$, we write the Laplace transform

$$\mathbb{E}\exp\big(\sum_{i=1}^n \langle X_T(t_i), \varphi_i\rangle\big). \tag{3.33}$$

The main observation is that, formally,

$$\sum_{i=1}^n \langle X_T(t_i), \varphi_i\rangle = \langle \tilde{X}_T, \Phi\rangle$$

if $\Phi = \sum_{i=1}^n \varphi_i \delta_{t_i}$ (which corresponds to $\Psi(x,s) = \sum_{i=1}^n \varphi_i(x) \mathbf{1}_{[0,t_i]}(s)$, recall the definition (3.3)).

It turns out that the Laplace transforms (3.18), (3.19) and formulae (3.8), (3.16) are still valid for Φ and Ψ. The proof for the Poisson-starting system is a simpler version of the one presented below and is left to the reader. We employ an approximation argument. Consider $\Phi_n \to \Phi$, where $\Phi_n \in \mathcal{S}(\mathbb{R}^{d+1})$, and additionally assume that the sequence $(\Phi_n)_n$ is chosen such that $\Psi^n(x,t) = \int_t^1 \Phi^n(x,s) ds$ is nondecreasing: $\Psi^n \leqslant \Psi^{n+1}$. To keep the proof short we adhere to the following notation: symbols with (without) the superscript n will denote functions defined for Φ^n and Ψ^n (respectively, Φ and Ψ) (e.g. $v^n := v_{\Psi^n}$ given by (3.8)). T is fixed, and hence is omitted where possible.

The first assertion is that $V(x,l)$ satisfies the equation (3.16). The definition (3.15) implies that $V^n(x,l) \to V(x,l)$ (pointwise), which follows immediately from $v^n \to v$ (left to the reader), the inequality $0 \leqslant v \leqslant 1$ and the dominated convergence theorem. By assumption $\Phi^n \in \mathcal{S}(\mathbb{R}^{d+1})$ and V^n satisfies the equation (3.16). Passing to the limit $n \to +\infty$ and employing the dominated convergence theorem to the right-hand side of the equation complete the proof.

Now we turn to the Laplace transform (3.19). It is obvious that

$$\lim_n \mathbb{E}\exp(-\langle \tilde{X}_T^{eq}, \Phi^n\rangle) = \mathbb{E}\exp(-\langle \tilde{X}_T^{eq}, \Phi\rangle).$$

One can see that formula (3.19) for Φ will be justified if only $A^n \to A$, $B^n \to B$. Proving the first one is left to the reader. It is straightforward to check that $\Phi^n \leqslant \Phi^{n+1}$ implies $V_n \leqslant V_{n+1}$ and that G is nondecreasing. A standard application of the monotone convergence theorem completes the proof. The finite distributions convergence is thus established. Indeed, the above argumentation allows the calculations from Section 3.1.1 to be repeated for $\Phi = \sum_{i=1}^n \varphi_i \delta_{t_i}$, which implies the convergence of the Laplace transform (3.33) and, consequently, the finite dimensional convergence in (1) of Theorems 2.1 and 2.2.

3.1.3. *Functional convergence.* In this subsection we present a general scheme of the proof of the functional convergence. The assertion follows immediately from the part (1) of Theorem 2.1 (Theorem 2.2) if we prove that $\{X_T, T > 2\}$ is tight in $C\big([0,1], \mathcal{S}'(\mathbb{R}^d)\big)$ (with no loss of generality we consider $\tau = 1$). Generally, we follow the lines of the proof of tightness in Theorem 2.2 in [9]. However, in our case new technical difficulties arise because of a more general branching law. Some estimates are more cumbersome and some extra terms appear. Moreover, we establish tightness for X_T^{eq} which was not investigated in [9]. This requires even more intricate computations than in the Poisson case. By the Mitoma theorem (see [13]) it suffices to show tightness of the real processes $\langle X_T, \varphi\rangle$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. This can be done by using the following criterion ([1], Theorem 12.3):

$$\mathbb{E}\big(\langle X_T(t), \varphi\rangle\,, \langle X_T(s), \varphi\rangle\big)^4 \leqslant C(t-s)^2. \tag{3.34}$$

Let $(\psi_n)_n$ be a sequence in $\mathcal{S}(\mathbb{R})$, and put $\chi_n(u) = \int_u^1 \psi_n(s)ds$. It is an easy exercise to show that the sequence $(\psi_n)_n$ can be chosen in such a way that

$$\psi_n \to \delta_t - \delta_s,$$

$$0 \leqslant \chi_n \leqslant \mathbf{1}_{[s,t]}. \tag{3.35}$$

A detailed construction can be found in [9].

Let us put $\Phi_n = \varphi \otimes \psi_n$. We have

$$\lim_{n\to+\infty} \langle X_T, \Phi_n \rangle = \langle X_T(t), \varphi\rangle - \langle X_T(s), \varphi\rangle;$$

thus by Fatou's lemma and the definition of ψ_n we will obtain (3.34) if we prove (C is a constant independent of n and T) that

$$\mathbb{E}\langle \tilde{X}_T, \Phi_n\rangle^4 \leqslant C(t-s)^2.$$

From now on we fix an arbitrary n and define $\Phi := \Phi_n$ and $\chi := \chi_n$. By properties of the Laplace transform we have

$$\mathbb{E}\langle \tilde{X}_T, \Phi\rangle^4 = \frac{d^4}{d\theta^4}\bigg|_{\theta=0} \mathbb{E}\exp(-\theta\langle \tilde{X}_T, \Phi\rangle).$$

Hence the proof of tightness will be completed if we show

$$\frac{d^4}{d\theta^4}\bigg|_{\theta=0} \mathbb{E}\exp(-\theta\langle \tilde{X}_T, \Phi\rangle) \leqslant C(t-s)^2. \tag{3.36}$$

The rest of the section is devoted to calculate the fourth derivative of the Laplace transforms (3.18) and (3.19). Here and subsequently $A(\theta,T)$ and $B(\theta,T)$ will denote (3.20) and (3.21) taken for $\Psi_{\theta,T} = \theta\varphi_T \otimes \chi_T$ (φ_T and χ_T are defined in (3.2)), i.e.,

$$A(\theta,T) = \int_{\mathbb{R}^d}\int_0^T \theta\varphi_T(x)\chi_T(T-s)v_{\Psi_{\theta,T}}(x,T-s,s) + VG\big(v_{\Psi_{\theta,T}}(x,T-s,s)\big)dsdx,$$

$$B(\theta,T) = V\int_0^{+\infty}\int_{\mathbb{R}^d} G\left(V_{\Psi_{\theta,T}}(x,t)\right)dxdt.$$

REMARK 3.1. This is the point where we need the existence of the fourth moment of the branching law. Note that in the case of the binary branching law (the model investigated in [9]) the fourth moment is obviously finite. The formulae derived below are consistent, but more complicated than the ones considered in [9]. This makes the computation here significantly longer and, moreover, some new technical difficulties arise especially in the case of critical dimensions. New arguments and estimations were required to cope with them.

A trivial verification shows that $A(0,T) = 0$, $A'(0,T) = 0$, $B(0,T) = 0$, $B'(0,T) = 0$. Hence

$$\frac{d^4}{d\theta^4}\bigg|_{\theta=0} \exp\big(A(\theta,T)\big) = A^{\mathrm{IV}}(0,T) + A''(0,T)^2,$$

$$\left.\frac{d^4}{d\theta^4}\right|_{\theta=0} \exp\big(A(\theta,T)+B(\theta,T)\big)$$
$$= A^{\mathrm{IV}}(0,T)+B^{\mathrm{IV}}(0,T)+\big(A''(0,T)+B''(0,T)\big)^2.$$

Now taking into account (3.36), to prove tightness, it suffices to show that

$$A''(0,T)\leqslant C(t-s),\quad B''(0,T)\leqslant C(t-s), \tag{3.37}$$

$$A^{\mathrm{IV}}(0,T)\leqslant C(t-s)^2,\quad B^{\mathrm{IV}}(0,T)\leqslant C(t-s)^2. \tag{3.38}$$

It will be convenient to put

$$v(\theta)=v(\theta)(x,T-u,u)=v_{\Psi_{\theta,T}}(x,T-u,u),$$
$$V(\theta)=V(\theta)(x,t)=V_{\Psi_{\theta,T}}(x,T-u,u),$$
$$k=G'''(0),\quad l=G^{\mathrm{IV}}(0).$$

Using the properties from Fact 3.1 we obtain

$$A''(0,T)=2\int_0^T\int_{\mathbb{R}^d}\varphi_T(x)\chi_T(T-u)v'(0)dxdu+Vm\int_0^T\int_{\mathbb{R}^d}\big(v'(0)\big)^2dxdu$$

$$\begin{aligned}A^{\mathrm{IV}}(0,T)=&\,4\int_0^T\int_{\mathbb{R}^d}\varphi_T(x)\chi_T(T-u)v'''(0)dxdu+Vl\int_0^T\int_{\mathbb{R}^d}\big(v'(0)\big)^4dxdu\\&+6Vk\int_0^T\int_{\mathbb{R}^d}\big(v'(0)\big)^2v''(0)dxdu+3Vm\int_0^T\int_{\mathbb{R}^d}\big(v''(0)\big)^2dxdu\\&+4Vm\int_0^T\int_{\mathbb{R}^d}v'(0)v'''(0)dxdu.\end{aligned}$$

Similarly,

$$B''(0,T)=Vm\int_0^T\int_{\mathbb{R}^d}\big(V'(0)\big)^2dsdx,$$

(3.39)

$$\begin{aligned}B^{\mathrm{IV}}(0,T)=&\,Vl\int_0^{+\infty}\int_{\mathbb{R}^d}\big(V'(0)\big)^4dxds+6Vk\int_0^{+\infty}\int_{\mathbb{R}^d}V''(0)\big(V'(0)\big)^2dxds\\&+3Vm\int_0^{+\infty}\int_{\mathbb{R}^d}\big(V''(0)\big)^2dxds+4Vl\int_0^{+\infty}\int_{\mathbb{R}^d}V'(0)V'''(0)dxds.\end{aligned}$$

Derivatives of $v(\theta)$ and $V(\theta)$ at $\theta=0$ are given by

$$v'(0)(x,T-u,u)=\int_0^u \mathcal{T}_{u-s}[\varphi_T(\cdot)\chi_T(T-s)](x)ds, \tag{3.40}$$

$$\begin{aligned}v''(0)(x,T-u,u)=&-2\int_0^u \mathcal{T}_{u-s}[\varphi_T(\cdot)\chi_T(T-s)v'(0)(\cdot,T-s,s)](x)ds\\&-mV\int_0^u \mathcal{T}_{u-s}\big[\big(v'(0)(\cdot,T-s,s)\big)^2\big](x)ds,\end{aligned}$$

$$\begin{aligned}v'''(0)(x,T-u,u)=&-3\int_0^u \mathcal{T}_{u-s}[\varphi_T(\cdot)\chi_T(T-s)v''(0)(\cdot,T-s,s)](x)ds\\&-kV\int_0^u \mathcal{T}_{u-s}\big[\big(v'(0)(\cdot,T-s,s)\big)^3\big](x)ds\\&-3mV\int_0^u \mathcal{T}_{u-s}[v'(0)(\cdot,T-s,s)v''(0)(\cdot,T-s,s)](x)ds,\\V'(0)(x,s)=&\,\mathcal{T}_s v'(0)(x,0,T),\end{aligned}$$

$$V''(0)(x,s)=\mathcal{T}_s v''(0)(x,0,T)-Vm\int_0^s \mathcal{T}_{t-u}\left(\left(V'(0)(\cdot,u)\right)^2\right)du, \tag{3.41}$$

$$\begin{aligned}&V^{\mathrm{IV}}(0)(x,s)=\\&\mathcal{T}_s v^{\mathrm{IV}}(0)(x,0,T)-V\int_0^s \mathcal{T}_{t-u}\Big(3mV'(0)(\cdot,u)V''(0)(\cdot,u)+k\left(V'''(0)(\cdot,u)\right)^3\Big)du.\end{aligned}$$

3.2. Proof of Theorem 2.1. We follow the scheme described in Section 3.1.1 for the large dimensions case. I_1 does not depend on F, so (3.26) can be obtained in the same way as (3.15) in [9].

We will turn now to (3.27) which is a little more intricate. Combining (3.24) and the decomposition of G from Fact 3.1 we obtain

$$I_2(T)=\frac{m}{2}I_{21}(T)+I_{22}(T), \tag{3.42}$$

where

(3.43)
$$I_{21}(T)=\int_0^T\int_{\mathbb{R}^d} v_T(x,T-s,s)^2-\Big(\int_0^s \mathcal{T}_u\Psi_T(\cdot,T+u-s)(x)\,du\Big)^2dxds,$$

$$I_{22}(T)=\int_0^T\int_{\mathbb{R}^d} g\big(v_T(x,T-s,s)\big)v_T(x,T-s,s)^2\,dxds. \tag{3.44}$$

We have the following inequalities (proofs are straightforward and can be found in [12], Section 3.1.3):

$$\begin{aligned}(3.45)\quad & 0 \leqslant n_T(x, T-s, s) - v_T(x, T-s, s) \\ & \leqslant C \int_0^s \mathcal{T}_{s-u}[\Psi_T(\cdot, T-u)\, n_T(\cdot, T-u, u) + n_T(\cdot, T-u, u)^2](x)\, du,\end{aligned}$$

$$n_T(x, T-s, s) + v_T(x, T-s, s) \leqslant 2n_T(x, T-s, s). \tag{3.46}$$

By (3.43) we have

$$0 \leqslant -I_{21}(T) \leqslant$$
$$\int_0^T \int_{\mathbb{R}^d} \big(n_T(x, T-s, s) - v_T(x, T-s, s)\big)\big(n_T(x, T-s, s) + v_T(x, T-s, s)\big) ds dx.$$

Using (3.45), (3.46) and (3.9) we obtain

$$-I_{21}(T) \leqslant C\big(I_{211}(T) + I_{212}(T)\big),$$

where

$$\begin{aligned} I_{211}(T) = \int_0^T \int_{\mathbb{R}^d} &\Big(\int_0^s \mathcal{T}_{s-u}\left[\Psi_T(\cdot, T-u)\, n_T(\cdot, T-u, u)\right](x)\, du\Big) \\ &\times \Big(\int_0^s \mathcal{T}_{s-u} \Psi_T(\cdot, T-u)(x) du\Big) dx ds,\end{aligned}$$

$$\begin{aligned} I_{212}(T) = \int_0^T \int_{\mathbb{R}^d} &\Big(\int_0^s \mathcal{T}_{s-u}[n_T(\cdot, T-u, u)^2](x)\, du\Big) \\ &\times \Big(\int_0^s \mathcal{T}_{s-u} \Psi_T(\cdot, T-u)(x) du\Big) dx ds.\end{aligned}$$

One can see that I_{211} and I_{212} coincide with J_1 and J_2 from [9] (see (3.20) and (3.21)). Hence by the proof therein we get

$$\lim_{T\to+\infty} I_{21}(T) = 0.$$

Next we show that $I_{22} \to 0$. Indeed, applying Facts 3.1 and 3.2 and the inequality (3.10) we see that for all $\epsilon > 0$ there exists T_0 such that for all $T > T_0$

$$0 \leqslant I_{22}(T) \leqslant \epsilon I_1(T),$$

which clearly implies $I_{22} \to 0$.

Finally we obtain (3.28). $I_3(T)$ can be split in the same way as (3.24) in [9]. The only difference is that

$$I_3'''(T) = \int_0^T \int_{\mathbb{R}^d} \varphi_T(x)\chi_T(T-u) \int_0^u \mathcal{T}_{u-s} G\big(v_{\Psi_T}(\cdot, T-s, s)\big)(x) ds dx du,$$

but $G(v)$ is comparable with v^2, so the rest of the proof goes along the same lines (see (3.27) in [9]).

Now we turn to the equilibrium case. As observed before, it suffices to prove (3.32). Using the Fourier transforms we get

$$B_1(T) = C \int_{\mathbb{R}^d} \frac{1}{|z|^\alpha} \big(\widehat{n}_T(z)\big)^2 dz$$

It is not hard to see that

$$|\widehat{n}_T(z)| \leqslant \frac{CT^{1-\beta/\alpha}}{F_T} \frac{|\widehat{\varphi}(z)|}{|z|^\beta}, \qquad \beta \in [0, \alpha].$$

Hence we obtain

$$|B_1(T)| \leqslant C \frac{T^{2(1-\beta/\alpha)}}{F_T^2} \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z)|^2}{|z|^\alpha} \frac{1}{|z|^{2\beta}} dz.$$

We take β such that $\frac{1}{2}\alpha < \beta$ but $\alpha + 2\beta < d$ (it can be done because $2\alpha < d$). The first condition gives us

$$\frac{T^{2(1-\beta/\alpha)}}{F_T^2} \to 0 \qquad \text{as } T \to +\infty,$$

and the second ensures that the integral is finite. This completes the proof of (3.32) and, consequently, part (1) of Theorem 2.1.

Now we proceed to part (2). Firstly, we follow the scheme from Section 3.1.3. The proof will be completed when we show inequalities (3.37) and (3.38). It can be done by applying the expressions derived in Section 3.1.3 repeatedly. This results in many terms which have to be estimated separately. As an example consider (3.39). Take only its third term, then substitute $V''(0, T)$ in it utilizing only the second term of (3.41), and finally eliminate $v'(0, T)$ using (3.40). In this way we obtain

$$R = \int_{\mathbb{R}^d} \int_0^{+\infty} \Big(\int_0^l \mathcal{T}_{l-s_1} \Big[\big[\mathcal{T}_{s_1} \big(\int_0^T \mathcal{T}_{T-s_3} [\varphi_T(\cdot)\chi_T(T-s_3)] ds_3 \big)^2 \big] \Big] ds_1 \Big)^2 dl dx.$$

Other terms can be derived analogously. They can be estimated in a similar way to that in [9] though some new difficulties arise and the number of terms is substantially bigger. To obtain estimates we need the following inequalities:

$$\int_u^1 \exp\big(-T(r-u)|z|^{\alpha}\big)\chi(r)dr \leqslant t-s, \quad 0 \leqslant u \leqslant 1, \tag{3.47}$$

$$\int_0^1\int_u^1 \exp\big(-T(r-u)|z|^{\alpha}\big)\chi(r)drdu \leqslant \frac{t-s}{T|z|^{\alpha}}, \tag{3.48}$$

$$\int_0^u \exp\big(-T(u-s)|z|^{\alpha}\big)du \leqslant \frac{1-\exp\left(-T|z|^{\alpha}\right)}{T|z|^{\alpha}}, \tag{3.49}$$

which are easily proved using the inequality (3.35).

Now, to illustrate techniques required in estimations, we will carry out the proof for the term R which is perhaps the most impressive one. Firstly, we apply the Fubini theorem multiple times in order to separate the "time part" and the "space part":

$$R = \int_0^{+\infty}\int_0^l\int_0^T\int_0^T\int_0^l\int_0^T\int_0^T \chi_T(T-s_3)\chi_T(T-s_4)\chi_T(T-s_5) \\ \times \chi_T(T-s_6)\, S\, ds_6ds_5ds_2ds_4ds_3ds_1dl,$$

where

$$S = \int_{\mathbb{R}^d} \mathcal{T}_{l-s_1}\Big[\mathcal{T}_{s_1}\big[\mathcal{T}_{T-s_3}\left[\varphi_T(\cdot)\right]\big]\mathcal{T}_{s_1}\big[\mathcal{T}_{T-s_4}\left[\varphi_T(\cdot)\right]\big]\Big] \\ \times \mathcal{T}_{l-s_2}\Big[\mathcal{T}_{s_2}\big[\mathcal{T}_{T-s_5}\left[\varphi_T(\cdot)\right]\big]\mathcal{T}_{s_2}\big[\mathcal{T}_{T-s_6}\left[\varphi_T(\cdot)\right]\big]\Big]dx.$$

Applying the Plancharel formula and the definition (3.2) we get

$$S = T^{-2}\int_{\mathbb{R}^{3d}} \exp\big(-(l-s_1)|z|^{\alpha} - s_1|z_1|^{\alpha} - (T-s_3)|z_1|^{\alpha} - s_1|z-z_1|^{\alpha} - s_2|z_2|^{\alpha}\big) \\ \times \exp\big(-(T-s_4)|z-z_1|^{\alpha} - (l-s_2)|z|^{\alpha} - (T-s_5)|z_2|^{\alpha} - s_2|z-z_2|^{\alpha}\big) \\ \times \exp\big(-(T-s_6)|z-z_2|^{\alpha}\big)\widehat{\varphi}(z_1)\widehat{\varphi}(z-z_1)\widehat{\varphi}(z_2)\widehat{\varphi}(z-z_2)dz_2dz_1dz.$$

The Fubini theorem yields

$$R = T^{-2}\int_{\mathbb{R}^{3d}} \widehat{\varphi}(z_1)\widehat{\varphi}(z-z_1)\widehat{\varphi}(z_2)\widehat{\varphi}(z-z_2) \\ \times \int_0^{+\infty}\int_0^l\int_0^T\int_0^T\int_0^l\int_0^T\int_0^T A ds_6ds_5ds_2ds_4ds_3ds_1dldz_2dz_1dz,$$

where

$$A = \exp\left(-(l-s_1)|z|^\alpha - s_1|z_1|^\alpha - (T-s_3)|z_1|^\alpha - s_1|z-z_1|^\alpha - (l-s_2)|z|^\alpha\right)$$
$$\times \exp\left(-(T-s_4)|z-z_1|^\alpha - (T-s_5)|z_2|^\alpha - s_2|z-z_2|^\alpha - (T-s_6)|z-z_2|^\alpha\right)$$
$$\times \exp\left(s_2|z_2|^\alpha\right)\chi_T(T-s_3)\chi_T(T-s_4)\chi_T(T-s_5)\chi_T(T-s_6).$$

A subsequent application of inequalities (3.47), (3.49) to integrals with respect to s_6, s_5, s_4, s_3 gives

(3.50)
$$R \leqslant (t-s)^2 \int_{\mathbb{R}^{3d}} \widehat{\varphi}(z_1)\widehat{\varphi}(z-z_1)\widehat{\varphi}(z_2)\widehat{\varphi}(z-z_2)\frac{1}{|z_2|^\alpha}\frac{1}{|z_1|^\alpha}S(z,z_1,z_2)dz_2dz_1dz,$$

where

$$S(z,z_1,z_2) = \int_0^{+\infty}\int_0^l\int_0^l \exp\left(-(l-s_1)|z|^\alpha - s_1|z_1|^\alpha - s_1|z-z_1|^\alpha\right) \times \exp\left(-(l-s_2)|z|^\alpha - s_2|z_2|^\alpha - s_2|z-z_2|^\alpha\right)ds_2ds_1dl. \tag{3.51}$$

A trivial verification shows that

$$S(z,z_1,z_2) = S_1(z,z_1,z_2) + S_2(z,z_1,z_2),$$

where

$$S_1(z,z_1,z_2) = \left(2\,|z|^\alpha\,(|z_1|^\alpha + |z-z_1|^\alpha + |z|^\alpha)(|z_1|^\alpha + |z-z_1|^\alpha + |z_2|^\alpha + |z-z_2|^\alpha)\right)^{-1},$$

$$S_2(z,z_1,z_2) = \left(2\,|z|^\alpha\,(|z_2|^\alpha + |z-z_2|^\alpha + |z|^\alpha)(|z_1|^\alpha + |z-z_1|^\alpha + |z_2|^\alpha + |z-z_2|^\alpha)\right)^{-1}.$$

Using the above considerations we write the right-hand side of (3.50) as $R_1 + R_2$, where R_1, R_2 have an obvious meaning. It is easy to see that

$$R_1 = (t-s)^2 \int_{\mathbb{R}^{3d}} \frac{\widehat{\varphi}(z_1)\widehat{\varphi}(z-z_1)}{|z_1|^\alpha\,|z-z_1|^{\alpha/2}}\frac{\widehat{\varphi}(z_2)\widehat{\varphi}(z-z_2)}{|z_2|^\alpha\,|z-z_2|^\alpha}\frac{1}{2|z|^{(3/2)\alpha}}dz_1dz_2dz.$$

Notice that the function $f(x) = \widehat{\varphi}(x)/|x|^\alpha$ is square-integrable. The integral with respect to z_2 is equal to $(f * f)(z)$. By Young's inequality (3.6) it is easy to see that it is bounded (take $q_1 = q_2 = 2$). Hence

$$R_1 \leqslant c_1(t-s)^2 \int_{\mathbb{R}^d} \frac{h(z)}{2|z|^{3/2\alpha}}dz,$$

where

$$h(z) = \int_{\mathbb{R}^d} \frac{\widehat{\varphi}(z_1)\widehat{\varphi}(z-z_1)}{|z_1|^\alpha |z-z_1|^{\alpha/2}} dz_1 = \left(\frac{\widehat{\varphi}(\cdot)}{|\cdot|^\alpha} * \frac{\widehat{\varphi}(\cdot)}{|\cdot|^{\alpha/2}} \right)(z).$$

We may apply Young's inequality (3.6) in two ways. Firstly, taking $q_1 = 2/3$ and $q_2 = 3$ proves that h is bounded; secondly, taking $q_1 = q_2 = 1$ shows that h is integrable. Hence

$$R_1 \leqslant c_2(t-s)^2.$$

The proof for R_2 goes along the same lines.

3.3. Proof of Theorem 2.2. As the proof for the critical dimensions in the Poisson-starting system case is similar to the one in Section 3.2, we present only a sketch of the proof. Once again we follow the scheme described in Section 3.1.1. The convergence (3.29) can be obtained in the same way as (3.31) in [9]. To prove the convergence (3.30) of $I_2(T)$ one can follow the proof for the large dimension case and estimate the arising terms I_{211} and I_{212} in a manner presented in [9] for J_1 and J_2 in the critical case. The limit for the I_3 is trivial.

Now we turn to the equilibrium case. We need to show (3.32). We have

$$\begin{aligned} B_1(T) = \int_0^{+\infty} \int_{\mathbb{R}^d} \big(\mathcal{T}_t \int_0^T \mathcal{T}_{T-s_1}\varphi_T(x)\chi_T(T-s_1)ds_1\big) \\ \times \big(\mathcal{T}_t \int_0^T \mathcal{T}_{T-s_2}\varphi_T(x)\chi_T(T-s_2)ds_2\big) dx dt \\ = \int_0^{+\infty}\int_0^T\int_0^T \chi_T(T-s_1)\chi_T(T-s_2) \int_{\mathbb{R}^d} \mathcal{T}_{t+T-s_1}\varphi_T(x)\mathcal{T}_{t+T-s_2}\varphi_T(x) dx ds_1 ds_2 dt. \end{aligned}$$

Applying the Fourier transform we obtain

$$\begin{aligned} B_1(T) = \frac{1}{(2\pi)^d} \int_0^{+\infty}\int_0^T\int_0^T \chi_T(T-s_1)\chi_T(T-s_2) \\ \times \int_{\mathbb{R}^d} \exp\big((t+T-s_1)|z|^\alpha + (t+T-s_2)|z|^\alpha\big) |\widehat{\varphi}_T(z)|^2 dz ds_1 ds_2 dt. \end{aligned}$$

Integrating with respect to t yields

$$B_1(T) = c_1 \frac{A_T}{F_T^2}, \tag{3.52}$$

where

$$A_T = \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z)|^2}{|z|^\alpha} \big(\int_0^T \exp(s|z|^\alpha)\, ds\big)^2 dz.$$

The derivative of A_T with respect to T is given by

$$A'_T = 2 \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z)|^2}{|z|^\alpha} \exp\left(-T|z|^\alpha\right) \frac{1-\exp\left(-T|z|^\alpha\right)}{|z|^\alpha} dz.$$

In the critical case $\alpha = d/2$, so substituting $T^{2/d}z = z'$ we obtain

$$A'_T = 2 \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z'/T^{2/d})|^2}{|z'|^\alpha} \exp(-|z'|^\alpha) \frac{1-\exp\left(-|z'|^\alpha\right)}{|z'|^\alpha} dz.$$

The term $\big(1-\exp\left(-|z'|^\alpha\right)\big)/|z'|^\alpha$ is bounded and $\big(\exp\left(-|z'|^\alpha\right)\big)/|z'|^\alpha$ is integrable, and hence there exists a constant c_2 such that

$$A'_T \leqslant c_2.$$

We obtain the limit of $B_1(T)$ using l'Hôpital's rule ($F_T^2 = T\log T$):

$$\lim_T B_1(T) = c_1 \lim_T \frac{A'_T}{(F_T^2)'} \leqslant \lim_T \frac{c_3}{\log T + 1} = 0.$$

This completes the proof of part (1). To show part (2) we follow, similarly to the proof of Theorem 2.1, the scheme from Section 3.1.3. In the same way we evaluate the terms arising from (3.37) and (3.38). Although the techniques of estimating them are similar to the ones presented in [9] we deal with more terms. To shorten the notation we introduce

$$\mathrm{Ex}(x) = 1 - \exp(-x). \tag{3.53}$$

We need the following estimates:

$$\frac{1}{\log T} \int_{\mathbb{R}^d} \frac{f(z)}{|z|^{2\alpha}} \mathrm{Ex}(T|z|^\alpha) dz \leqslant c(f) \tag{3.54}$$

for f bounded and integrable;

$$\frac{1}{\log T} \int_{\mathbb{R}^d} \frac{\widehat{\varphi}(z-z_1)}{|z-z_1|^\alpha} \mathrm{Ex}(T|z-z_1|^\alpha) \frac{\mathrm{Ex}(T|z_1|^\alpha)}{|z_1|^\alpha} dz_1 \leqslant c(\varphi) \tag{3.55}$$

for φ rapidly decreasing;

$$\frac{1}{\log T} \int_{\mathbb{R}^d} \frac{\widehat{\varphi}(z-z_1)}{|z-z_1|^\alpha} \mathrm{Ex}(T|z-z_1|^\alpha) \frac{\widehat{\varphi}(z_1)}{|z_1|^\alpha} \mathrm{Ex}(-T|z_1|^\alpha) dz_1 \leqslant f(z) \tag{3.56}$$

where f is integrable and bounded.

Inequalities (3.54) and (3.55) follow easily from l'Hôpital's rule. To show (3.56) it suffices to observe that boundedness is a direct consequence of (3.55). The fact that $f \in \mathcal{L}^1$ follows from Young's inequality applied to

$$\int_{\mathbb{R}^d} \frac{\widehat{\varphi}(z-z_1)}{|z-z_1|^\alpha} \frac{\widehat{\varphi}(z_1)}{|z_1|^\alpha} dz_1.$$

Finally, to illustrate problems arising in the critical dimension case, we show one example. Let us take the fourth term in $B^{\mathrm{IV}}(0)$ (see (3.39))

$$\int_0^{+\infty} \int_{\mathbb{R}^d} V'''(0)(x,l)V'(0)(x,l)dxdl.$$

One of the terms resulting from its evaluation is

$$R = \int_{\mathbb{R}^d} \int_0^{+\infty} \mathcal{T}_l \Big[\int_0^T \mathcal{T}_{T-s_1} [v'(0)(x,s_1) \int_0^{s_1} \mathcal{T}_{s_1-s_2}[v'(0)(x,s_2)v'(x,s_2)]ds_2 \big] ds_1 \Big] \\ \times \mathcal{T}_l[v'(0)(x,T)](x)dldx.$$

We substitute $v'(0)$ and change the order of integration:

$$R = \int_0^{+\infty} \int_0^T \int_0^{s_1} \int_0^{s_1} \int_0^{s_2} \int_0^{s_2} \int_0^T \chi_T(T-s_5)\chi_T(T-s_3) \\ \times \chi_T(T-s_4)\chi_T(T-s_6) S ds_6 ds_4 ds_3 ds_2 ds_5 ds_1 dl,$$

where

$$S = \int_{\mathbb{R}^d} \mathcal{T}_l \Big\{ \mathcal{T}_{T-s_1} \Big[\mathcal{T}_{s_1-s_5} [\varphi_T(\cdot)] \, \mathcal{T}_{s_1-s_2} \big[\mathcal{T}_{s_2-s_3} [\varphi_T(\cdot)] \, \mathcal{T}_{s_2-s_4} [\varphi_T(\cdot)] \big] \Big] \Big\} \\ \times \mathcal{T}_l \big[\mathcal{T}_{T-s_6} [\varphi_T(\cdot)] \big] dx.$$

Applying the Fourier transform we obtain

$$S = \int_{\mathbb{R}^{3d}} \exp\big(-l|z|^\alpha - (T-s_1)|z|^\alpha - (s_1-s_5)|z_1|^\alpha - (s_1-s_2)|z-z_1|^\alpha\big) \\ \times \exp\big(-(s_2-s_3)|z_2|^\alpha - l|z|^\alpha - (s_2-s_4)|z-z_1-z_2|^\alpha - (T-s_6)|z|^\alpha\big) \\ \times \widehat{\varphi}(z_1)\widehat{\varphi}(z_2)\widehat{\varphi}(z-z_1-z_2)\widehat{\varphi}(z)dz_2dz_1dz.$$

Once again we change the order of integration:

$$R = T^{-2}\log T^{-2} \int_{\mathbb{R}^{3d}} \widehat{\varphi}(z_1)\widehat{\varphi}(z_2)\widehat{\varphi}(z-z_1-z_2)\widehat{\varphi}(z) Q dz_2 dz_1 dz, \tag{3.57}$$

where

$$Q = \int_0^{+\infty} \int_0^T \int_0^{s_1} \int_0^{s_1} \int_0^{s_2} \int_0^{s_2} \int_0^T \exp\left(-2l|z|^\alpha - (T-s_1)|z|^\alpha - (s_1-s_5)|z_1|^\alpha\right)$$
$$\times \exp\left(-(T-s_6)|z|^\alpha\right)$$
$$-(s_1-s_2)|z-z_1|^\alpha - (s_2-s_3)|z_2|^\alpha - (s_2-s_4)|z-z_1-z_2|^\alpha$$
$$\times \chi_T(T-s_5)\chi_T(T-s_3)\chi_T(T-s_4)\chi_T(T-s_6)ds_6ds_4ds_3ds_2ds_5ds_1dl.$$

Applying the inequality (3.47) to the integral with respect s_6 we get

$$Q \leqslant c_1 T(t-s) \int_0^{+\infty} \int_0^T \int_0^{s_1} \int_0^{s_1} \int_0^{s_2} \int_0^{s_2} \exp\left(-2l|z|^\alpha - (T-s_1)|z|^\alpha - (s_1-s_5)|z_1|^\alpha\right)$$
$$\times \exp\left(-(s_1-s_2)|z-z_1|^\alpha - (s_2-s_3)|z_2|^\alpha - (s_2-s_4)|z-z_1-z_2|^\alpha\right)$$
$$\times \chi_T(T-s_5)\chi_T(T-s_3)\chi_T(T-s_4)ds_4ds_3ds_2ds_5ds_1dl.$$

Next we utilise (3.49) to eliminate the integral with respect to s_5:

$$Q \leqslant c_2 T(t-s)\frac{\mathrm{Ex}(T|z_1|^\alpha)}{|z_1|^\alpha} \int_0^{+\infty} \int_0^T \int_0^{s_1} \int_0^{s_2} \int_0^{s_2} \exp\left(-2l|z|^\alpha - (T-s_1)|z|^\alpha\right)$$
$$\times \exp\left(-(s_1-s_2)|z-z_1|^\alpha - (s_2-s_3)|z_2|^\alpha - (s_2-s_4)|z-z_1-z_2|^\alpha\right)$$
$$\times \chi_T(T-s_3)\chi_T(T-s_4)ds_4ds_3ds_2ds_1dl.$$

Once again we use (3.47) this time to the integral with respect to s_4:

$$Q \leqslant c_3 (t-s)^2 T^2 \frac{\mathrm{Ex}(T|z_1|^\alpha)}{|z_1|^\alpha} \int_0^{+\infty} \int_0^T \int_0^{s_1} \int_0^{s_2} \exp\left(-2l|z|^\alpha - (T-s_1)|z|^\alpha\right)$$
$$\times \exp\left(-(s_1-s_2)|z-z_1|^\alpha - (s_2-s_3)|z_2|^\alpha\right)\chi_T(T-s_3)ds_3ds_2ds_1dl.$$

Finally we apply (3.49) to the integrals with respect to s_3, s_2, s_1 consequently and integrate with respect to l:

$$Q \leqslant c_4 (t-s)^2 T^2 \frac{\mathrm{Ex}(T|z_1|^\alpha)}{|z_1|^\alpha} \frac{1}{|z_2|^\alpha} \frac{\mathrm{Ex}(T|z-z_1|^\alpha)}{|z-z_1|^\alpha} \frac{\mathrm{Ex}(T|z|^\alpha)}{|z|^{2\alpha}}.$$

We return to (3.57) and we obtain

$$R \leqslant c_5(t-s)^2 \log T^{-2} \int_{\mathbb{R}^{3d}} \widehat{\varphi}(z_1)\widehat{\varphi}(z_2)\widehat{\varphi}(z-z_1-z_2)\widehat{\varphi}(z)$$
$$\times \frac{1}{|z_1|^\alpha}\left[1-\exp(-T|z_1|^\alpha)\right] \frac{1}{|z_2|^\alpha}\frac{1}{|z-z_1|^\alpha}\left[1-\exp(-T|z-z_1|^\alpha)\right]$$
$$\times \frac{1}{|z|^{2\alpha}}\left[1-\exp(-T|z|^\alpha)\right] dz_2 dz_1 dz.$$

The integral with respect to z_2 is bounded:

$$R \leqslant c_6(t-s)^2 \log T^{-2} \int_{\mathbb{R}^{2d}} \frac{\widehat{\varphi}(z_1)}{|z_1|^\alpha}\mathrm{Ex}(T|z_1|^\alpha)\frac{\mathrm{Ex}(T|z-z_1|^\alpha)}{|z-z_1|^\alpha}\frac{\widehat{\varphi}(z)}{|z|^{2\alpha}}\mathrm{Ex}(T|z|^\alpha)dz_1 dz.$$

Using the inequality (3.55) we obtain

$$R \leqslant c_7(t-s)^2 \log T^{-1} \int_{\mathbb{R}^d} \frac{\widehat{\varphi}(z)}{|z|^{2\alpha}}\mathrm{Ex}(T|z|^\alpha)dz.$$

We complete the proof by applying (3.54) and arriving at

$$R \leqslant c_8(t-s)^2.$$

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Insitute of Mathematics
Polish Academy of Sciences
Śniadeckich 8, Warsaw, Poland
E-mail: pmilos@mimuw.edu.pl

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