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LIMIT THEOREMS FOR STOCHASTIC DYNAMICAL SYSTEM ARISING IN ISING MODEL ANALYSIS

BY

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Abstract. A simple stochastic dynamical system defined on the space of doubly-infinite sequences of real numbers is considered. Limit theorems for this system are proved. The results are applied to the physical model of wetting of the flat heterogeneous wall.

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1. INTRODUCTION

In this paper we investigate the asymptotic behaviour of a Markov chain whose state space is the set of doubly-infinite sequences of real numbers. We define it as a random dynamical system. Suppose that $F_1, F_2, \ldots$ is a sequence of independent identically distributed random transformations. The Markov chain $\{Z^{(n)}\}$ is defined by choosing the initial condition $Z^{(0)}$ and by setting

$$Z^{(n)} = F_n \circ F_{n-1} \circ \ldots \circ F_1(Z^{(0)}),$$

where $\circ$ denotes composition. In our case F_1 with probability p transforms a sequence $\{x_k\}_{-\infty}^{+\infty}$ into the sequence $\{y_k = \max(x_k, x_{k+1})\}_{-\infty}^{+\infty}$ and with probability $1-p$ into the sequence $\{y_k = \min(x_k, x_{k+1})\}_{-\infty}^{+\infty}$.

The paper is organized in the following way. The main results dealing with asymptotics of the sequence $\{Z^{(n)}\}$ are given in Section 2, for the case $p = \frac{1}{2}$, in Theorems 2.1 and 2.3, and in Section 3, for the case $p \neq \frac{1}{2}$, in Theorem 3.1. The main result of Section 4 is the concavity property of the wall tension known as Cassie's law. Theorem 2.1 states that for $p = \frac{1}{2}$, assuming that the sequence

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$\{X_k^2\}_{-\infty}^{+\infty}$ is uniformly integrable, the terms of the sequence $\{Z_k^{(n)}\}$ come close to each other as the number of iterations tends to infinity. This means that if a weak limit of $\{Z_k^{(n)}\}$ exists, it is distributed on constant sequences. Theorem 3.1 states that for $p \neq \frac{1}{2}$ and a stationary sequence $\{X_k\}_{-\infty}^{+\infty}$ such that

$$P\{\sup_k |X_k| < \infty\} = 1$$

the terms of the sequence $\{Z_k^{(n)}\}$ come close to each other as the number of iterations tends to infinity.

Section 4 is devoted to applications of the obtained results to the analysis of wetting on heterogeneous substrata. In order to describe the wetting phenomenon let us consider a binary system which can be in two different phases called $+phase$ and $-phase$. The physical parameters are chosen in such a way that the system is in the coexistence region. Let us introduce a horizontal wall in the system which adsorbs preferentially one of the phases. There are two possibilities. In the first situation we can observe the formation of a microscopic thick film of adsorbing phase between the wall and the second phase. This is a *partial wetting* of the wall. In the second situation the thickness of the film is macroscopic. This is a *complete wetting* of the wall. This phenomenon as well as related one can be analyzed in terms of the semi-infinite Ising model with an external field acting only on the spin on one "surface". If the temperature of the system is below the critical temperature of the Ising model, then the bulk is in the coexistence region. We may force the system to be in the $-phase$ by choosing an appropriate boundary condition. On the other hand, by applying a positive external field the wall will adsorb preferentially the $+phase$. In this situation the competition between a $-phase$ in the bulk and a $+phase$ adsorbed at the wall leads to wetting. Numerous studies have been devoted to discuss different aspects of the statistical mechanics of surface phenomena of wetting the wall. The further details of this topic may be found, for example, in the review papers [1], [2], [4], [11] and references quoted therein. Here we are interested in study of the influence of heterogeneities on wetting. We prove the inequalities which generalize the heuristic equation known among chemists as Cassie's law (see, e.g., [5] or [2]).

2. ISOTROPIC CASE

Let $\{X_k\}_{-\infty}^{+\infty}$ and $\{Y_n\}_1^{\infty}$ be mutually independent sequences of random variables defined on a common probability space. We assume additionally that $\{Y_n\}_1^{\infty}$ is a sequence of 0-1 valued i.i.d. random variables. This will be assumed in the whole paper.

Denote by $\{Z_k^{(n)}\}_{k=-\infty}^{+\infty}$ the sequences of random variables defined by the sequences $\{X_k\}_{-\infty}^{+\infty}$ and $\{Y_n\}_1^{\infty}$ and the following recurrence relation:

$$
\begin{aligned}
&Z_k^{(0)} = X_k, \qquad -\infty < k < \infty,\\
&Z_k^{(n)} = \begin{cases} \min\{Z_k^{(n-1)},\, Z_{k+1}^{(n-1)}\} & \text{if } Y_n = 1,\\ \max\{Z_k^{(n-1)},\, Z_{k+1}^{(n-1)}\} & \text{if } Y_n = 0.\end{cases}
\end{aligned}
\tag{2.1}
$$

The following theorem states that with probability one the terms of the sequence $Z_k^{(n)}$ come close to each other when n tends to infinity.

THEOREM 2.1. *Assume that* $P\{Y_n = 1\} = \frac{1}{2}$ *and the sequence* $\{X_k^2\}_{-\infty}^{+\infty}$ *is uniformly integrable. Then for every* $0 < \varepsilon < 1$ *and* $k, l \in \mathbb{Z}$ *we have*

$$
P\{|Z_k^{(n)} - Z_l^{(n)}| > \varepsilon\} \leqslant \frac{\delta_n}{\varepsilon}|k-l|, \tag{2.2}
$$

where $\delta_n \to 0$ *as* $n \to \infty$ *and* δ_n *is independent of* ε. *Moreover, if for some constant* $C < \infty$ *we have* $P\{\sup_k |X_k| \leqslant C\} = 1$, *then*

$$
P\{|Z_k^{(n)} - Z_l^{(n)}| > \varepsilon\} \leqslant 2C\frac{|k-l|}{\varepsilon\sqrt{n}}. \tag{2.3}
$$

The proof of Theorem 2.1 is based on the following lemmas. For a fix real number r and $k > l$, let

$$
q_n^{(r)}(k,l) = P\{Z_k^{(n)} < r \leqslant Z_l^{(n)}\} + P\{Z_l^{(n)} < r \leqslant Z_k^{(n)}\}. \tag{2.4}
$$

LEMMA 2.1. *The quantities* $q_n^{(r)}(k,l)$ *satisfy*

$$
\begin{aligned}
q_n^{(r)}(k,l) &\leqslant \frac{1}{2}q_{n-1}^{(r)}(k+1,l) + \frac{1}{2}q_{n-1}^{(r)}(k,l+1),\\
q_n^{(r)}(k+1,k) &= \frac{1}{2}q_{n-1}^{(r)}(k+2,k).
\end{aligned}
$$

The second lemma is a simple consequence of the Chebyshev inequality.

LEMMA 2.2. *Assume that the sequence* $\{X_k^2\}_1^{\infty}$ *is uniformly integrable. Then there exists a function* $\delta(c)$ *such that* $\delta(c) \to 0$ *when* $c \to \infty$ *and*

$$
\sup_{k \geqslant 1} P\left(|X_k| > \delta(c)\sqrt{c}\right) \leqslant \frac{\delta(c)}{c}.
$$

P r o o f o f L e m m a 2.1. Let $A_{k,l}^{(n,r)}$ denote the following event:

$$
\{Z_k^{(n)} < r \leqslant Z_l^{(n)}\} \cup \{Z_l^{(n)} < r \leqslant Z_k^{(n)}\}.
$$

For $l-k \geqslant 2$ we have

$$A_{k,l}^{(n,r)} \subset A_{k+1,l}^{(n-1,r)} \cup A_{k,l+1}^{(n-1,r)}.$$

Furthermore,

$$\begin{aligned} P(A_{k,l}^{(n,r)} | A_{k+1,l}^{(n-1,r)} \setminus A_{k,l+1}^{(n-1,r)}) &= \frac{1}{2}, \\ P(A_{k,l}^{(n,r)} | A_{k,l+1}^{(n-1,r)} \setminus A_{k+1,l}^{(n-1,r)}) &= \frac{1}{2}, \\ P(A_{k,l}^{(n,r)} | A_{k+1,l}^{(n-1,r)} \cap A_{k,l+1}^{(n-1,r)}) &\leqslant 1. \end{aligned}$$

Hence by the total probability rule we have

$$\begin{aligned} q_n^{(r)}(k,l) &\leqslant \frac{1}{2} P(A_{k+1,l}^{(n-1,r)} \setminus A_{k,l+1}^{(n-1,r)}) + \frac{1}{2} P(A_{k,l+1}^{(n-1,r)} \setminus A_{k+1,l}^{(n-1,r)}) \\ &\quad + P(A_{k+1,l}^{(n-1,r)} \cap A_{k,l+1}^{(n-1,r)}) \\ &= \frac{1}{2} q_{n-1}^{(r)}(k+1,l) + \frac{1}{2} q_{n-1}^{(r)}(k,l+1). \end{aligned}$$

For $l-k=1$ we have $A_{k,k+1}^{(n,r)} \subset A_{k,k+2}^{(n-1,r)}$ and

$$P(A_{k,k+1}^{(n,r)} | A_{k,k+2}^{(n-1,r)}) = \frac{1}{2}.$$

Hence

$$q_n^{(r)}(k,k+1) = \frac{1}{2} q_{n-1}^{(r)}(k,k+2).$$

This completes the proof of Lemma 2.1 ■

P r o o f o f L e m m a 2.2. By the Chebyshev inequality we have

$$d\,P(|X_k| > \sqrt{d}) \leqslant \int_{\sqrt{d}}^{+\infty} x^2 dF_{|X_k|}(x).$$

By uniform convergence there is a continuous strictly increasing function $h(d)$ such that $h(d) \to \infty$ when $d \to \infty$ and

$$d\,P(|X_k| > \sqrt{d}) \leqslant \frac{1}{h(d)^3}. \tag{2.5}$$

Let

$$c = dh(d)^2 \quad \text{and} \quad \delta(c) = \frac{1}{h(d)}.$$

Then $\delta(c) \to 0$ when $c \to \infty$. Substituting $d = c\delta(c)^2$ into (2.5) gives

$$c\delta(c)^2 P\left(|X_k| > \delta(c)\sqrt{c}\right) \leqslant \delta(c)^3. \quad ■$$

P r o o f o f T h e o r e m 2.1. First we prove (2.3). Let $\{u_n(k,l),\ k,\ l \geqslant 0\}$ be a sequence of matrices defined in the following way:

$$
\begin{aligned}
&u_0(k,l) = 0 && \text{for } k \leqslant l,\\
&u_0(k,l) = q_0^{(r)}(k,l) && \text{for } k > l,\\
&u_n(k,l) = 0 && \text{for } k < l,\\
&u_n(k,l) = \frac{1}{2}u_{n-1}(k+1,l) + \frac{1}{2}u_{n-1}(k,l+1) && \text{for } k \geqslant l.
\end{aligned}
\tag{2.6}
$$

With this notation by Lemma 2.1 we have

$$q_n^{(r)}(k,l) \leqslant u_n(k,l). \tag{2.7}$$

Now iterating n times the equality (2.6) we get

$$u_n(k,l) = \frac{1}{2^n} \sum_{i<\frac{1}{2}(n+k-l)} d_i^{(n,k,l)} u_0(k+n-i, l+i), \tag{2.8}$$

where the coefficient $d_i^{(n,k,l)}$ is the number of paths of length n from point (k,l) to point $(k+n-i, l+i)$ which are lying under the main diagonal for a walk with steps only one unit to the right or one unit up. By the reflection principle (see [8], Chapter III.2) we obtain

$$d_i^{(n,k,l)} = \begin{cases} \dbinom{n}{i} - \dbinom{n}{l+i-k} & \text{if } \ l+i-k \geqslant 0,\\[2ex] \dbinom{n}{i} & \text{if } \ l+i-k < 0. \end{cases} \tag{2.9}$$

Since $u_0(k,l) \leqslant 1$, we get

$$
\begin{aligned}
u_n(l+1,l) \leqslant \frac{1}{2^n} \sum_{i<\frac{1}{2}(n+1)} d_i^{(n,l+1,l)} &= \frac{1}{2^n} \sum_{i<\frac{1}{2}(n+1)} \left[\binom{n}{i} - \binom{n}{i-1}\right]\\
&= \frac{1}{2^n}\binom{n}{[(n+1)/2]} \leqslant \frac{1}{\sqrt{n}}.
\end{aligned}
$$

The last inequality follows easily by Stirling's formula. Hence by (2.7) we obtain

$$q_n^{(r)}(l+1,l) \leqslant u_n(k,l) \leqslant \frac{k-l}{\sqrt{n}} \quad \text{for } k \geqslant l+1. \tag{2.10}$$

For fixed $\varepsilon > 0$ let $N = [2C/\varepsilon]$. Then

$$\{|Z_k^{(n)} - Z_l^{(n)}| > \varepsilon\} = \bigcup_{i=1}^{N} A_{k,l}^{(n,i\varepsilon-C)}.$$

Thus
$$P\{|Z_k^{(n)} - Z_l^{(n)}| > \varepsilon\} \leqslant \frac{N(k-l)}{\sqrt{n}} \leqslant \frac{2C(k-l)}{\varepsilon\sqrt{n}}.$$
This proves (2.3) of Theorem 2.1.

In order to show (2.2), let us assume that $k = l + m$ and notice that $Z_l^{(n)}$ and $Z_{l+m}^{(n)}$ depend only on $X_l, X_{l+1}, \ldots, X_{l+m+n+1}$. Hence
$$\{|Z_l^{(n)}| > \delta_n\sqrt{n}\} \cup \{|Z_{l+m}^{(n)}| > \delta_n\sqrt{n}\} \subset \bigcup_{i=0}^{n+m+1} \{|X_{l+i}| > \delta_n\sqrt{n}\},$$
and by Lemma 2.2 there exists a sequence δ_n such that $\delta_n \to 0$ as $n \to \infty$, and
$$P\Big(\bigcup_{i=0}^{n+m} \{|X_{l+i}| > \delta_n\sqrt{n}\}\Big) \leqslant (n+m+1)\frac{\delta_n}{n}. \tag{2.11}$$
Now from (2.11) and (2.3) we infer that for any $0 < \varepsilon < 1$
$$\begin{aligned}
&P\{|Z_{l+m}^{(n)} - Z_l^{(n)}| > \varepsilon\} \\
&\leqslant P\{|Z_{l+m}^{(n)} - Z_l^{(n)}| > \varepsilon,\ \max_{0\leqslant i\leqslant n+m} |X_{l+i}| \leqslant \delta_n\sqrt{n}\} + \left(1 + \frac{m+1}{n}\right)\delta_n \\
&\leqslant \frac{2m\delta_n\sqrt{n}}{\varepsilon\sqrt{n}} + \left(1 + \frac{m+1}{n}\right)\delta_n \\
&\leqslant \frac{2m\delta_n}{\varepsilon} + (m+2)\delta_n \leqslant \frac{5m\delta_n}{\varepsilon}.
\end{aligned}$$
This completes the proof of Theorem 2.1. ∎

REMARK 2.1. The estimate (2.10), which immediately implies (2.3), is sharp. Indeed, let X_k be a sequence of constant random variables such that
$$X_k = \begin{cases} 0 & \text{for } k \leqslant n, \\ 1 & \text{for } k > n. \end{cases}$$
Then
$$q_{2n}^{(1/2)}(1,0) = P\left(Z_{n+1}^{(2n)} < \frac{1}{2} \leqslant Z_n^{(2n)}\right) = \frac{1}{2^{2n}}\binom{2n}{n} \geqslant \frac{1}{2\sqrt{n}}.$$

REMARK 2.2. We can derive a more explicit estimate in (2.2). Indeed, by assumptions there exists a continuous strictly decreasing function $g(x)$ such that $g(x) \to 0$ when $x \to +\infty$, and
$$\int_{\sqrt{a}}^{+\infty} x^2 dF_{|X_k|}(x) \leqslant g(a)$$

for all $k \in \mathbb{Z}$ and $a \in \mathbb{R}$. Fix $\alpha > 0$ and let

$$h(d) = \min\{g(d^{1/2})^{-1/3},\, d^{\alpha}\}.$$

Then using the notation from the proof of Lemma 2.2 gives

$$\frac{1}{h(d)^3} \leqslant \int\limits_{\sqrt{a}}^{+\infty} x^2 dF_{|X_k|}(x), \qquad c = dh(d)^2 \leqslant d^{1+2\alpha},$$

$$\delta(c) = \frac{1}{h(d)} \leqslant \frac{1}{h(c^{1/(1+2\alpha)})} = \max\{g(c^{1/(1+2\alpha)}),\, c^{-\alpha/(1+2\alpha)}\}.$$

Thus by (2.2) we get

$$P\{|Z_k^{(n)} - Z_l^{(n)}| > \varepsilon\} \leqslant \frac{1}{\varepsilon} \max\{g(c^{1/(1+2\alpha)}),\, c^{-\alpha/(1+2\alpha)}\}.$$

Now we determine whether the random variables $Z_k^{(n)}$ have limit distribution when n tends to infinity. Let

$$G_k^{(n)}(r) = P(Z_k^{(n)} < r);$$

then the following result holds.

LEMMA 2.3. *For any $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $r \in \mathbb{R}$ we have*

$$G_k^{(n)}(r) = \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} G_k^{(0)}(r).$$

P r o o f. Let $B_k^{(n,r)} = \{Z_k^{(n)} < r\}$. Then

$$B_k^{(n,r)} \subset B_k^{(n-1,r)} \cup B_{k+1}^{(n-1,r)}.$$

Moreover, we have

$$P(B_k^{(n,r)} | B_k^{(n-1,r)} \setminus B_{k+1}^{(n-1,r)}) = \frac{1}{2},$$
$$P(B_k^{(n,r)} | B_{k+1}^{(n-1,r)} \setminus B_k^{(n-1,r)}) = \frac{1}{2},$$
$$P(B_k^{(n,r)} | B_k^{(n-1,r)} \cap B_{k+1}^{(n-1,r)}) = 1.$$

Thus

$$\begin{aligned} G_k^{(n)}(r) &= \frac{1}{2} P(B_{k+1}^{(n-1,r)} \setminus B_k^{(n-1,r)}) + \frac{1}{2} P(B_{k+1}^{(n-1,r)} \setminus B_k^{(n-1,r)}) \\ &\quad + P(B_k^{(n-1,r)} \cap B_{k+1}^{(n-1,r)}) \\ &= \frac{1}{2} G_k^{(n-1)}(r) + \frac{1}{2} G_{k+1}^{(n-1)}(r). \end{aligned}$$

Now the lemma follows by induction from the above formula. ∎

As a direct consequence of Lemma 2.3 we get the following result.

THEOREM 2.2. *Assume that random variables in the sequence* $\{X_k\}_{-\infty}^{+\infty}$ *have common probability distribution function* F. *Then for every* $n \in \mathbb{N}$ *and* $k \in \mathbb{Z}$ *random variables* $Z_k^{(n)}$ *have probability distribution function* F.

EXAMPLE 2.1. Assume that the sequence $\{X_k\}_{-\infty}^{+\infty}$ is deterministic, which means that $\{X_k\}_{-\infty}^{+\infty} = \{a_k\}_{-\infty}^{+\infty}$, where $\{a_k\}_{-\infty}^{+\infty}$ is a sequence of real numbers. We would like to determine for which sequences $\{a_k\}_{-\infty}^{+\infty}$ the variables $Z_k^{(n)}$ have limit distribution when $n \to \infty$. Let us choose a sequence $\{a_k\}_{-\infty}^{+\infty}$ in the following way. First fix a probability measure μ on $\mathbb{R}$ and then pick up the number a_k at random with respect to probability measure μ. Assume that the choices for different k's are independent. In other words, X_k are independent random variables with common distribution μ. We will show that for almost all choices of the sequence $\{a_k\}_{-\infty}^{+\infty}$ the variables $Z_k^{(n)}$ have limit distribution μ when $n \to \infty$.

Let f be a continuous and bounded function on $\mathbb{R}$. Then for a fixed choice of a sequence $\{a_k\}_{-\infty}^{+\infty}$

$$Ef(Z_k^{(n)}) = \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} f(a_{k+i}).$$

According to [12], the sum in this formula is the Euler summation of order 1. The random variables $f(X_k)$ are independent and identically distributed, so by [6], Theorem 4, with probability 1 we have

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} f(X_{k+i}) = Ef(X_0) \equiv \int_{-\infty}^{+\infty} f(x)d\mu(x).$$

EXAMPLE 2.2. Assume again that $\{X_k(\omega)\}_{-\infty}^{+\infty}$ is a deterministic 0-1 sequence. Then the sequences $Z_k^{(n)}$ take also the values 0 or 1. By Theorem 2.1 we get

$$P(Z_k^{(n)} \neq Z_l^{(n)}) \to 0 \quad \text{as } n \to \infty.$$

We can identify doubly-infinite 0-1 sequences with the interval $[0, 1]$. Then, by reasoning made in Example 2.1, the distribution of $Z_k^{(n)}$ tends to $\frac{1}{2}(\delta_0 + \delta_1)$ for almost every 0-1 sequence with respect to the Lebesgue measure.

EXAMPLE 2.3. The random variables $\{X_k(\omega)\}_{-\infty}^{+\infty}$ can be unbounded, but it is necessary that the second moments EX_k^2 are uniformly bounded for Theorem 2.1 to hold. Indeed, let X_k be constantly equal to k. Then $Z_k^{(n)} = k + S_n$, where $S_n = \sum_{i=1}^{n} Y_i$. Hence

$$Z_k^{(n)} - Z_l^{(n)} = k - l,$$

which contradicts the conclusion of Theorem 2.1.

When we are interested in evolution of the whole sequence $\mathbf{Z}^{(n)} = \{Z_k^{(n)}\}_{-\infty}^{+\infty}$ instead of its fixed coordinates, it is convenient to use the following description. Let θ denote the shift transformation which transforms the sequence $\mathbf{x} = \{x_k\}_{-\infty}^{+\infty}$ into the sequence $\theta\mathbf{x}$ whose kth coordinate is equal to x_{k+1}. Let $\mathbf{x} \vee \mathbf{y}$ and $\mathbf{x} \wedge \mathbf{y}$ denote sequences whose kth coordinates are equal to $\max\{x_k,\, y_k\}$ and $\min\{x_k,\, y_k\}$, respectively. Define transformation F on $\mathbb{R}_{-\infty}^{\infty}$ by the equality

$$F(\mathbf{x}, i) = \begin{cases} \mathbf{x} \vee \theta\mathbf{x} & \text{if } i = 1, \\ \mathbf{x} \wedge \theta\mathbf{x} & \text{if } i = 0. \end{cases} \tag{2.12}$$

Now the sequence $\mathbf{Z}^{(n)}$ may be interpreted as the random dynamical system

$$\mathbf{Z}^{(n)} = F_n \circ F_{n-1} \circ \ldots \circ F_1(\mathbf{X}),$$

where $F_i(\mathbf{x}) = F(\mathbf{x}, Y_i)$ and $\mathbf{X} = \{X_k\}_{-\infty}^{+\infty}$ is an initial state of the system. Limit behaviour of this system can be determined immediately from Theorems 2.1 and 2.2.

THEOREM 2.3. *Assume that* $\{X_k\}_{-\infty}^{+\infty}$ *is a sequence of identically distributed random variables with finite second moment. Then*, *as* $n \to \infty$,

$$\mathbf{Z}^{(n)} \xrightarrow{\mathcal{D}} \mathbf{Z} \quad \textit{in } \mathbb{R}_{-\infty}^{\infty}, \tag{2.13}$$

where $\mathbf{Z}$ *is a random element of* $\mathbb{R}_{-\infty}^{\infty}$, *with equal coordinates*, *i.e.* $Z_k = Z_l$ *for all* $k, l \in \mathbb{Z}$.

Proof. Assume that X_k has the same distribution μ for all k. Then, by Theorem 2.2, for all k the random variable $Z_k^{(n)}$ is distributed according to μ. By Theorem 2.1 we see that for any $k, l \in \mathbb{Z}$

$$(Z_l^{(n)}, Z_l^{(n)} - Z_{l+1}^{(n)}, \ldots, Z_l^{(n)} - Z_{l+k}^{(n)}) \xrightarrow{\mathcal{D}} (X_0, 0, \ldots, 0) \quad \text{as } n \to \infty. \tag{2.14}$$

Now, applying the continuous mapping theorem ([3], Theorem 5.1) to (2.14) and to the mapping

$$h(x_1, x_2, \ldots, x_l) = (x_1, x_1 - x_2, \ldots, x_1 - x_l)$$

we get

$$(Z_l^{(n)}, Z_{l+1}^{(n)}, \ldots, Z_{l+k}^{(n)}) \xrightarrow{\mathcal{D}} (X_0, X_0, \ldots, X_0) \quad \text{as } n \to \infty.$$

Let $\mathbf{Z}$ denote the random element of the space $\mathbb{R}_{-\infty}^{\infty}$ for which all coordinates are equal to the same random variable X_0. Since weak convergence in $\mathbb{R}^{\infty}$ is equivalent to convergence of finite-dimensional distributions, we prove (2.13). ∎

3. ANISOTROPIC CASE

In this section we will admit random variables Y_n for which $P\{Y_n = 1\} \neq P\{Y_n = 0\}$. Methods which we use allow us to consider stationary bounded sequences of random variables X_k.

THEOREM 3.1. *Let $P\{Y_n = 1\} = p$, where $p < \frac{1}{2}$, and $\{X_k\}_{-\infty}^{+\infty}$ be a stationary sequence of bounded random variables. Then for every $\varepsilon > 0$ and $k, l \in \mathbb{Z}$ we have*

$$\lim_{n\to\infty} P\{|Z_k^{(n)} - Z_l^{(n)}| > \varepsilon\} = 0. \tag{3.1}$$

Proof. First observe that for every n the random variables $Z_k^{(n)}$ are identically distributed. Define the quantities $q_n^{(r)}(l+1, l)$ as in (2.4) and let

$$G_l^{(n)}(r) = P(Z_l^{(n)} < r).$$

Let

$$A = \{Z_l^{(n)} < r\} \quad \text{and} \quad B = \{Z_{l+1}^{(n)} < r\}.$$

By the relation between $Z_l^{(n+1)}$, $Z_l^{(n)}$ and $Z_{l+1}^{(n)}$ we get

$$G_l^{(n+1)}(r) = P(A \cap B) + p\,[P(A \setminus B) + P(B \setminus A)].$$

On the other hand, we have

$$q_n^{(r)}(l+1, l) = P(A \setminus B) + P(B \setminus A),$$
$$G_l^{(n)}(r) = P(A) = P(B).$$

Using the formula

$$P(A \cap B) = \frac{1}{2}\,[P(A) + P(B)] - \frac{1}{2}\,[P(A \setminus B) + P(B \setminus A)]$$

we get

$$q_n^{(r)}(l+1, l) = \frac{2}{1-2p}\big(G_l^{(n)}(r) - G_l^{(n+1)}(r)\big). \tag{3.2}$$

Thus for any r and l the sequence $G_l^{(n)}(r)$ is non-increasing relative to n. Hence it is convergent and

$$\sum_{n=0}^{\infty} q_n^{(r)}(l+1, l) = \frac{2}{1-2p}\big(G_l^{(0)}(r) - \lim_{n\to\infty} G_l^{(n)}(r)\big).$$

In particular, $q_n^{(r)}(l+1, l) \to 0$ when $n \to \infty$. This implies $q_n^{(r)}(k, l) \to 0$ when $n \to \infty$ for every k and l. Proceeding as in the proof of Theorem 2.1 we conclude that (3.1) holds. ■

4. APPLICATION TO WETTING IN THE ISING MODEL

Consider a plane substrate made of two species **a** and **b**. Assume that the species **a** and **b** show up at the surface with concentration p and $1-p$, respectively, in an ordered or disordered fashion. Let us model a fluid phase on top of the substrate by a semi-infinite ferromagnetic Ising model, and represent the interaction of the fluid with the compound substrate by a boundary field taking two different values: a where there is **a**, and b where there is **b**. The wetting properties of the system are described by the differential wall tension $\Delta\tau$. Let us denote by $\Delta\tau(\mathbf{a})$ and $\Delta\tau(\mathbf{b})$ the differential wall tensions associated with pure **a** and pure **b**, respectively. In paper [7] it has been proved that for periodic or random substrates, in any dimension, under the condition $a+b\geqslant 0$ the following convexity property holds:

$$\Delta\tau \geqslant p\Delta\tau(\mathbf{a}) + (1-p)\Delta\tau(\mathbf{b}). \tag{4.1}$$

In this section, using results of the previous section, we prove this inequality for the two-dimensional Ising system and for more general external field.

First we describe our model and recall some definitions (for more details see [7] or [9]). Consider the two-dimensional Ising model with the nearest-neighbour interactions defined on the semi-infinite lattice $\mathbf{L}\subset\mathbb{Z}^2$, $\mathbf{L}=\mathbb{Z}\times\mathbb{Z}^+$. The points of $\mathbf{L}$ are denoted by (i,j), where $i\in\mathbb{Z}$ and $j\in\mathbb{Z}^+$. For each $(i,j)\in\mathbf{L}$, $\sigma(i,j)=\pm1$ denotes an Ising spin. The system of spins $\sigma^\Lambda=\{\sigma(i,j)\}_{(i,j)\in\Lambda}$, confined in a box Λ,

$$\Lambda=\{i\in\mathbf{L}\colon |i|\leqslant L,\, 0\leqslant j\leqslant M\},$$

interacts by the Hamiltonian

$$\begin{aligned} &H_\Lambda(\{\sigma^\Lambda\}) \\ &= -\sum_{j=1}^{M}\Big[\sum_{i=-(L+1)}^{L} J\sigma(i,j)\sigma(i+1,j) + \sum_{i=-L}^{L} J\sigma(i,j)\sigma(i,j+1)\Big] - \sum_{i=-L}^{L} h_i\sigma(i,1). \end{aligned} \tag{4.2}$$

The right-hand side spin of (4.2) contains spins which lie outside the box Λ. We treat them as the boundary condition. The partition function associated with Hamiltonian H_Λ is defined as

$$\mathcal{Z}_\Lambda(J,h)=\sum_{\{\sigma^\Lambda\}}\exp\big(-\beta H_\Lambda(\{\sigma^\Lambda\})\big).$$

Now we define the differential wall tension. This notion has been studied by Fröhlich and Pfister [9], [10] for uniform substrates. Let us recall some definitions and results. Consider two Hamiltonians $H_\Lambda^{(+)}(\{\sigma^\Lambda\})$ and $H_\Lambda^{(-)}(\{\sigma^\Lambda\})$ which differ only by the choice of boundary conditions. For the first one we choose $+$boundary condition ($\sigma_i=+1$ for $i\in\Lambda^c$), while for the second we choose $-$boundary condition ($\sigma_i=-1$ for $i\in\Lambda^c$). These boundary conditions will be denoted simply by $+$ and $-$, respectively, while partition functions associated with these Hamiltonians will be denoted by $\mathcal{Z}_\Lambda(J,h,+)$ and $\mathcal{Z}_\Lambda(J,h,-)$, respectively. The surface field $\mathbf{h}=\{h_i\}_{i\in W}$ describes the properties of the wall. The wall adsorbs preferentially the $+phase$ or $-phase$ at the point i according to the sign of h_i.

The differential wall tension $\Delta\tau_\Lambda(\mathbf{h})$ is first defined in finite volume, by the formula

$$\Delta\tau_\Lambda(\mathbf{h})=-\frac{1}{\beta(2L+1)}\ln\frac{\mathcal{Z}_\Lambda(J,h,-)}{\mathcal{Z}_\Lambda(J,h,+)},$$

and

$$\Delta\tau(\mathbf{h})=\lim_{L\to\infty}\lim_{M\to\infty}\Delta\tau_\Lambda(\mathbf{h}), \tag{4.3}$$

whenever the limit exists.

Existence of this limit has been shown in [10] for nearest neighbour positive interactions ($J_{ij}=0$ if $|i-j|>1$ and $J_{ij}>0$ if $|i-j|=1$) and deterministic uniform $\mathbf{h}$. In [7] it has been proved that for translation invariant ferromagnetic finite range coupling ($J_{ij}=J(i-j)\geqslant 0$ and, for some r, $J(k)=0$ whenever $|k|>r$) and for stationary ergodic non-negative random field $\mathbf{h}=\{h_i\}_{i\in W}$ for which $Eh_1<\infty$ the limit (4.3) exists with probability 1 and in mean.

Let $\mathbf{x}=\{x_n\}_{-\infty}^{+\infty}$ be a sequence of real numbers, by θ we denote the shift transformation defined as $\theta\mathbf{x}=\mathbf{y}$ for which $y_n=x_{n+1}$. We will use the following result, which was proved for more general situation in [7].

LEMMA 4.1. *Let $J>0$ and $\mathbf{h}=\{h_n\}_{-\infty}^{+\infty}$ be a sequence of positive real numbers. Then*

$$\Delta\tau_\Lambda(\mathbf{h})+\Delta\tau_\Lambda(\theta\mathbf{h})\geqslant\Delta\tau_\Lambda(h\vee\theta\mathbf{h})+\Delta\tau_\Lambda(h\wedge\theta\mathbf{h}).$$

Proof. For a proof see [7], Proposition 1. ∎

Now we state and prove the main result of this section.

THEOREM 4.1. *Let $J>0$ and $\mathbf{h}=\{h_n\}_{-\infty}^{+\infty}$ be a stationary ergodic sequence of positive random variables such that $Eh_0^2<\infty$. Then*

$$\Delta\tau(\mathbf{h})\geqslant\int_0^\infty\Delta\tau(x)\,dF(x),$$

where F is the distribution function of h_0 and $\Delta\tau(x)$ is the differential wall tension for the constant surface field $\mathbf{x}=\{x\}_{-\infty}^{+\infty}$.

Proof. For fixed realization $\mathbf{h}(\omega)$ of the surface field from Lemma 4.1 we have the inequality

$$\Delta\tau_\Lambda\big(\mathbf{h}(\omega)\big)+\Delta\tau_\Lambda\big(\theta\mathbf{h}(\omega)\big)\geqslant\Delta\tau_\Lambda\big(h\vee\theta\mathbf{h}(\omega)\big)+\Delta\tau_\Lambda\big(h\wedge\theta\mathbf{h}(\omega)\big).$$

Observe that the last inequality may be written as

$$\frac{1}{2}\big[\Delta\tau_\Lambda\big(\mathbf{h}(\omega)\big) + \Delta\tau_\Lambda\big(\theta\mathbf{h}(\omega)\big)\big] \geqslant E_{h(\omega)}\Delta\tau_\Lambda(\mathbf{Z}^{(1)}),$$

where $E_{h(\omega)}$ denotes that expectation taken for $\mathbf{Z}^{(1)}$ with initial condition $\mathbf{h}(\omega)$. Iterating the last inequality n times we get

$$\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\Delta\tau_\Lambda\big(\theta^i\mathbf{h}(\omega)\big) \geqslant E_{h(\omega)}\Delta\tau_\Lambda(\mathbf{Z}^{(n)}). \tag{4.4}$$

Taking expectation of both sides of the inequality (4.4) we get for all $n \geqslant 1$

$$E\Delta\tau_\Lambda(\mathbf{h}) \geqslant E\Delta\tau_\Lambda(\mathbf{Z}^{(n)}).$$

From Theorem 2.3 we know that $\mathbf{Z}^{(n)} \xrightarrow{\mathcal{D}} \mathbf{Z}$ as $n \to \infty$. Because $\mathbf{Z}$ is distributed on constant sequences and $\Delta\tau_\Lambda(\mathbf{h})$ is the function of only finite number of terms of the sequence $\mathbf{h} = \{h_n\}_{-\infty}^{+\infty}$ we have $P(\mathbf{Z} \in D_{\Delta\tau_\Lambda}) = 0$, where $D_{\Delta\tau_\Lambda}$ is the set of discontinuities of the mapping $\Delta\tau_\Lambda(\mathbf{h})$. Hence from [3], Theorem 5.1, we conclude that

$$\Delta\tau_\Lambda(\mathbf{Z}^{(n)}) \xrightarrow{\mathcal{D}} \Delta\tau_\Lambda(\mathbf{Z}) \quad \text{as } n \to \infty.$$

Now we apply [3], Theorem 5.3, and get

$$E\Delta\tau_\Lambda(\mathbf{h}) \geqslant \liminf_{n\to\infty} E\Delta\tau_\Lambda(\mathbf{Z}^{(n)}) \geqslant E\Delta\tau_\Lambda(\mathbf{Z}) = \int_0^\infty \Delta\tau_\Lambda(x)\,dF(x), \tag{4.5}$$

where F is the distribution function of h_0 and $\Delta\tau(x)$ is the differential wall tension for the constant surface field $\mathbf{x} = \{x\}_{-\infty}^{+\infty}$. The last equality follows from the form of the distribution of $\mathbf{Z}$. From [7], Proposition 1, we know that, under the condition of Theorem 4.1, $\Delta\tau(\mathbf{h})$ exists and that

$$\lim_{\Lambda\to\infty} E\Delta\tau_\Lambda(\mathbf{h}) = \Delta\tau(\mathbf{h}). \tag{4.6}$$

In order to complete the proof we notice that $\int_0^\infty x\,dF(x) < \infty$ and, for $x \geqslant 0$, $\Delta\tau_\Lambda(x) \leqslant 2x$. From [10] or [7] it follows that the limit

$$\lim_{\Lambda\to\infty} \Delta\tau_\Lambda(x) = \Delta\tau(x)$$

exists, so we may apply the Lebesgue theorem to get

$$\lim_{\Lambda\to\infty}\int_0^\infty \Delta\tau_\Lambda(x)\,dF(x) = \int_0^\infty \lim_{\Lambda\to\infty}\Delta\tau_\Lambda(x)\,dF(x) = \int_0^\infty \Delta\tau(x)\,dF(x). \tag{4.7}$$

Finally, from (4.5)–(4.7) we have

$$\Delta\tau(\mathbf{h}) \geqslant \int_0^\infty \Delta\tau(x)\,dF(x).$$

This completes the proof of Theorem 4.1. ■

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